SAND2013-10342C

Implicit Anisotropic Thermal Diffusion

Thomas A. Gardiner

Sandia National Laboratories

November 12, 2013

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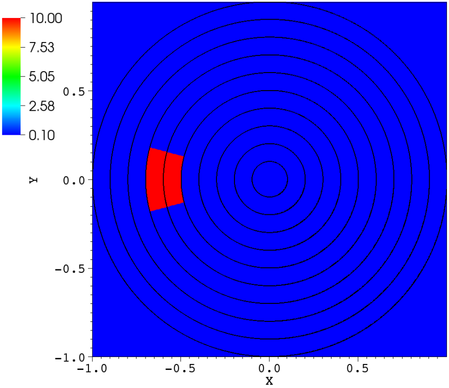
Motivation

- Thermal diffusion in magnetized systems is anisotropic.
- This results in important physical phenomena, e.g.
  - Magnetothermal Instability
  - Inhibiting radial energy transport in MAGLIF
- $\chi_\perp \simeq 10^{-5}\chi_\parallel$ in the compressed fuel in MAGLIF point design.
- For $\chi_\perp \leq 10^{-2}\chi_\parallel$ solutions can violate the entropy condition.
- Heat flows from cold to hot zones!
- Implicit algorithms are presented for solving highly anisotropic thermal diffusion that satisfy the entropy condition.

Outline

- Motivate this work and characterize solution accuracy:
  1. Heat flow in a flux tube
  2. Evolution of a sinusoidal temperature perturbation
- Tests will be run with 4 different algorithms
  1. Centered Difference
  2. Implicit version of Sharma & Hammett JCP **227**, 123 (2007).
  3. Flux limited Finite Volume
  4. Flux limited Discontinuous Galerkin
- Flux based diffusion
- Monotonicity Constraint

Heat Flow in a Flux Tube



- $\chi_{\parallel} = 10^{-2}$, $\chi_{\perp} = 0$
- $T_{\mathrm{cold}} = 0.1$, $T_{\mathrm{hot}} = 10$
- Cross-field diffusion is unresolved and temperature is discontinuous
- Heat should remain confined by magnetic field.
- L1 error in thermal energy outside of flux tube captures cross-field diffusion.

Parrish & Stone ApJ **633**, 334 (2005).

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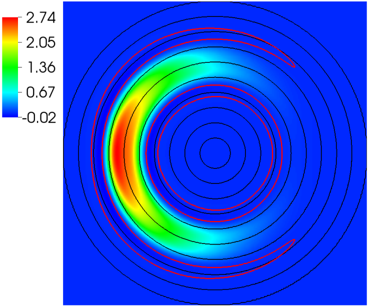
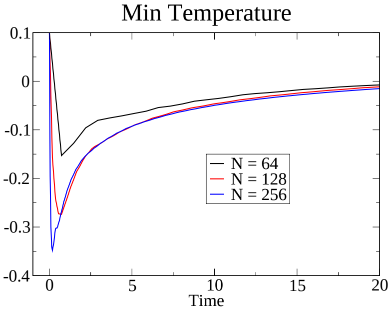
Centered Difference $\mathrm{T_{min}}$ Error

- Regions interior to red contours have $\mathrm{T} < \mathrm{T_{min}}$.
- $\mathrm{T_{min}}$ / $\mathrm{T_{max}}$ solution errors increase with resolution.

Time = 20, CFL = 10, $\theta = 1$

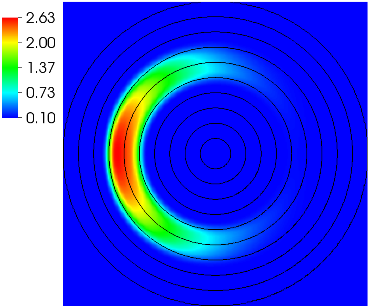


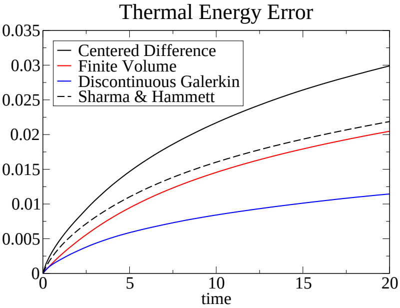


Perpendicular Heat Flux Error

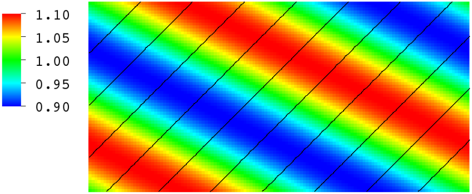
- Perpendicular diffusion is reduced by $2-3\times$ relative to Centered Difference.
- Sharma & Hammett JCP **227**, 123 (2007) algorithm is comparable to the limited Finite Volume algorithm.

Time $=20$, CFL $=10$, $\theta=1$





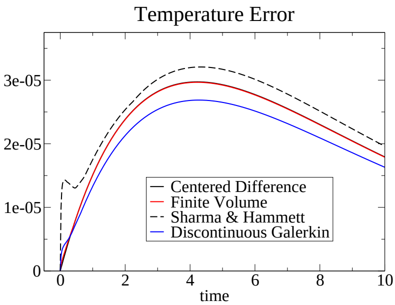
Sinusoidal Temperature Perturbation

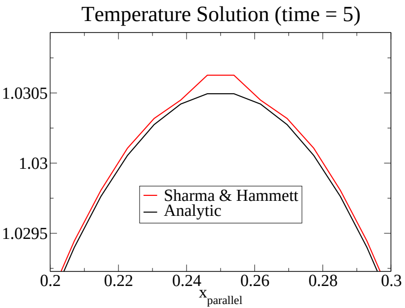


- $\chi_{\parallel} = 10^{-2},\ \chi_{\perp} = 0$
- $B_x = B_y = 1,\ B_z = 0$
- $T = T_0 + T_1 \sin(\mathbf{k} \cdot \mathbf{x})$

- $k_x = 2\pi \cos(\alpha),\ k_y = 2\pi \sin(\alpha)$ where $\alpha \sim 63.4^\circ$
- Solution $T_1 = \delta T \exp(-\gamma t)$ where $\gamma = \frac{\chi_{\parallel} (\mathbf{k} \cdot \mathbf{b})^2}{\rho C_v}$
- Temperature error is easily computed as a function of time.
- We'll use Crank-Nicholson integration at CFL=1.
- Assess the impact of monotonicity constraints by varying $(\mathbf{B}, \mathbf{k})$.

Sinusoidal Temperature Evolution





- Discontinuous Galerkin solution is an improvement over Centered Difference.
- Flux Limited Finite Volume is nearly identical to Centered Difference.
- Sharma & Hammett's algorithm shows slightly higher errors than Centered Difference due to limiting at extrema.

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Flux Based Diffusion

- Starting with the thermal diffusion equations

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot q = 0$$
$$q = -\Big(\chi_{\parallel} bb + \chi_{\perp}(1 - bb)\Big) \cdot \nabla T$$

- Define the time-integrated heat flux

$$Q = \int_{t^n}^{t} q(t')dt'$$

- The thermal diffusion equations change roles, becoming

$$\epsilon - \epsilon^n + \nabla \cdot Q = 0$$
$$\frac{\partial Q}{\partial t} = -\Big(\chi_{\parallel} bb + \chi_{\perp}(1 - bb)\Big) \cdot \nabla T$$

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Relaxation Operator

- Construct a predictor / corrector relaxation scheme as

$$\frac{1}{\chi_\parallel}\frac{\partial Q}{\partial t} + \nabla T^c = \left(\frac{\chi_\parallel - \chi_\perp}{\chi_\parallel}\right)(1 - bb)\cdot\nabla T^p$$

- Formally consider $\partial_t Q \to 0$

$$\begin{aligned} \nabla T^c &= \left(\frac{\chi_\parallel - \chi_\perp}{\chi_\parallel}\right)(1 - bb)\cdot\nabla T^p \\ b\cdot\nabla T^c &= 0 \\ (1 - bb)\cdot\nabla T^c &= \left(\frac{\chi_\parallel - \chi_\perp}{\chi_\parallel}\right)(1 - bb)\cdot\nabla T^p \end{aligned}$$

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Relaxation with Flux Limiting

- At an interface with normal n and tangential τ directions

$$\frac{1}{\chi_\parallel}\frac{\partial Q_n}{\partial t} + \nabla_n T^c = S_n + S_\tau$$

$$S_n = \left(\frac{\chi_\parallel - \chi_\perp}{\chi_\parallel}\right)(1 - b_n^2)\nabla_n T^p$$

$$S_\tau = -\left(\frac{\chi_\parallel - \chi_\perp}{\chi_\parallel}\right) b_n b_\tau \cdot \nabla T^p$$

- S_n is evaluated with the same discretization as $\nabla_n T^c$.
- S_τ is evaluated using limited Finite Volume gradients, Discontinuous Galerkin gradients, Moments, etc.
- S_τ is limited to enforce monotonicity.
  - Temperature Monotonicity $\rightarrow$ Q^{n+1} bounds $\rightarrow$ S_τ bounds

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Conclusions

- Implicit algorithms for anisotropic thermal diffusion that satisfy the entropy condition were presented.
- These methods behave well for $0 \leq \chi_\perp \leq \chi_\parallel$.
- These are effective for wide ranging CFL $\sim (10, 10^2, 10^3, \ldots)$
- Discontinuous Galerkin formulation is generally more accurate than the Finite Volume formulation.
- This solution algorithm is generally applicable to any conductivity tensor.
- Multigrid methods are effective for acceleration.