DEPARTMENT OF ENERGY RESEARCH PROGRAM
FINAL TECHNICAL REPORT
Analysis of Self-Similar Solutions of Multidimensional Conservation Laws

PI Name: Barbara Lee Keyfitz
PI Institution: The Ohio State University
PI Address: Columbus, Ohio 43210-1174

Award Number: DE-SC0001285
Period: September 1, 2009 – August 31, 2013
Date of Report: February 15, 2014

The funding period of this grant was three years, beginning September 1, 2009. It was extended (no-cost extension) for an additional year. This is the final report. A significant impact of the funding has been the involvement of a postdoctoral visitor, Charis Tsikkou, in the project. Charis spent two years in Columbus, beginning in September 2010; she worked with me on the singular shock problems described below, and is now an assistant professor (tenure-track) at West Virginia University.

Before the original proposal was funded, both Keyfitz and Tesdall moved to different universities. For administrative reasons, the proposal was funded as two separate grants, and hence two separate reports are being filed. Nonetheless, the collaboration between Keyfitz and Tesdall has continued.

1 Report on Research Accomplished on the Grant

1.1 Two-Dimensional Riemann Problems

Five papers have appeared, and two more are in draft form.

The first paper, Reference 1, with co-author Hao Ying, a graduate student at Ohio State who was supported on the grant, contains a list of open problems in multidimensional conservation laws, and also proves a nice result on the location of discontinuities in self-similar problems with initial discontinuity at a single point. The result, which is rigorously proved only for a scalar equation but is expected to hold for systems also, is that the locus of discontinuities is bounded at each time, with a bound that can be determined precisely.

The second paper, Reference 2, combines a numerical study of a simple problem for a simple model equation – interaction of a rarefaction wave with a boundary in the unsteady transonic small disturbance equation – with some elementary analysis explaining the numerical results. Two features stand out in the computed solution. The first is that when there is a sonic line across which the solution is continuous then the influence of the subsonic region on the supersonic region is quite as large as its theoretical maximum. The second is that downstream the sonic line merges with a shock. This feature can be anticipated from the linearized solution, computed in the paper, and is clearly seen in careful numerical simulations. Very detailed calculations, performed by Tesdall and reported in a separate paper by Tesdall, appear to show that the shock actually forms away from the sonic line.

The problem considered in this paper is of interest because it may shed light on the phenomenon of Guderley Mach Reflection. For this reason, we began to look at an even simpler problem: a transonic problem with continuous flow at the sonic line that can be regarded as a perturbation of a one-dimensional transonic problem (which trivially has a unique solution). In that project, Reference 5, we have designed such a problem and have found some solutions. The one-dimensional problem and its perturbations turn out to be amenable to analysis via the hodograph transform followed by a partial Fourier transformation. This results in an elegant use of both techniques (uncommon in conservation laws), but with the limitation that the perturbations we can use are rather narrow. Perhaps this is not important, since the problem was designed only to give some insight into appropriate boundary conditions at the sonic line, and we are moving on to more realistic problems.

Another longstanding research problem has recently been resolved in work of Keyfitz with Jegdić and Čanić, Reference 6. In this paper, now in final revision, we prove a local result (near the shock interaction point) for regular reflection for the isentropic gas dynamics equations. This continues a program we started several years ago, proving existence of the free-boundary-shock solution of the self-similar equations, first for the transonic small disturbance equation and later for the nonlinear wave system. This paper contains the result only for transonic reflection – that is, the less physically interesting (and less mathematically difficult) case that the state behind the reflected shock is strictly subsonic. An extension to the supersonic case, following our results for the two simpler systems we have solved, is in progress.

1.2 A Linear Problem

In work that is loosely related to the quasilinear problems that are the focus of this research, Keyfitz and Popivanov are completing a paper on the so-called Morawetz-Protter problem, Reference 7. The name refers to a collection of boundary-value problems for Tricomi and Tricomi-like equations. Letting t represent the change-of-type variable, so the equation is hyperbolic when $t < 0$ and elliptic when $t > 0$, one obtains a well-posed problem by applying Dirichlet boundary conditions on the elliptic part of the boundary and, as Morawetz proved for the case of two independent variables (denoting the second variable as x), a Goursat condition on a part of the characteristic boundary in the hyperbolic region. The Protter problem is to identify well-posed problems when x is higher-dimensional. It is known that there are null solutions, and singular solutions. We find a correspondence between one boundary-value problem and a non-local problem which can be shown to be well-posed. This provides a partial answer to Protter's conjecture.

1.3 Change of Type and Singular Shocks

We have been inspired by the work of Marco Mazotti, a chemical engineer at ETH Zurich, on anti-Langmuir isotherm models for two-component chromatography, in which he found, numerically, experimentally, and via some analysis on a viscous approximation, singular shocks. The investigation began with updating Keyfitz's earlier research in this area, Reference 3.

A graduate student, Ting-Hao Hsu, partially supported on the grant, studied Mazotti's papers and verified the analytic results. We also performed some numerical calculations,

using our own code, which produced consistent results. We attempted to find asymptotic approximations, using the technique that had worked with the so-called Keyfitz-Kranzer model, but were unsuccessful. This puzzling situation suggests that the repertory of singular solutions may be richer and less uniform than the distribution solutions catalogued in Sever's *AMS Memoir* on the subject. Both because of the intriguing application and because of the inherent mathematical interest of a phenomenon that appears inconsistent with 50 years of research in conservation laws, we have looked further at this problem. With postdoctoral visitor Charis Tsikkou, we completed an examination of the "velocity-entropy" model for isentropic gas dynamics with $1 < \gamma < 5/3$. This is the closest relative of the Keyfitz-Kranzer model (which corresponds to the not-completely-physical case of $\gamma = 1$). Unlike that model, it changes type, with a non-hyperbolic region corresponding to negative density. We consider data in the hyperbolic region and on the degenerate boundary (vacuum). Solutions appear to exist along the lines of the $\gamma = 1$ prototype, with a minor novelty in how vacuum states appear in the solution. We are able to copy Stephen Schecter's construction [JDE, 2006], that proves the existence of viscous profiles for the model system using geometric singular perturbation theory. This was published as Reference 4.

Our goal is analysis of the chromatography system, including both asymptotic descriptions of the singular shocks and, if possible, an existence theorem along the lines of Schecter's result for the model system.

2 Honours, Outreach and Synergistic Activities

2.1 Honors and Awards

- Selected to be the first Charles Saltzer Professor of Mathematics at the Ohio State University.
- Elected SIAM Fellow, 2010; selected as Fellow of the American Mathematical Society, 2012; selected as a Fields Institute Fellow, 2013.
- Awarded honorary degree, Doctor of Mathematics, by the University of Waterloo, 2010.
- AWM Noether Lecturer, 2012 Joint Mathematics Meetings; AWM-SIAM Kovalevsky Lecturer, 2012 SIAM Annual Meeting.
- SIAM award for Distinguished Service to the Profession, 2012.

2.2 Outreach and Synergistic Activities

Continuing outreach and networking activities include

SIAM Co-chaired (with Nick Trefethen) the scientific organizing committee for the 2010 Annual Meeting of the Society for Industrial and Applied Mathematics.

ICIAM President of the International Council for Industrial and Applied Mathematics, from October, 2011 through September 2015.

AWM PI on the AWM-NSF Travel Grant (2008-2015), and Chair of the Long Range Planning Committee (2009-2011).

AMS Elected Vice-President for a three-year term (2011-2014); serving on Math Reviews Editorial Committee (2009-2018), Steele Prize Committee (2009-2012).

Fields Member of the Board of Directors of the Fields Institute (2004-2012); member of committee planning 20th anniversary events for Fields (2011).

ICERM Chair of the Board of Directors of the new NSF-funded research institute at Brown, the Institute for Computational and Experimental Research in Mathematics (2011-2015).

ICWM Chair of the Scientific Program Committee for the International Congress of Women Mathematicians, Seoul 2014.

3 Researchers Partially Supported by the Grant

Three people have received partial support:

- Charis Tsikkou (PhD, Brown University, 2009) a student of Constantine Dafermos, received half-salary from this grant for two years, (the remaining salary came from teaching as a Visiting Assistant Professor at Ohio State).
- Hao Ying, now an advanced graduate student at Ohio State, is completing a dissertation under my direction. His project involves shock formation in a transonic problem.
- Ting-Hao Hsu, also an advanced graduate student, has completed a project on applying singular shocks to an incompressible two-phase flow model, which will form a part of his disseration under my direction.

4 Publications Citing the Grant

1. H. Ying and B. L. Keyfitz, 'A Two-Dimensional Riemann Problem for Scalar Conservation Laws', in *IMA Volume 153: Nonlinear Conservation Laws and Applications*, (A Bressan, G-Q Chen, M Lewicka, D-H Wang, editors), Springer Science + Business Media, LLC, New York, NY, 2011, 447-455.
2. A. M. Tesdall and B. L. Keyfitz, *A Continuous, Two-Way Free Boundary in the Unsteady Transonic Small Disturbance Equations*, Journal of Hyperbolic Differential Equations, **Vol. 7, No. 2**, 2010, pages 317-338.
3. B. L. Keyfitz, 'Singular Shocks: Retrospective and Prospective', Confluentes Mathematici, **3** (2011), 445-470.
4. B. L. Keyfitz and Charis Tsikkou, 'Conserving the Wrong Variables in Gas Dynamics: A Riemann Solution with Singular Shocks', Quarterly of Applied Mathematics, **LXX** (2012), 407-436.

5. B. L. Keyfitz, Allen M. Tesdall, Kevin Ray Payne, Nedyu I. Popivanov, 'The Sonic Line as a Free Boundary', Quarterly of Applied Mathematics, **LXXI** (2013), 119-133.

6. Katarina Jegdić, Barbara Lee Keyfitz Hao Ying and Sunčica Čanić, *A Free Boundary Value Problem for the Isentropic Gas Dynamics Equations – Transonic Regular Reflection*, draft, 35 pages.

7. Barbara Keyfitz and Nedyu Popivanov, *Nonlocal Regularization for the 3-D Morawetz-Protter Problem* (working title), draft, 48 pages.