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Is submodularity testable?

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Joint work with
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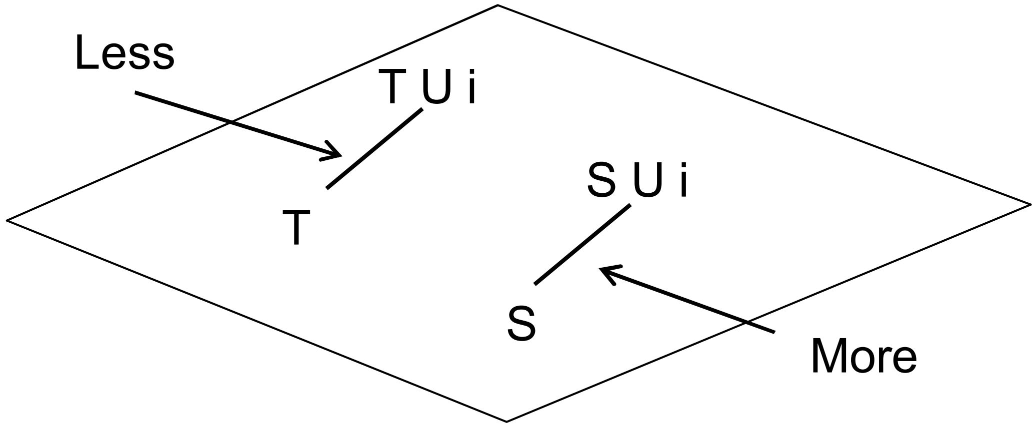
Submodularity



- f: {0,1}^n -> R (domain is subset of universe [n])

$$f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$$

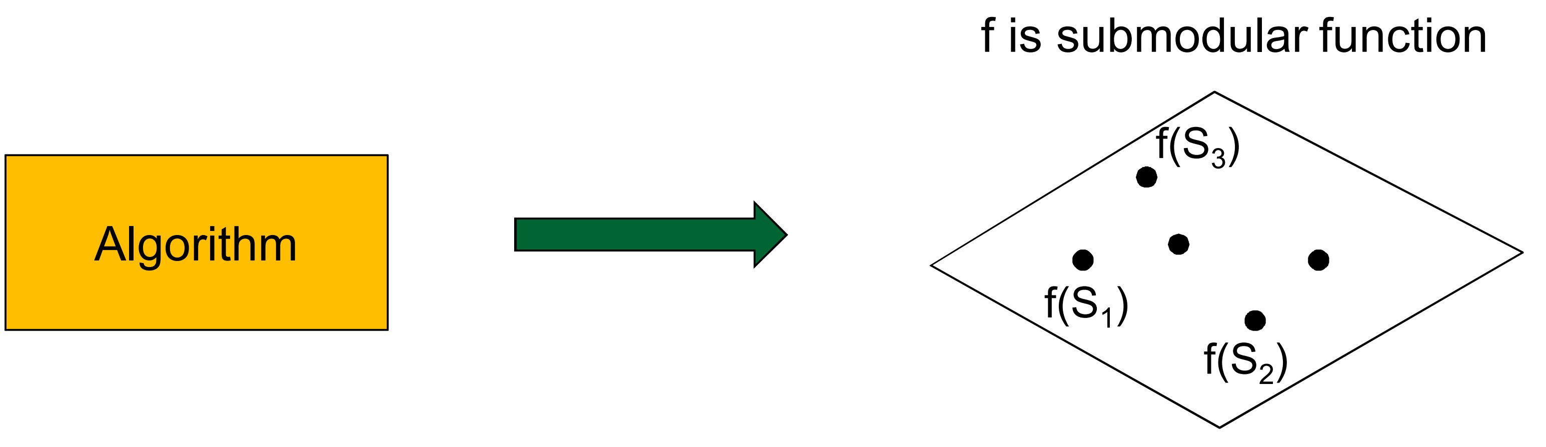
- Monotonicity (decreasing) of marginal utilities

  For S $\subseteq$ T, i not in S, T

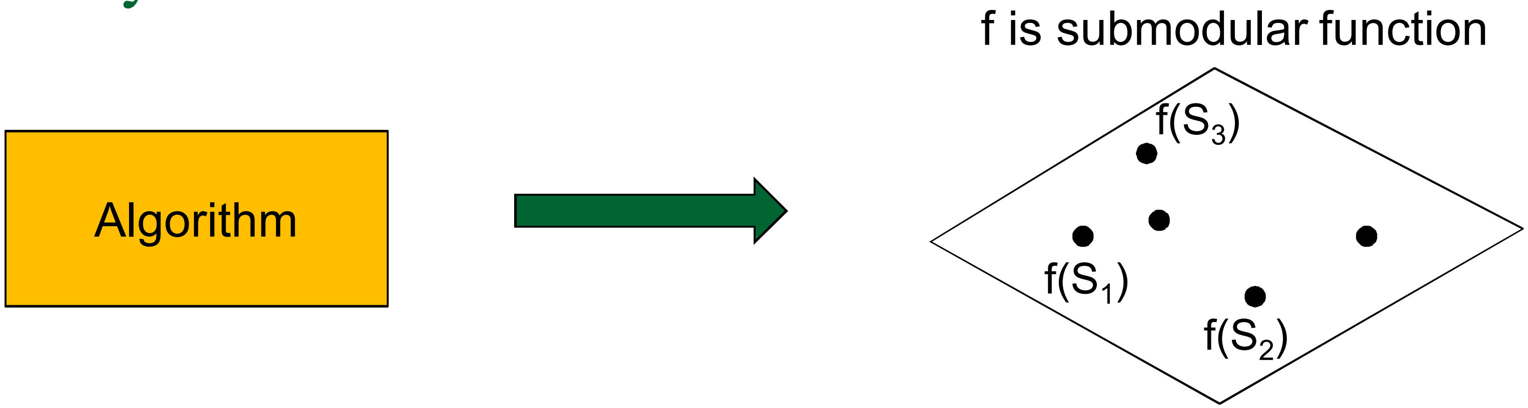
$$f(T \cup i) - f(T) \leq f(S \cup i) - f(S)$$

Use of submodularity

- Submodularity comes up a LOT
  - Combinatorial optimization, modelling utilities…
- Host of algorithms that use submodular functions
  - Maximization, minimization, etc.

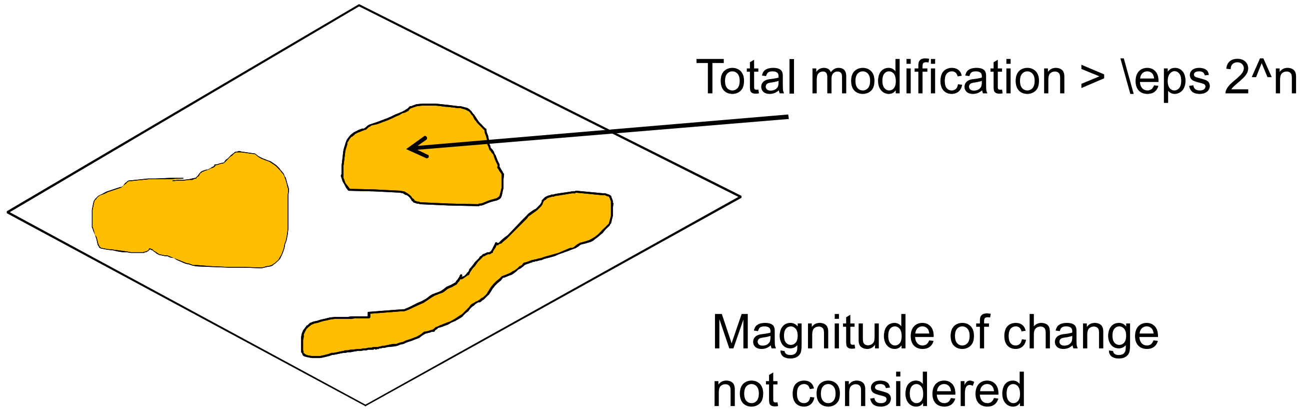


Everyone is sublinear!



- Domain of f is {0,1}^n
- Algorithms run in poly(n) time
  - This is sublinear in the “size” of f, which is 2^n
- Let’s study submodularity from sublinear algorithms perspective

Distance to submodularity



Magnitude of change
not considered

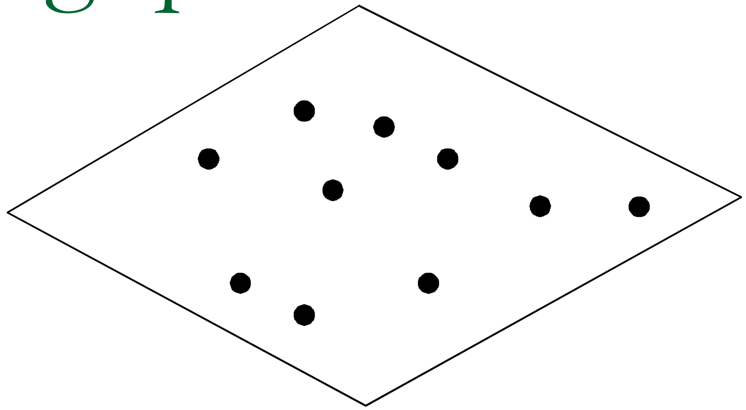
- f is eps-far from being submodular if:

  f has to be modified at an \eps-fraction (\eps 2^n) of domain to make f submodular
- We’re looking at Hamming distance

  dist(f,g) = (# S, s.t. f(S) \neq g(S)) / 2^n
- distance of f to submodularity

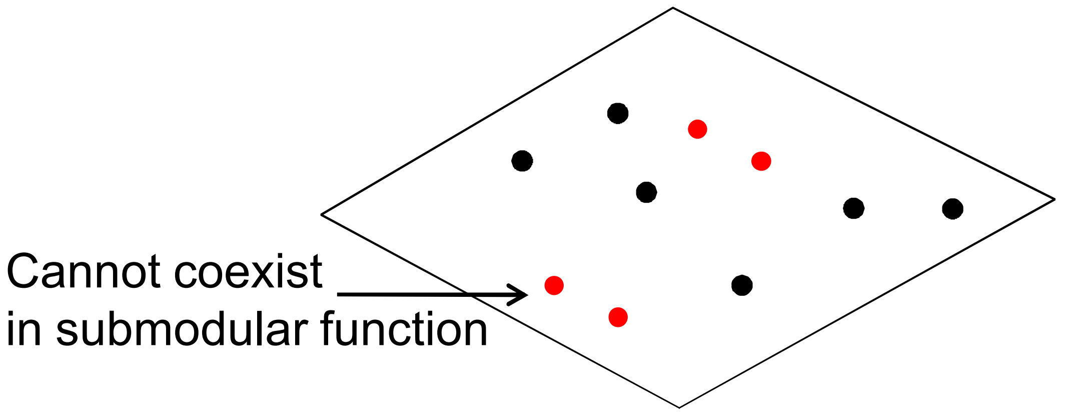
  min_{g submod} dist(f,g)

The testing question



- Is there a property tester for submodularity?
  - Studied by [Parnas-Ron-Rubinfeld 04] over grids
- Given f that is eps-far from submodular, is there a $(n/eps)^{O(1)}$ procedure that certifies f is not submodular?
  - Procedure is randomized
  - One-sided testers more interesting here

The testing question



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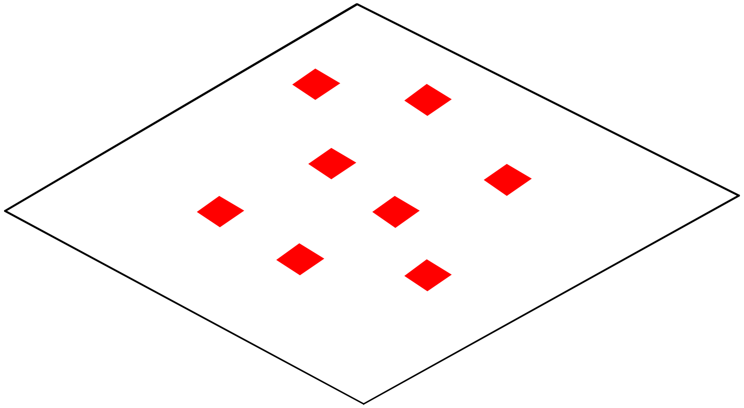
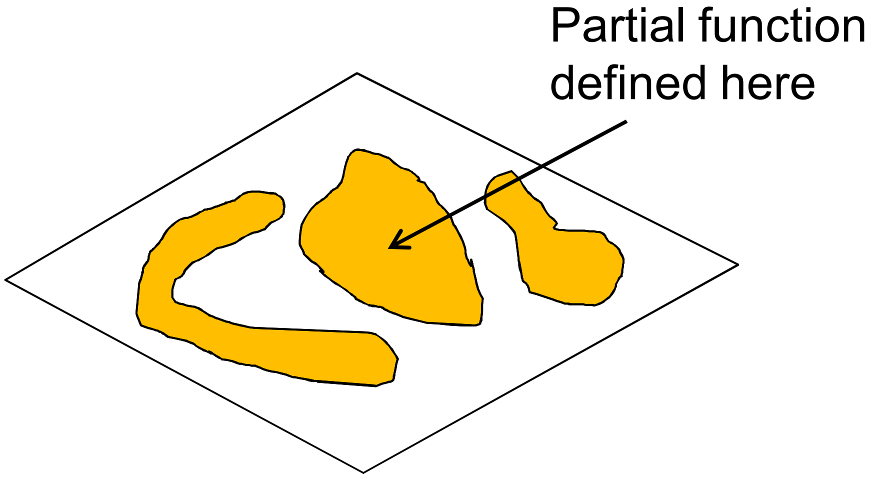
Structural questions

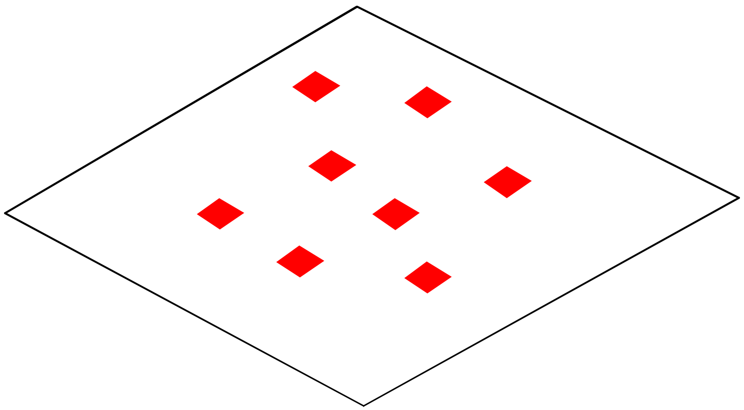


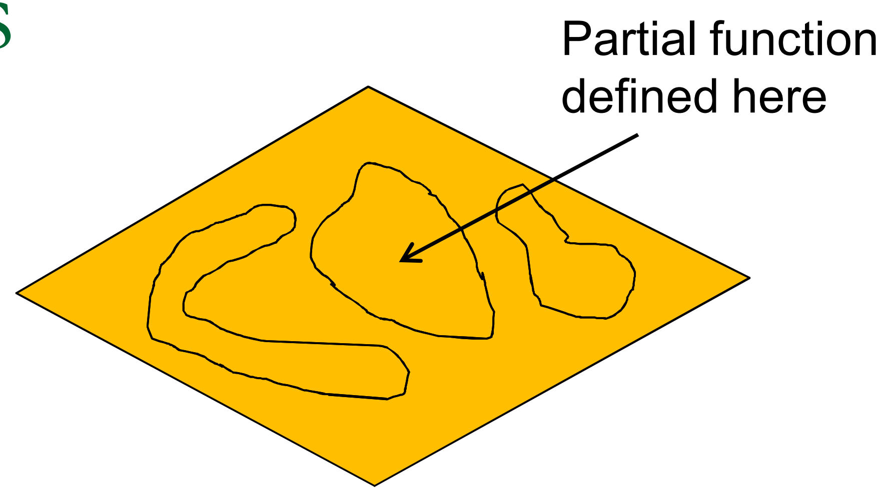


- How do local violations to submodularity relate to distance?
  - Does eps-far mean many violations?

- When can partial function be completed?
  - Fill in remaining values to get a submodular function

Structural questions





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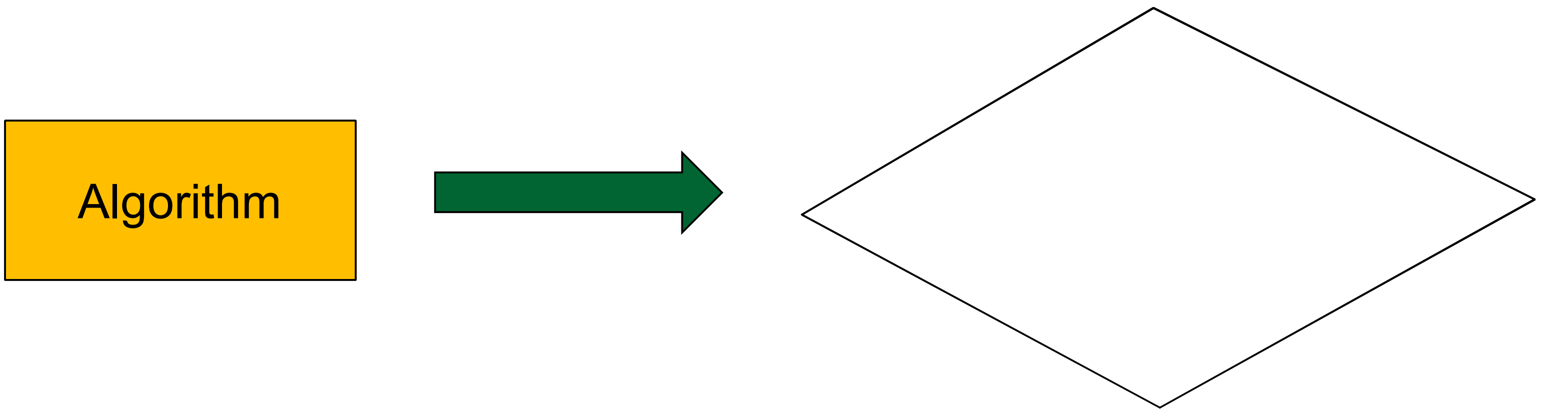
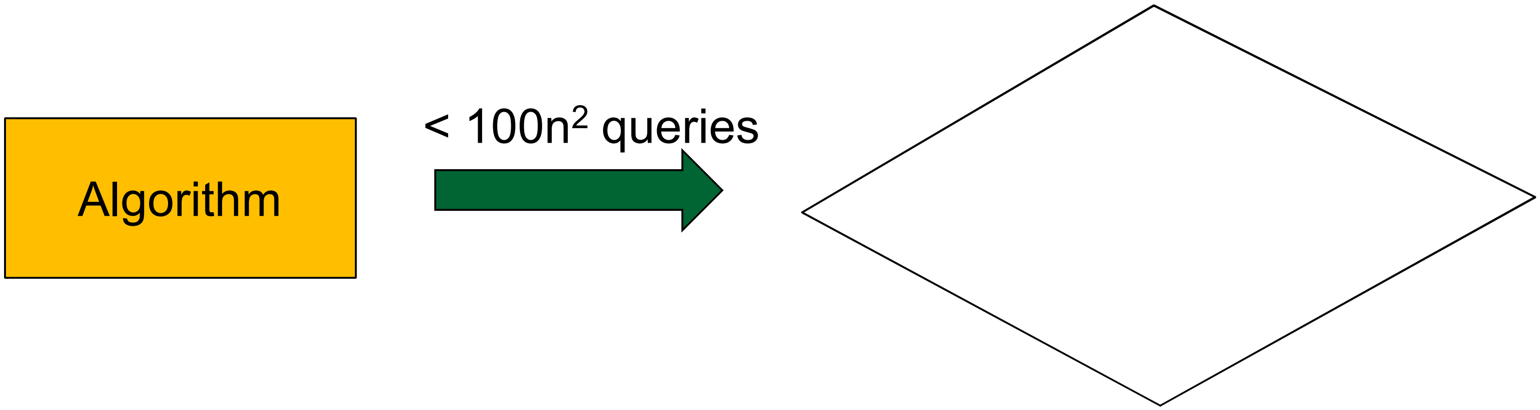
Some motivation…?



- All algorithms on submodular functions are sublinear
- If submodularity not testable, then how can these algorithms use submodularity?
  - Suppose (n/eps)^2 lower bound for testing

Some motivation…?



- If submodularity not testable, then how can these algorithms use submodularity?
  - Suppose $(n/eps)^2$ lower bound for testing
- How does this algorithm use submodularity?
  - Could get fooled by f that is 1/10-far from submodular

Some motivation…?

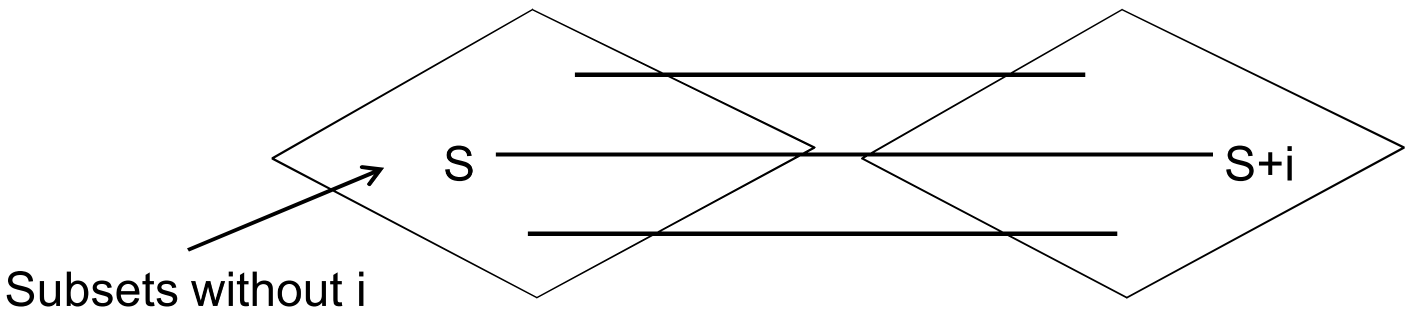
Algorithm

Testing submodularity intimately related to the "use" of submodularity

The "sublinear lens" might shed light on structure of submodularity

- If subm… these algorith…
  - Suppose (n… …g
- How does this al… …submodularity?
  - Could get fooled … that is 1/10-far from submodular

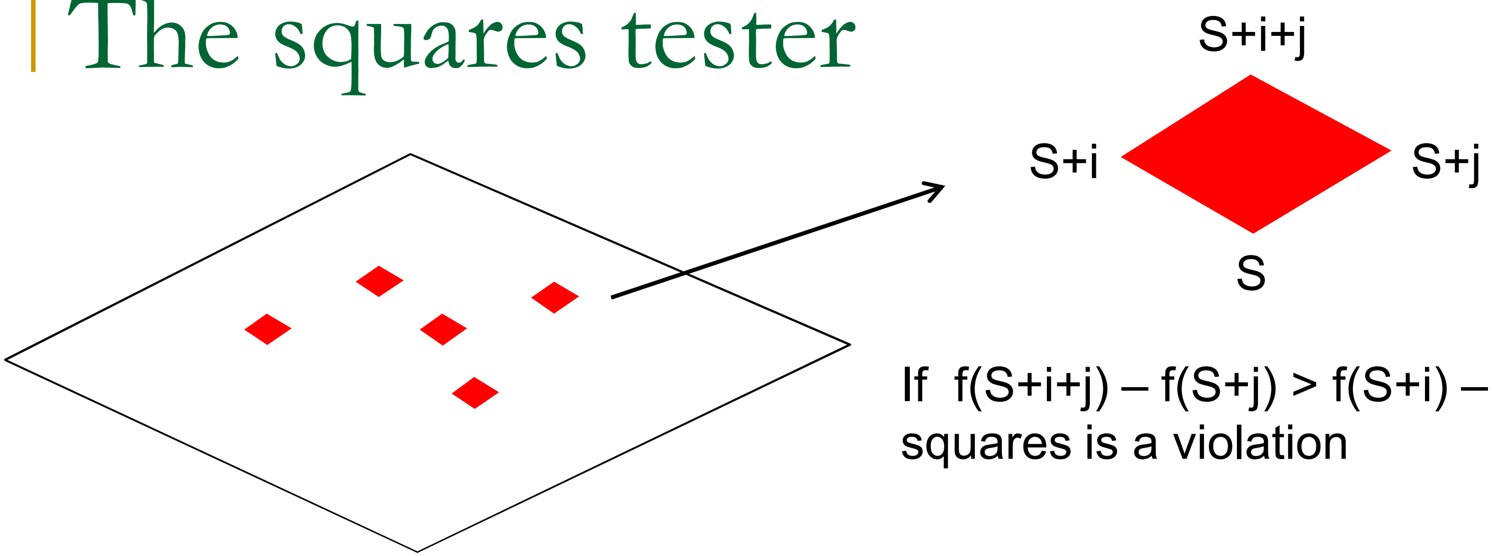
The basic tester



- $f_i(S)$ is defined as f(S U i) – f(S)
  - Marginal utility functions, for each i
  - Domain for fi is {0,1}^{n-1}

- f is submodular iff all f_i are monotone decreasing
  - So use the monotonicity tester for each f_i

The squares tester



If $f(S+i+j) - f(S+j) > f(S+i) - f(S)$
squares is a violation

- Take small uniform sample of squares
  - Check submodularity constraint on each squares

- Density of violated squares = (# violated sq)/ (total sq)
- If dist to submod is eps, what is density of violated squares?

Results

- If f is eps-far from submodular

  Violated squares density > eps

  - There is sublinear tester for submodularity ($o(2^n)$).

- [Surprise 1] For any eps > 2^{-n/10}, there is an f such that

  Violated squares density < eps^4.8

- Testing monotonicity reduces to testing submodularity

What Surprise 1 means

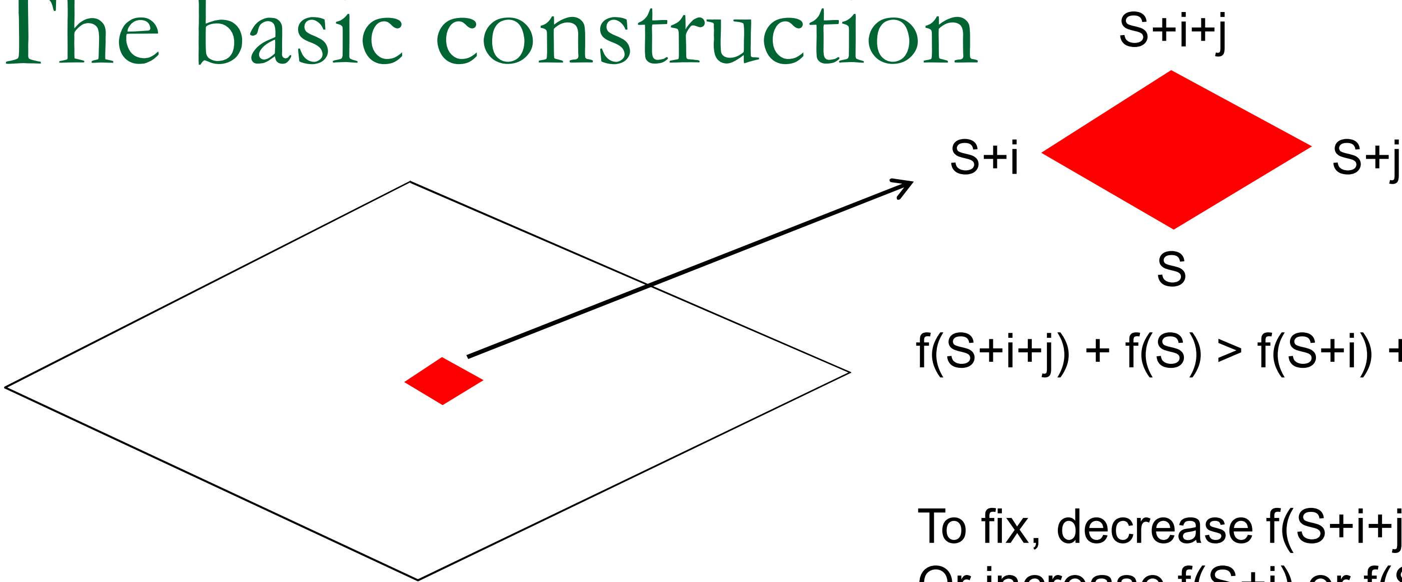
- [Surprise 1] For any eps > 2^{-n/10}, there is an f such that

  Violated squares density < eps^4.8

- For monotonicity, violated edges > eps/n^2
- Major difference between testing both

- Distance of f to submodularity is “large”, but all f_i are “close” to being monotone
  - Marginal utility functions, for each i: $f_i(S)$ is defined as f(S U i) – f(S)

The basic construction

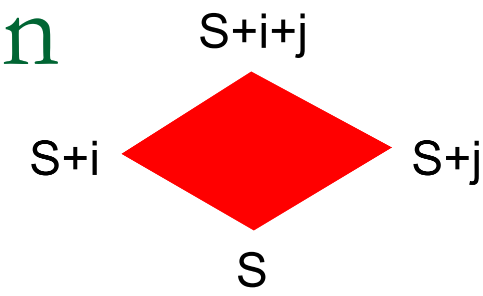


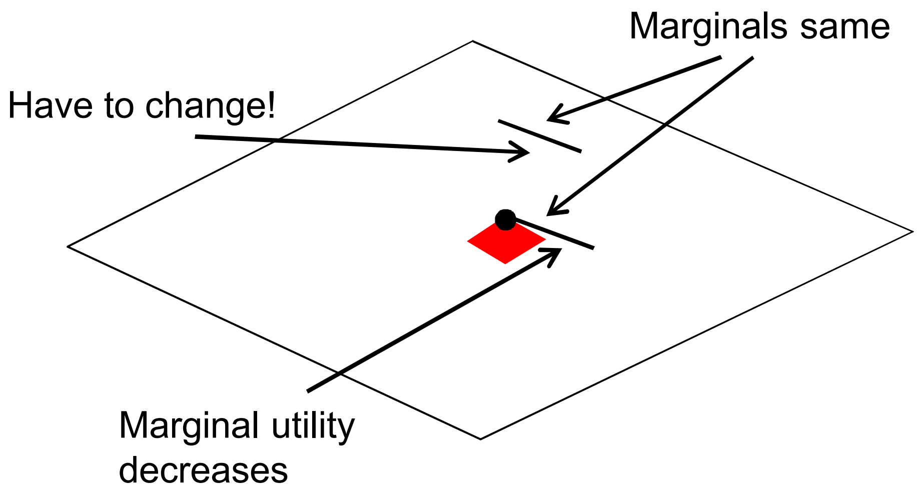
f(S+i+j) + f(S) > f(S+i) + f(S+j)

To fix, decrease f(S+i+j) or f(S)
Or increase f(S+i) or f(S+j)

- There is f s.t.: f has ONE violated square, but to make f submodular 2^{n/2} values must be changed

The basic construction





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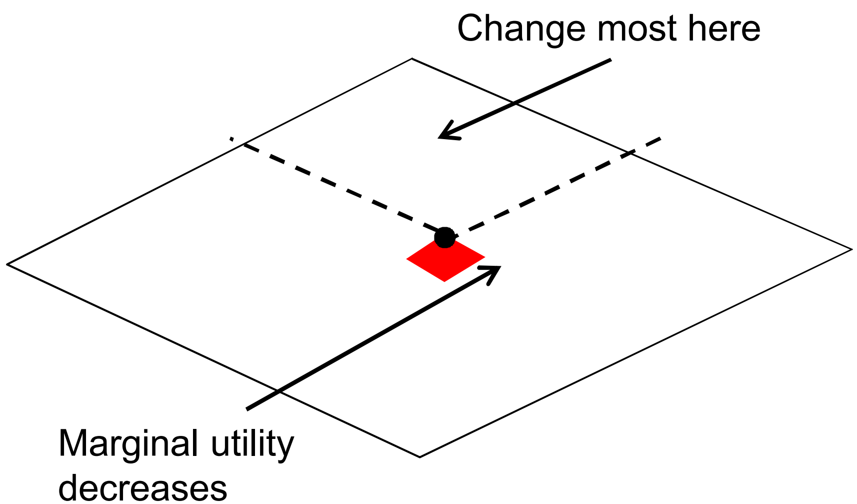
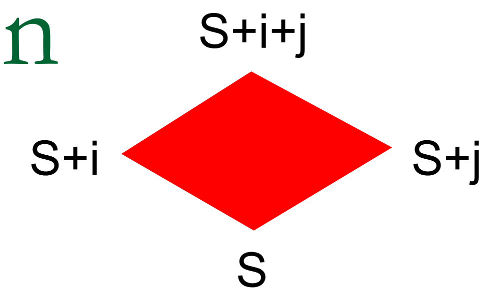


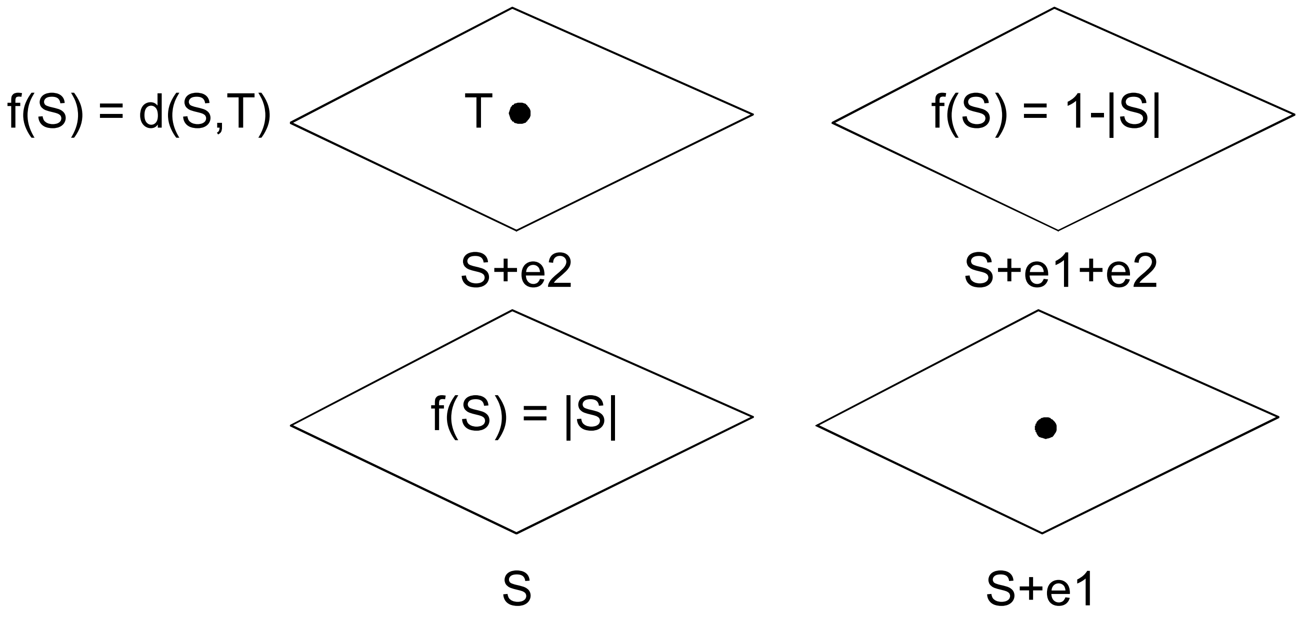


f(S+i+j) + f(S) > f(S+i) + f(S+j)

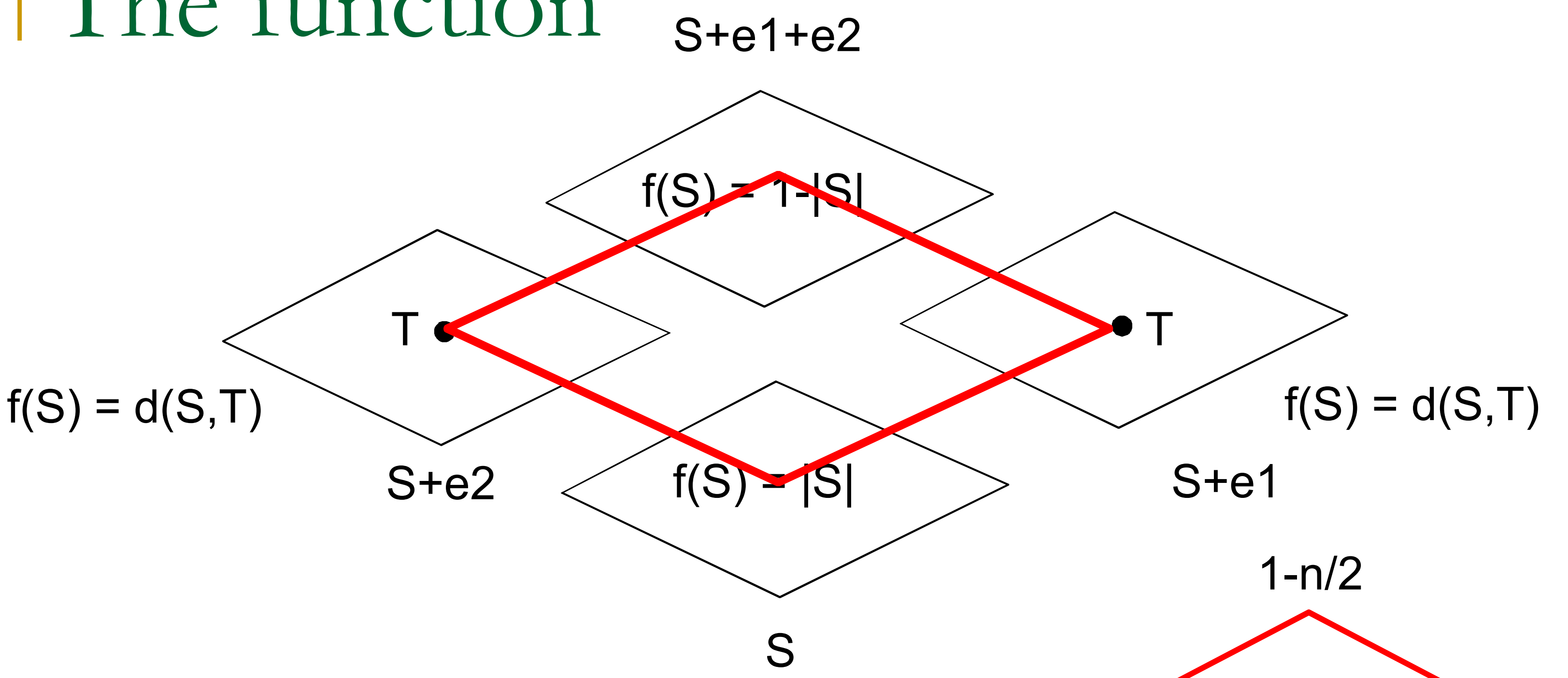
To fix, decrease f(S+i+j) or f(S)
Or increase f(S+i) or f(S+j)

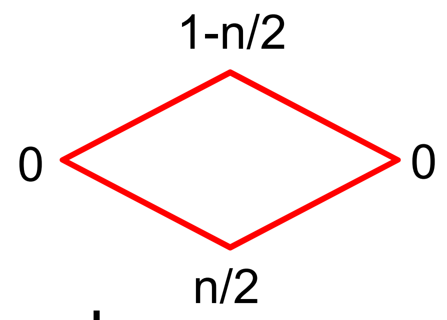
- There is f s.t.: f has ONE violated square, but to make f submodular 2^{n/2} values must be changed
- Plant a single violated square in middle, with most marginal values same

The function



The function



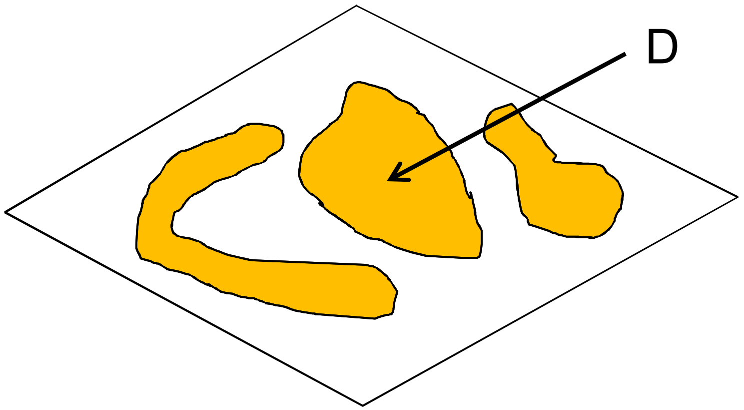


- Only 1 violated square
- Have to change 2^n/2 values to make submodular

Generalize

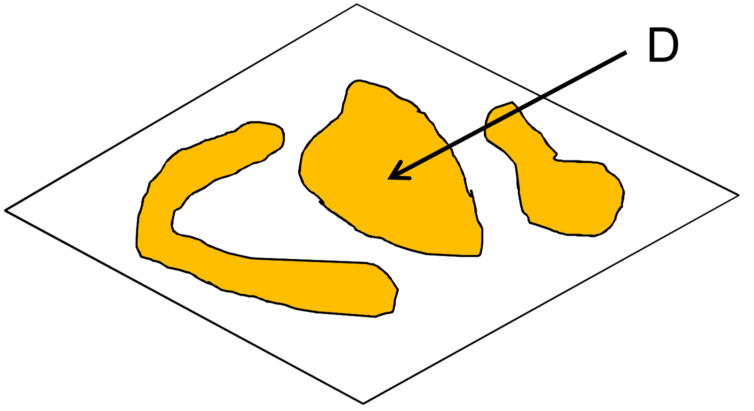
- Few violated squares
  - Have to change many values to make submodular
- Connection between lattice structure and submodular functions
  - Paste together many linear functions over hypercubes to ensure most marginal values same
- Density of violated squares < eps^{4.8}

Extendable functions



- f is defined on some subset D of domain
- Can we fill in rest of the values to get submodular function?
  - If yes, f is submodular-extendable
- If not, is there small certificate of that fact?

Contrast: monotonicity



- f is monotone if for S subset T, f(S) \leq f(T)
- f is monotone-extendable, if for all defined S subset T, f(S) \leq f(T)
- If f not monotone-extendable, there is certificate of size 2 (S subset T, f(S) > f(T))

Extendability in general

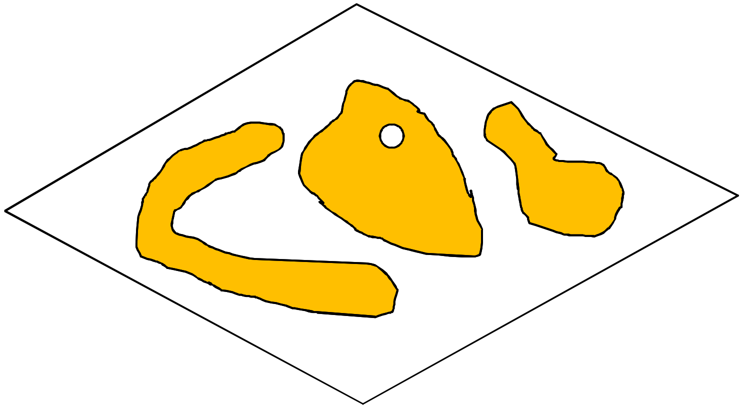
- Element of property testing proofs
  - Minimally modifying an input that passes tester to satisfy property
- Monotonicity tester works because certificates are so small
  - Happens for many other properties

- Is there analog for submodularity?

  Maybe if f not extendable, there are defined S, T, S U T, S T

  s.t.

No!

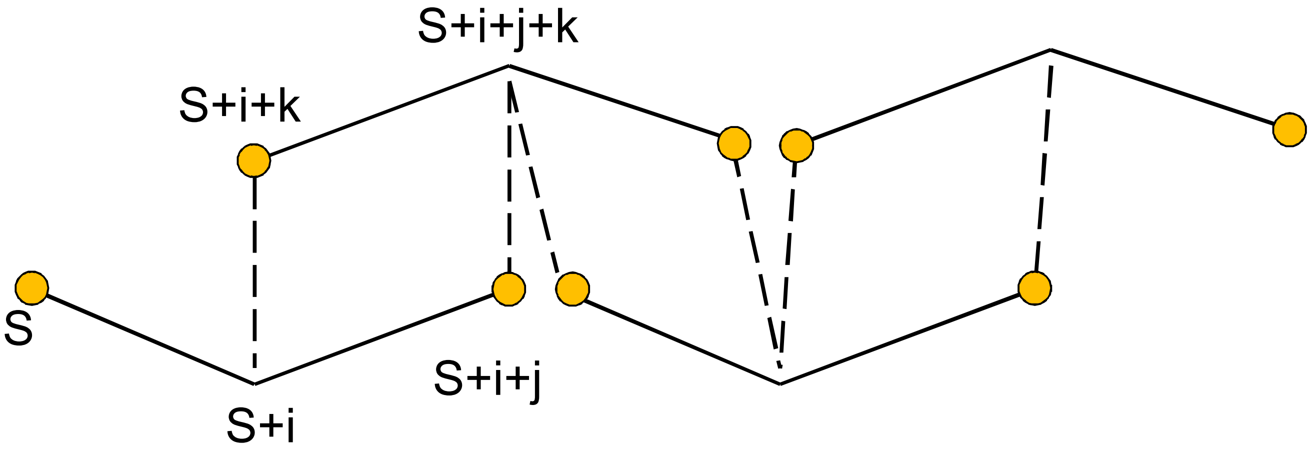


Not extendable
Extendable!

- f is defined on 2^{n/4} domain points
- f is not submodular-extendable
- But remove ANY defined point, f becomes extendable

- No small certificates!
  - Bad news for property testing…?

A small glimpse



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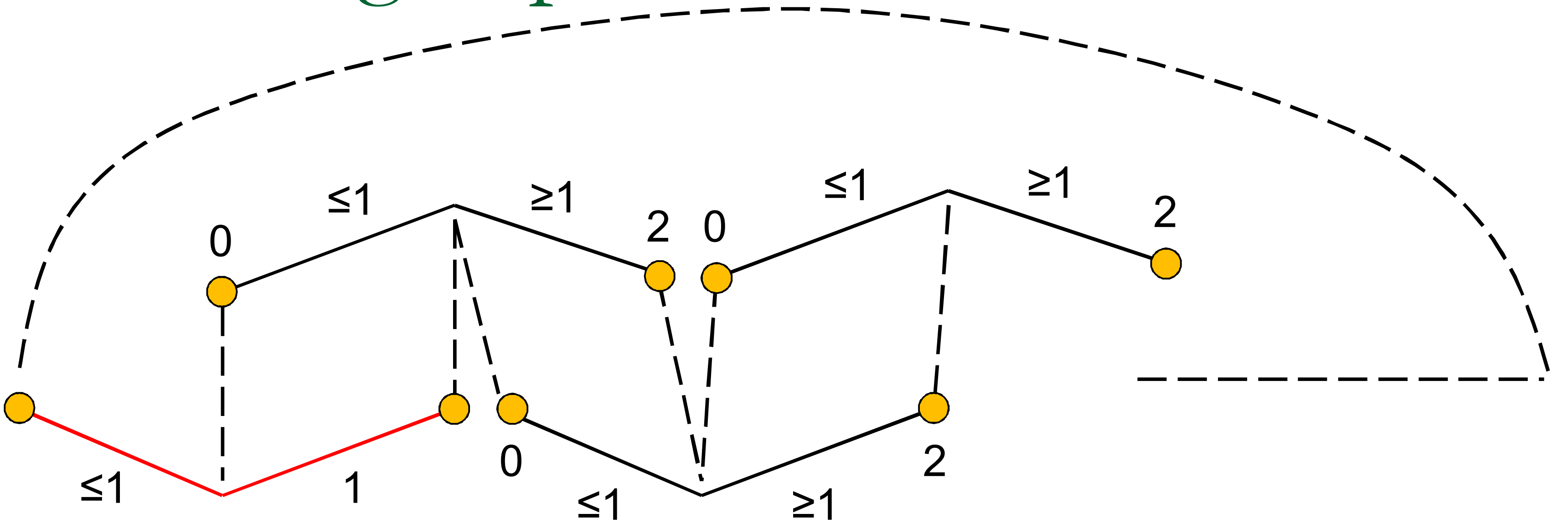

- Long unsatisfied cycle of linear inequalities
  - Break anywhere, and satisfy at ease!

So…

- Study of testing submodularity
  - Surprisingly different from monotonicity
  - Relation between violated squares and distance
  - Testing seems to be a difficult question. We don't know the answer. Help!

- Restricted versions
  - When marginals are 0-1: rank function of matroid
  - Testing whether a set system is matroid

- We don't have good understanding of structure in submodular functions