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A Modified Error in Constitutive Equations Approach for Inverse Problems in Frequency Domain Elastodynamics

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Outline

- Motivation
- Formulation and Solution methods
  - Classical L2 Minimization Approach
  - Error in Constitutive Equation (ECE) Methods
- Numerical Examples
- Concluding Remarks

Direct Problem

Boundary-Value Problem

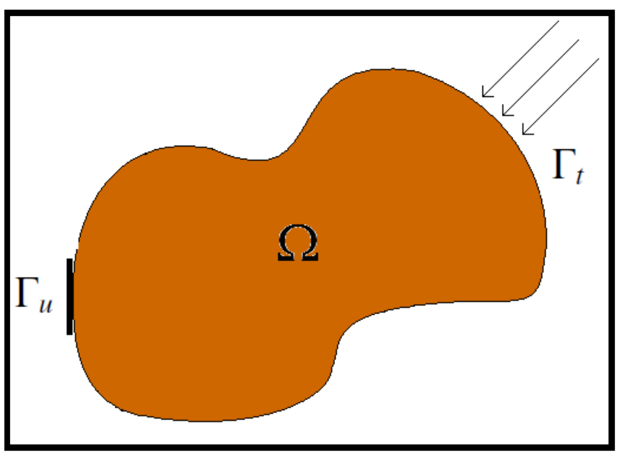
$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} - \rho\omega^2\mathbf{u} = \mathbf{0} \quad \text{in} \quad \Omega$$

$$\mathbf{u} = \mathbf{u_0} \quad \text{on} \quad \Gamma_u$$

$$\boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\tau} \quad \text{on} \quad \Gamma_\tau$$

$$\boldsymbol{\sigma} = \boldsymbol{C} : \mathbf{E}(\mathbf{u})$$

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right)$$



Discrete Form

$$\left([K] + i\omega[D] - \omega^2[M]\right)\{u\} = \{P\}$$

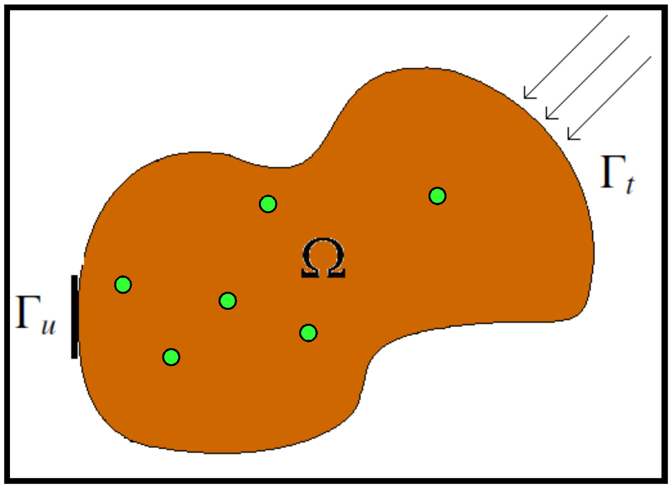
$$\mathbf{u} \approx [N]\{u\}$$

$$[K] = \int_\Omega [B]^{\mathrm{T}}[C][B]d\Omega$$

$$[M] = \int_\Omega \rho[N]^{\mathrm{T}}[N]d\Omega$$

$\{P\}$ is the force vector
$[D]$ is the damping matrix

Inverse Problem Statement



Sensor point

Find $\mathbf{u}$ and $\mathbf{C}$ such that governing dynamics equations are satisfied and $\mathbf{u} = \mathbf{u}_m$ in Ω_M.

$$\Omega_M = \{\mathbf{x}_i \in \Omega \cup \Gamma : \mathbf{x}_i \text{ is a sensor point}, i = 1, 2, ..., m\}$$

L2-Functional in Elastodynamics

Define

$$\|\mathbf{u} - \mathbf{u}_m\|^2_{L_2(\Omega_M)} = \int_{\Omega_M} |\mathbf{u} - \mathbf{u}_m|^2 d\Omega_M$$

Minimization Problem

$$\left(\hat{\mathbf{u}}, \hat{\boldsymbol{C}}\right) = \arg\min \tfrac{1}{2}\|\mathbf{u}(\mathbf{C}) - \mathbf{u}_m\|^2_{L_2(\Omega_M)}$$
$$\text{such that}$$
$$\left([K] + i\omega[D] - \rho\omega^2[M]\right)\{u\} = \{P\}$$

Modified Error in Constitutive Equations (MECE)

Modified Error in Constitutive Equations

$$J(\boldsymbol{\sigma}, \mathbf{u}, G, B) = \frac{1}{2}\Big[\int_\Omega \Big(\boldsymbol{\sigma} - [C]\mathbf{E}(\mathbf{u})\Big)^T [C]^{-1} \Big(\boldsymbol{\sigma} - [C]\mathbf{E}(\mathbf{u})\Big) d\Omega + \kappa\Big(\{u\} - \{u_m\}\Big)^T [Q]\Big(\{u\} - \{u_m\}\Big)\Big]$$

$$[C] - B[C_v] + G[C_d]$$

Dynamically Admissible Set

$$\mathcal{D} = \Big\{\boldsymbol{\sigma} : -\rho\omega^2[M]\{u\} + \{I(\boldsymbol{\sigma})\} = \{F\}\Big\}$$

$$\{I(\boldsymbol{\sigma})\} - \int_\Omega [B]^T \boldsymbol{\sigma} d\Omega$$

Kinematically Admissible Set

$$\mathcal{K} = \{\mathbf{u} : \mathbf{u} = \mathbf{g} \text{ on } \Gamma_u\}$$

Original method published by Feissel P and Allix O, Modified constitutive relation error identification strategy for transient dynamics with corrupted data: the elastic case *Computer Methods in Applied Methods in Engineering.* 196 1968–83, 2007.

MECE Elastodynamics (2)

Minimization problem

$$\left(\hat{\boldsymbol{\sigma}}, \{\hat{u}\}, \hat{G}, \hat{B}\right) = \arg\min J\left(\boldsymbol{\sigma}, \{u\}, G, B\right) \text{ s.t. } \boldsymbol{\sigma} \in \mathcal{D}, \mathbf{u} \in \mathcal{K}$$

Lagrangian

$$\mathcal{L}\left(\boldsymbol{\sigma}, \{u\}, G, B, \{w\}\right) = J\left(\boldsymbol{\sigma}, \{u\}, G, B\right) - \{w\}^{\mathrm{T}}\left(-\rho\omega^2[M]\{u\} + \{I(\boldsymbol{\sigma})\} - \{F\}\right)$$

KKT optimality conditions

$$\delta\mathcal{L} = 0 \quad \Rightarrow \quad \begin{aligned} D_{\boldsymbol{\sigma}}\mathcal{L} \cdot \delta\boldsymbol{\sigma} &= 0 \\ D_{\{u\}}\mathcal{L} \cdot \{\delta u\} &= 0 \\ D_{\{w\}}\mathcal{L} \cdot \{\delta w\} &= 0 \\ D_G\mathcal{L} \cdot \delta G &= 0 \\ D_B\mathcal{L} \cdot \delta B &= 0 \end{aligned}$$

MECE Elastodynamics (3)

Full KKT System

$$D_{\boldsymbol{\sigma}}\mathcal{L}\cdot\delta\boldsymbol{\sigma}=0 \implies \boldsymbol{\sigma}=[C]\mathbf{E}(\mathbf{u}+\mathbf{w})$$ Modified constitutive equation

$$D_{\{w\}}\mathcal{L}\cdot\{\delta w\}=0 \implies [K]\left(\{u\}+\{w\}\right)-\rho\omega^2[M]\{u\}=\{F\}$$ Direct problem

$$D_{\{u\}}\mathcal{L}\cdot\{\delta u\}=0 \implies \Big([K]-\rho\omega^2[M]\Big)\{w\}=\kappa[Q]^{\mathrm{T}}(\{u\}-\{u_m\})$$ Adjoint problem

$$D_G J\cdot\delta G=0 \implies \frac{\|\boldsymbol{\sigma}_d\|^2}{4G^2}-\|\mathbf{E_d}(\mathbf{u})\|^2=0$$

$$D_B J\cdot\delta B=0 \implies \frac{p^2}{B^2}-e^2=0$$

Notation

$$\boldsymbol{\sigma}_d-\boldsymbol{\sigma}-p\mathbf{I}$$

$$p=\frac{1}{3}\boldsymbol{\sigma}_{ii}$$

$$\mathbf{E_d}(\mathbf{u})=\mathbf{E}(\mathbf{u})-\frac{1}{3}e_u$$

$$e_u=\mathbf{E}(\mathbf{u})_{ii}$$

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MECE Elastodynamics (4)

Alternating Directions MECE Algorithm

At each iteration

i) Solve the coupled system

$$\begin{bmatrix} [K]-\rho\omega^2[M] & [K] \\ -\kappa[Q]^{\mathrm{T}} & [K]-\rho\omega^2[M] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ -\kappa[Q]^{\mathrm{T}}\{u_m\} \end{Bmatrix}$$

ii) Update the moduli

$$G_{k+1} = \frac{||\boldsymbol{\sigma}_d||_k}{2||\mathbf{E_d}(\mathbf{u})||_k}$$

$$B_{k+1} = \frac{||p||_k}{||e_u||_k}$$

MECE Elastodynamics (5)

Parallel Solution Strategy

At each MECE iteration we need to solve the coupled system

$$\begin{bmatrix} [K] - \rho\omega^2[M] & [K] \\ -\kappa[Q]^{\mathrm{T}} & [K] - \rho\omega^2[M] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ -\kappa[Q]^{\mathrm{T}}\{u_m\} \end{Bmatrix}$$

- We use successive over-relaxation (SOR) combined with Parallel Helmholtz solver FETI-H (Farhat et al)
- Allows to leverage existing parallel Helmholtz solver

Block SOR Combined with FETI-H

$$\begin{bmatrix} [K]-\rho\omega^2[M] & [K] \\ -\kappa[Q]^{\mathrm{T}} & [K]-\rho\omega^2[M] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ -\kappa[Q]^{\mathrm{T}}\{u_m\} \end{Bmatrix}$$

$$\begin{bmatrix} A & 0 \\ \omega C & A \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

SOR iterations

2 parallel Helmholtz solves: one for u, one for w

FETI-H algorithm

MECE Elastodynamics (4)

Alternating Directions MECE Algorithm

At each MECE iteration

At each SOR iteration

2 FETI-H solves for forward, adjoint solutions

End

Update the moduli

End

Numerical Examples

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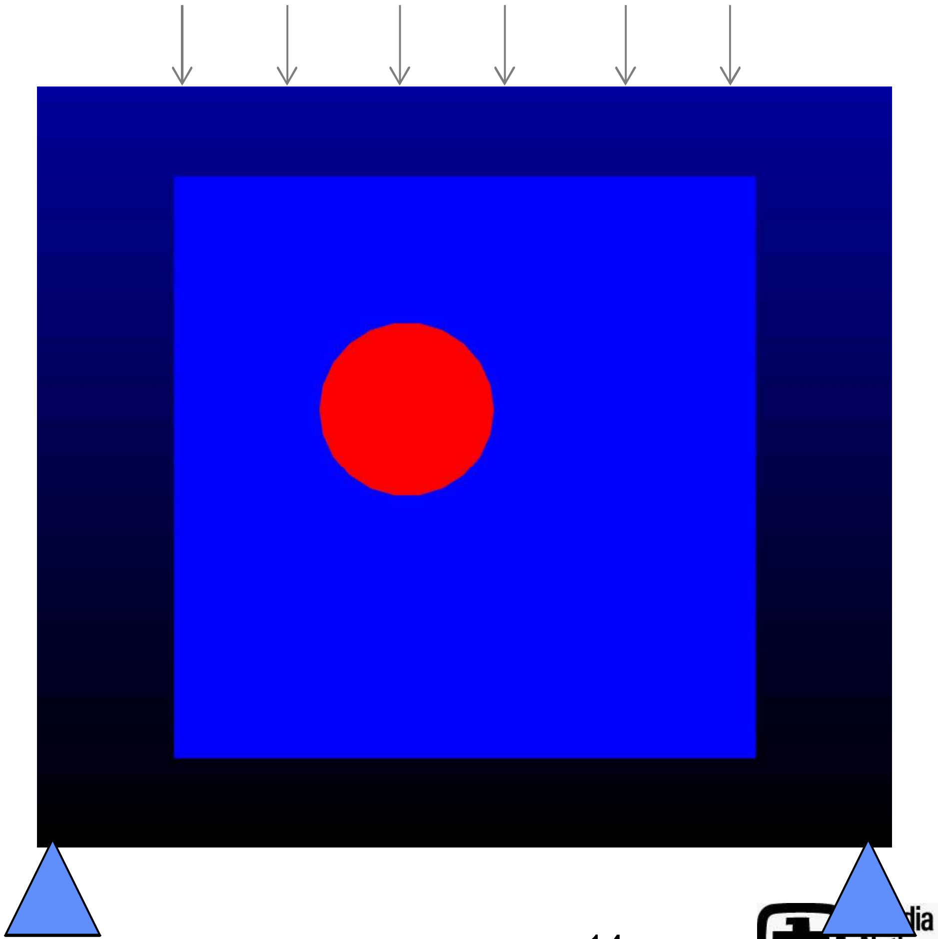
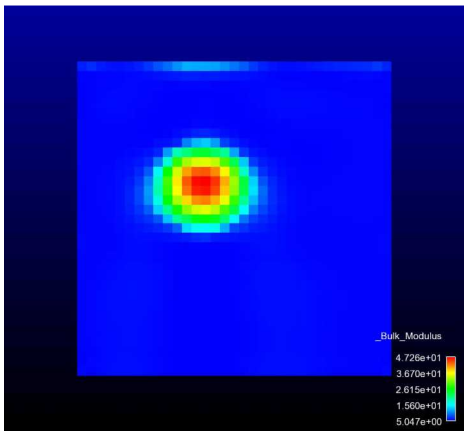
Linear ElastoDynamics Problem

- Pressure applied at the top.
- Elastic moduli of inclusion 10 times those of the matrix.
- Measurements taken over entire body.

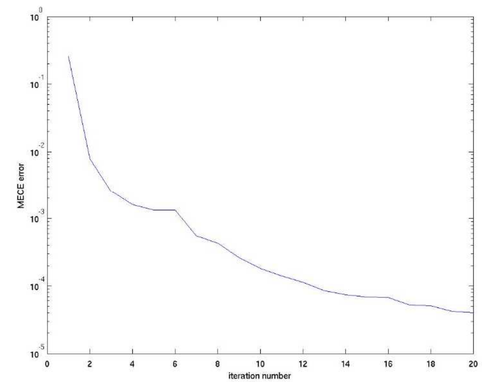


MECE in Inverse Elasticity using Low-Frequency Data

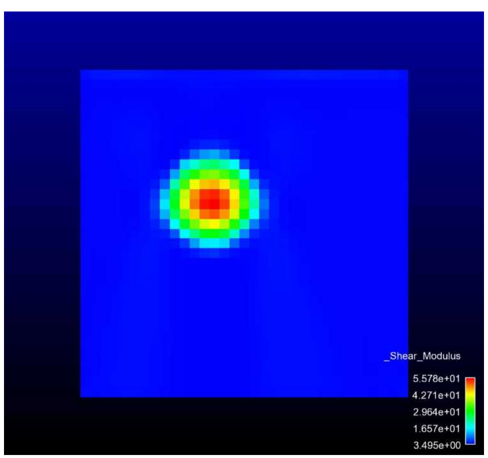
Bulk modulus



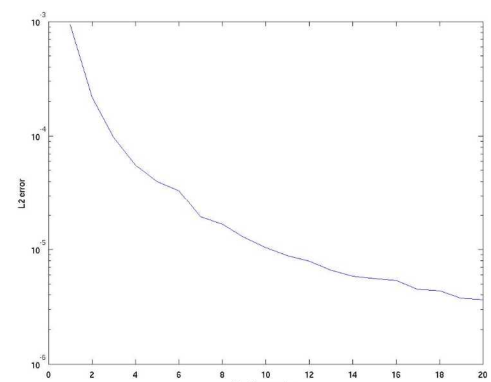
MECE error



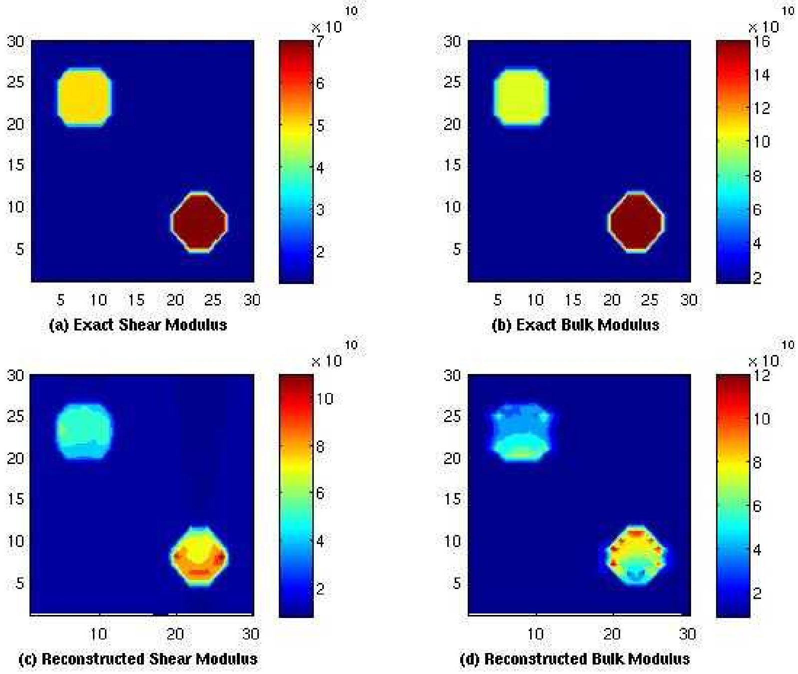
Shear modulus

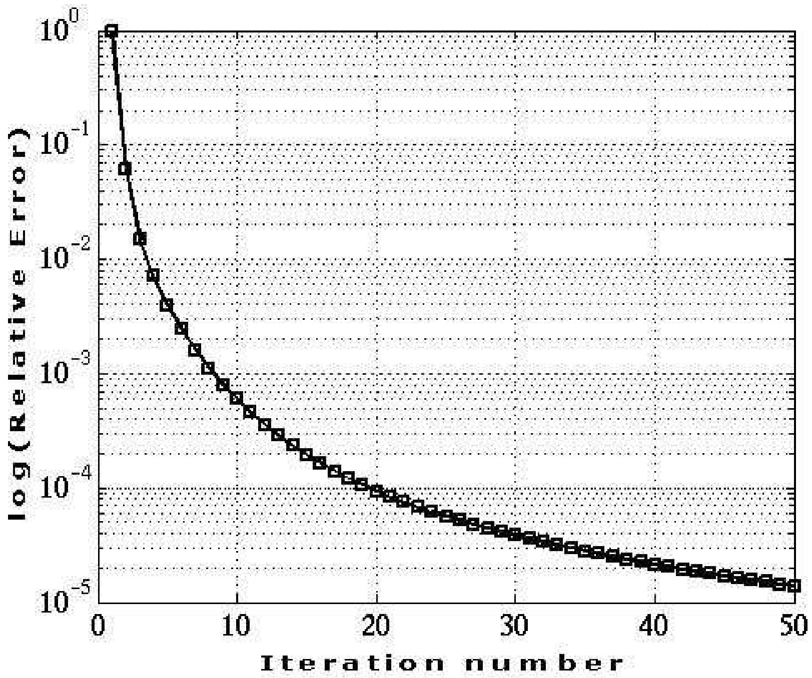


L2 error



MECE Frequency Domain Elastodynamics





Concluding Remarks

- **Our preliminary results have shown that ECE functionals can deliver better accuracy and faster convergence for inverse elasticity and viscoelasticity problems when compared to L2 functionals.**
- **The main drawback of using a MECE functional is that it leads to coupled forward and adjoint problems.**
- **However, we expect that for many problems its potential advantages outweigh the computational cost of the coupled problems.**
- **Mathematical analysis of the properties of MECE functionals is still needed to better understand their true advantages and limitations.**

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Error in Constitutive Equations (ECE) Methods

- Use error functionals based on energy norms or discrepancies in the constitutive equations.
- Stresses belong to a dynamically or statically admissible set.
- Displacements belong to a kinematically admissible set.
- The constitutive equations are not satisfied strongly in the inverse problem.
- Similar energy functionals have been shown to be convex (given certain conditions) with respect to material parameters in elliptic problems.

Energy Functional

$$J\left(\boldsymbol{\sigma}_N, \mathbf{u_D}, [C]\right) = \frac{1}{2}\int_\Omega \Big(\boldsymbol{\sigma}_N - [C]\mathbf{E}(\mathbf{u_D})\Big)^T [C]^{-1}\Big(\boldsymbol{\sigma}_N - [C]\mathbf{E}(\mathbf{u_D})\Big)\,d\Omega$$

Dynamically Admissible Set

$$\mathcal{D} = \Big\{\boldsymbol{\sigma}_N \colon \boldsymbol{\sigma}_N \in L_2\left(\Omega\right),\ \int_\Omega \left(\nabla\mathbf{v}^* \colon \boldsymbol{\sigma}_N - \rho\omega^2\mathbf{v}^*\cdot\mathbf{u}_N\right)d\Omega - \int_{\Gamma_t}\mathbf{v}^*\cdot\left(\boldsymbol{\sigma}_N\mathbf{n}_s\right)d\Gamma_t = 0\ \forall\mathbf{v}\in V\Big\}$$

Kinematically Admissible Set

$$\mathcal{K} = \left\{\mathbf{u}_D \colon \mathbf{u}_D \in H^1\left(\Omega\right),\ \mathbf{u}_D = \mathbf{u}_0 \text{ on } \Gamma_u,\ \mathbf{u}_D = \mathbf{u}_m \text{ in } \Omega_M\right\}$$