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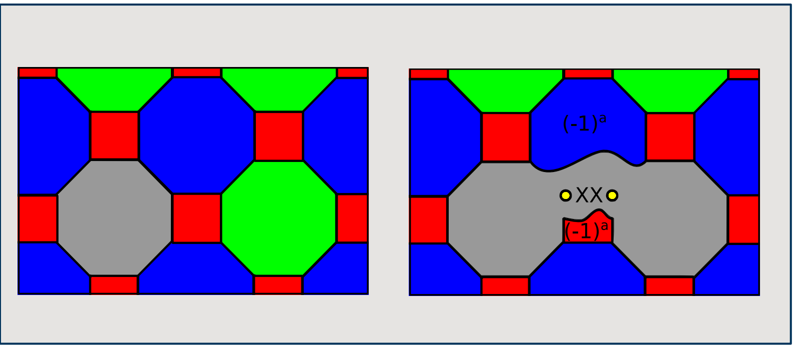

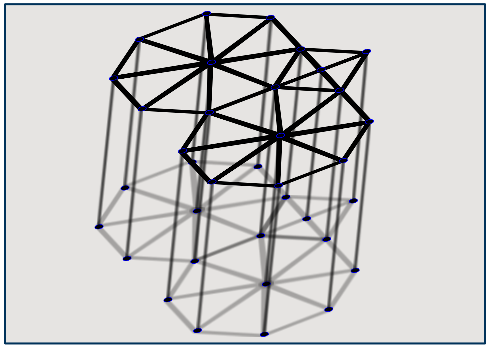

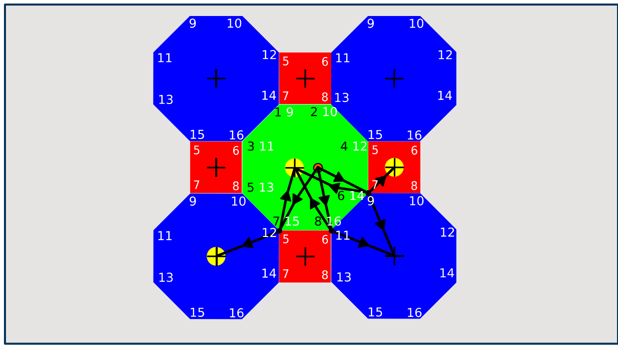

Fault-tolerant quantum computing with color codes

Andrew J. Landahl

with Jonas T. Anderson and Patrick R. Rice.

arXiv:1108.5738

12/13/11

UNM NSF U.S. DEPARTMENT OF ENERGY NNSA National Nuclear Security Administration LDRD Laboratory Directed Research & Development

This work was supported in part by the Laboratory Directed Research and Development program at Sandia National Laboratories.

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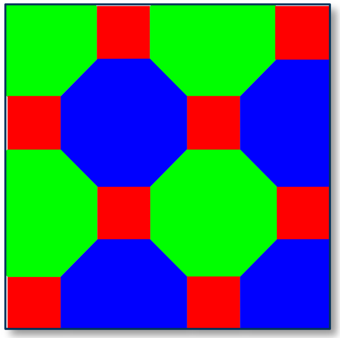
Color codes

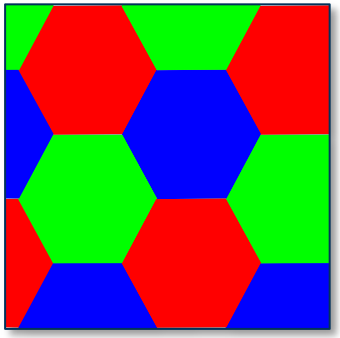
The three semiregular 2D topological color codes

[Bombin & Martin-Delgado, PRL **97**, 180501 (2006)]

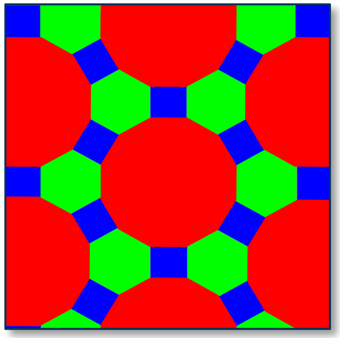


4.8.8

S is transversal
Fewest qubits/distance



6.6.6

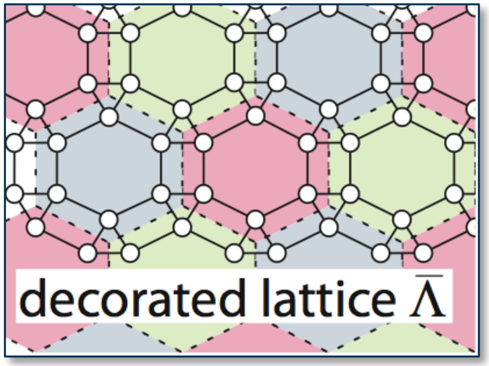


4.6.12

Checks: X_f, Z_f

The 2D topological *subsystem* color code

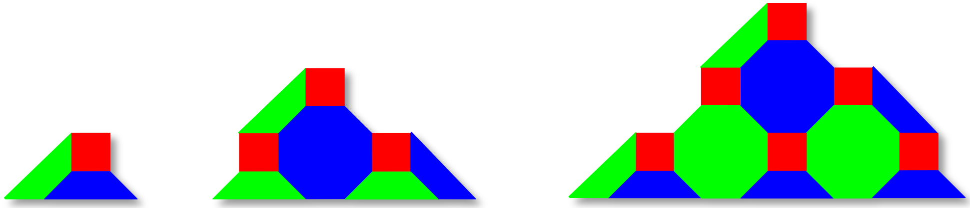
[Bombin, PRA **80**, 032301 (2010)]



3.4.6.4

S is transversal
Two-body checks suffice

Planar color codes: 3*m* corners



These codes are naturally suited to 2D quantum technologies in which long-distance quantum transport is impractical.

The 2D surface code has many promising features for fault-tolerant quantum computing, including a *high accuracy threshold* and *no need for syndrome ancilla distillation.*
How do 2D color codes compare?

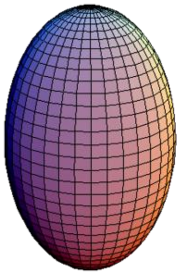
Control & noise models

Control model:

- **(Faulty) gate basis:** $\{I, X, Z, H, S, S^\dagger, CNOT, M_Z, M_X, |0\rangle, |+\rangle, |\pi/4\rangle\}$
- **Standard assumptions:** Parallel operation, refreshable ancilla, fast classical computation, equal-time gates.
- **Locality assumptions:** 2D layout, local quantum processing.

BP channel: *Bit-flip channel* $B(p)$ *followed by phase-flip channel* $\Phi(p)$.

DP channel: *Applies each two-qubit ("double Pauli") product with probability* $p/16$.



Noise model:

- **Standard assumptions:** No leakage, reliable classical computation.

1. **Circuit-based noise model**
   - Each preparation and one-qubit gate followed by $BP(p)$.
   - Each $CNOT$ gate followed by $DP(p)$.
   - Each measurement preceded by $BP(p)$ and result flipped with probability p.
2. **Phenomenological noise model**
   - Same, except each syndrome-bit extraction circuit modeled "phenomenologically" as a measurement that fails with probability p; ignores noise propagation between data and ancilla. Gates only appear in encoded computation.
3. **Code-capacity noise model**
   - Same as phenomenological model, except syndrome measurements are perfect.

Decoders & thresholds

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Optimal decoder: Returns recovery most likely to succeed given the syndrome.

MLE decoder: Returns most likely error that occurred given the syndrome.

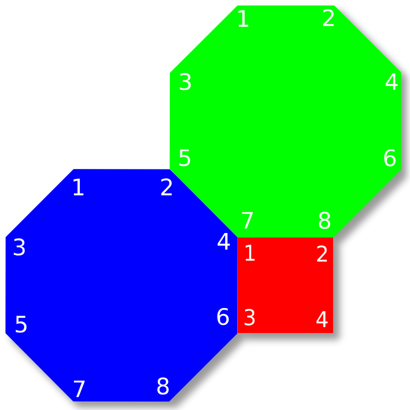
	Code Capacity			Phenomenological		Circuit-based		
Code	Other	MLE	Optimal	MLE	Optimal	Other	MLE	
4.8.8	8.87 %[a,b] [22] 8.7 %[d] [24]	[redacted] (Our result)	10.9(2) % [23] 10.925(5) % [25]	[redacted] (Our result)		"∼ 0.1 %"[a,b,c] [22]	[redacted] (Our result)	
6.6.6	7.8% [A]		10.9(2) % [23] 10.97(1) % [25]		4.5(2) % [26]			
4.6.12								
4.4.4.4 Kitaev		10.31(1) % [27]	10.9187 % [28] 10.939(6) % [30]	2.93(2) % [27]	3.3 % [29]		0.75 %[a] [17] 1.1 %[a] [19]	0.9% [B]
3.4.6.4 TSCC	1.3 %[a,c] [24]							
"SBT" [31]	1.3 %[a,c] [31]							

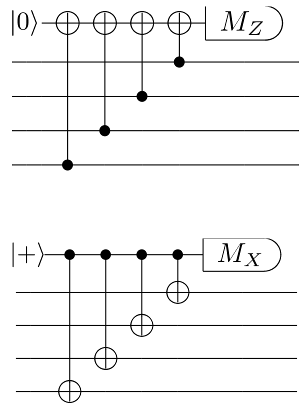
TABLE I: Numerically-estimated accuracy thresholds for several topological quantum error-correcting codes, noise models, and decoding algorithms. The first three codes (4.8.8, 6.6.6, 4.6.12) are the color codes described in Fig. 1 and its preceding text. The last three codes are the Kitaev surface code on the square lattice [15], a topological subsystem color code on the 3.4.6.4 lattice [32], and a hypergraph-based topological subsystem code proposed by Suchara, Bravyi, and Terhal [31]. The details of the noise models (code capacity, phenomenological, and circuit-based) and decoders (MLE, optimal, and other) are discussed in the text; when possible, results from other references have been translated into one of these models. The notation "$x.y_1 \cdots y_k(z)\%$" means $x.y_1 \cdots y_k\% \pm (z \times 10^{-k})\%$. When such notation is not used, it means that the no error analysis was reported in the reference from which the value was drawn.

[A: Sarvepalli & Raussendorf, arXiv:1111.0831]
[B: Fowler, Whiteside, and Hollenberg, arXiv:1110.5133]
[See our paper 1108.5738 for other references.]

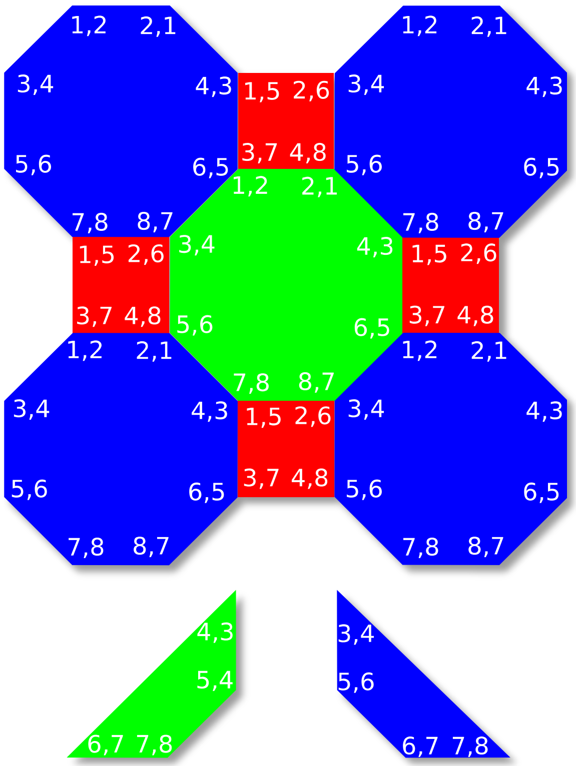
Syndrome extraction

XZ sequential schedule

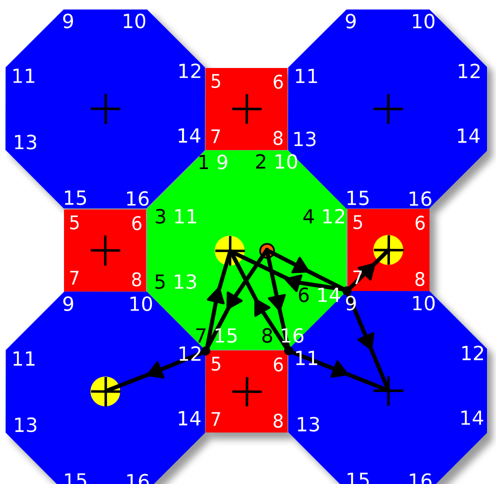




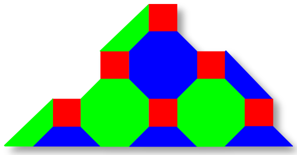
XZ interleaved schedule



Example of error propagation



- *X* error on *X*-check bit (red circle) between time steps 5 and 6.
- Propagates to 3 data errors; detected correctly by 3 *Z*-check bits (yellow)



Decoding

Code-capacity MLE decoder: (Works for all CSS codes.)

Optimization problem

$$\begin{aligned} \min \; & \sum_v x_v \\ \text{sto} \; & \bigoplus_{v \in f} x_v = s_f \qquad \forall f \\ & x_v \in \mathbb{B} := \{0,1\} \end{aligned}$$

Integer program over GF(2)

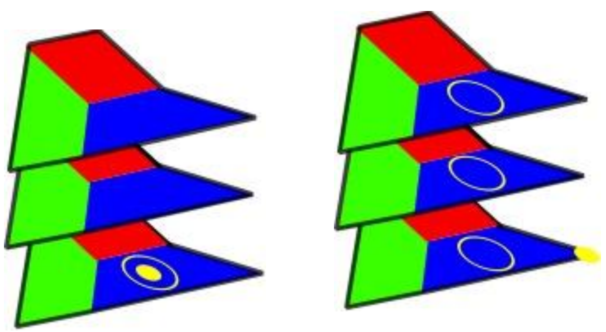
$$\begin{aligned} \min \; & \mathbf{1}^T \mathbf{x} \\ \text{sto} \; & H\mathbf{x} = \mathbf{s} \bmod 2 \\ & \mathbf{x} \in \mathbb{B}^n \end{aligned}$$

Integer program over the reals

$$\begin{aligned} \min \; & \mathbf{c}^T \mathbf{y} \\ \text{sto} \; & A\mathbf{y} = \mathbf{z} \\ & \mathbf{y} \in \mathbb{B}^n \end{aligned}$$

$$\begin{aligned} \mathbf{z} &:= \mathbf{s} + 2\mathbf{t}_1 + 4\mathbf{t}_2 + 8\mathbf{t}_3 \\ \mathbf{y} &:= (\mathbf{x}^T, \mathbf{t}_1^T, \mathbf{t}_2^T, \mathbf{t}_3^T)^T \\ \mathbf{c} &:= \left(\mathbf{1}^T, \mathbf{0}^T, \mathbf{0}^T, \mathbf{0}^T\right)^T \\ A &:= (H | -2I | -4I | -8I) \end{aligned}$$

Phenomenological MLE decoder: (Works for all CSS codes.)



Measurement error

Data error

Integer program over the reals

$$\begin{aligned} \min \; & \mathbf{c}^T \mathbf{y} \\ \text{sto} \; & A\mathbf{y} = \Delta\mathbf{z} \\ & \mathbf{y} \in \mathbb{B}^n \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Delta}\mathbf{z} &:= \boldsymbol{\Delta}\mathbf{s} + 2\mathbf{t}_1 + 4\mathbf{t}_2 + 8\mathbf{t}_3 \\ \mathbf{y} &:= (\mathbf{x}_{\text{data}}^T, \mathbf{x}_{\text{synd}}^T, \mathbf{t}_1^T, \mathbf{t}_2^T, \mathbf{t}_3^T)^T \\ \mathbf{c} &:= \left(\mathbf{1}^T, \mathbf{1}^T, \mathbf{0}^T, \mathbf{0}^T, \mathbf{0}^T\right)^T \end{aligned}$$

$$A := \left(\begin{array}{cccc|cccc|c|c|c} H & & & & I & & & & & & \\ & H & & & I & I & & & & & \\ & & \ddots & & & & \ddots & & -2I & -4I & -8I \\ & & & H & & & I & I & & & \end{array}\right).$$

Code-capacity threshold

Exact curves found up to *d* = 7.

$$p_{\mathrm{fail}} = \sum_{\text{failing patterns } E} p^{|E|}(1-p)^{n-|E|},$$

Monte-Carlo estimate for *d* = 9.

$$p_{\mathrm{fail}}^{(\mathrm{est})} = \frac{N_{\mathrm{fail}}}{N} \qquad (\sigma^2_{\mathrm{fail}})^{(\mathrm{est})} = \frac{p_{\mathrm{fail}}^{(\mathrm{est})}\left(1-p_{\mathrm{fail}}^{(\mathrm{est})}\right)}{N}$$

Threshold by scaling *Ansatz* fit, not curve crossing estimate.

[Wang, Harrington, & Preskill, Ann. Phys. **303**, 31 (2003)]

- **Theory:**

$$\xi \sim |p-p_c|^{-\nu_0}$$

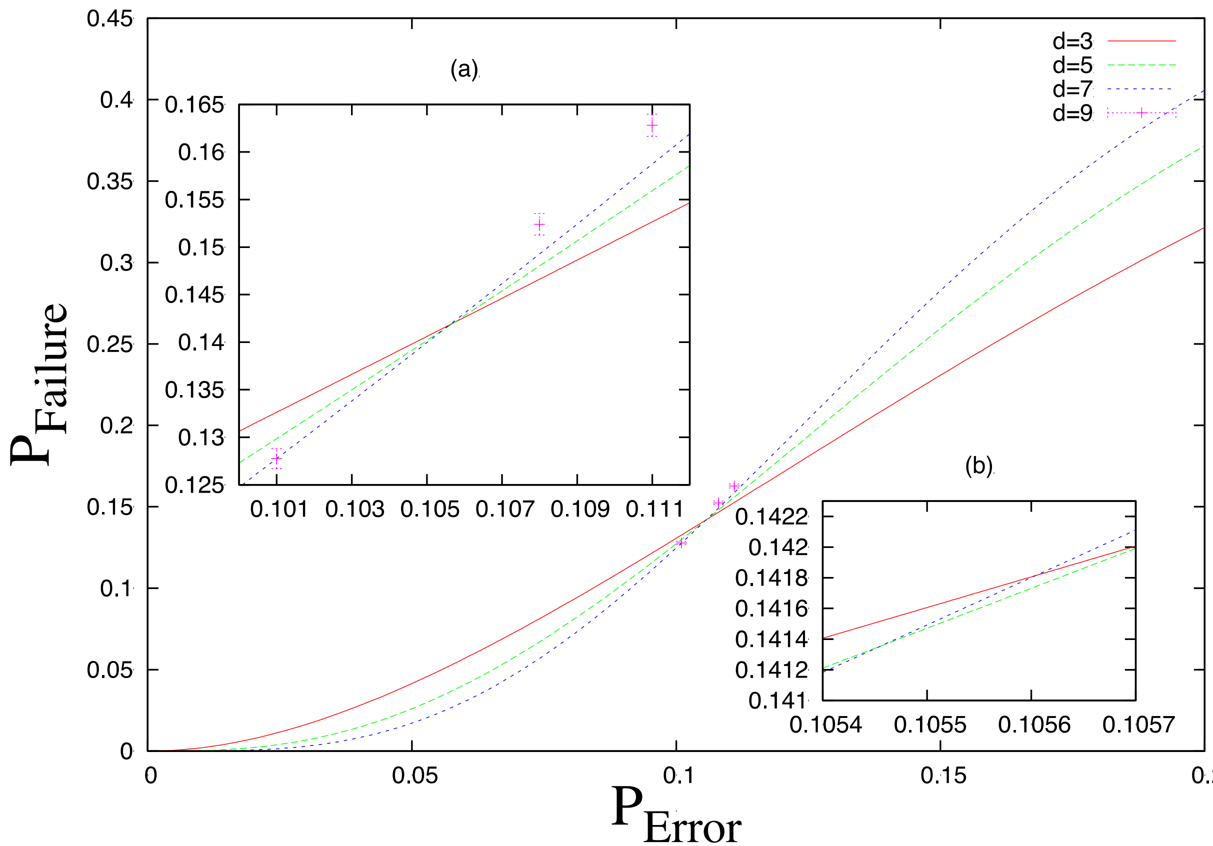
$$p_{\mathrm{fail}} = (p-p_c)d^{1/\nu_0}$$

- **Fit:**

$$p_{\mathrm{fail}} = A + B(p-p_c)d^{1/\nu_0}$$

$$p_{\mathrm{th}} = 10.56(1)\%$$

N.B. Finite size effects may matter.

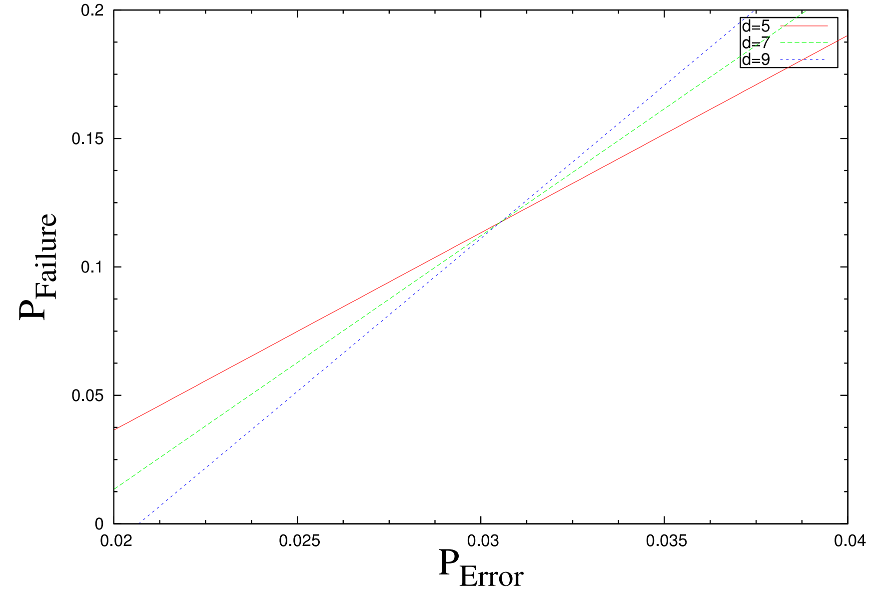


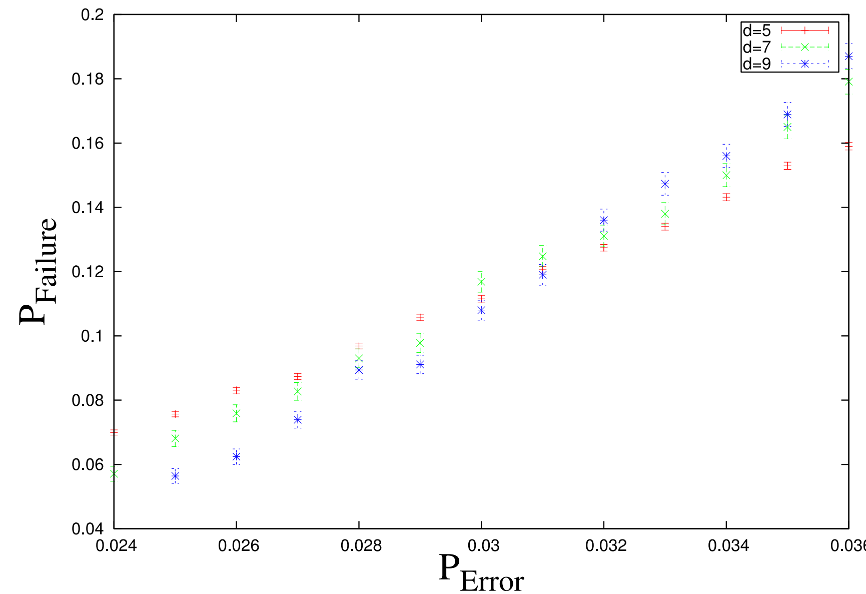
Phenomenological threshold

Algorithm 1 : $p_{\text{fail}}(p)$ by Monte Carlo

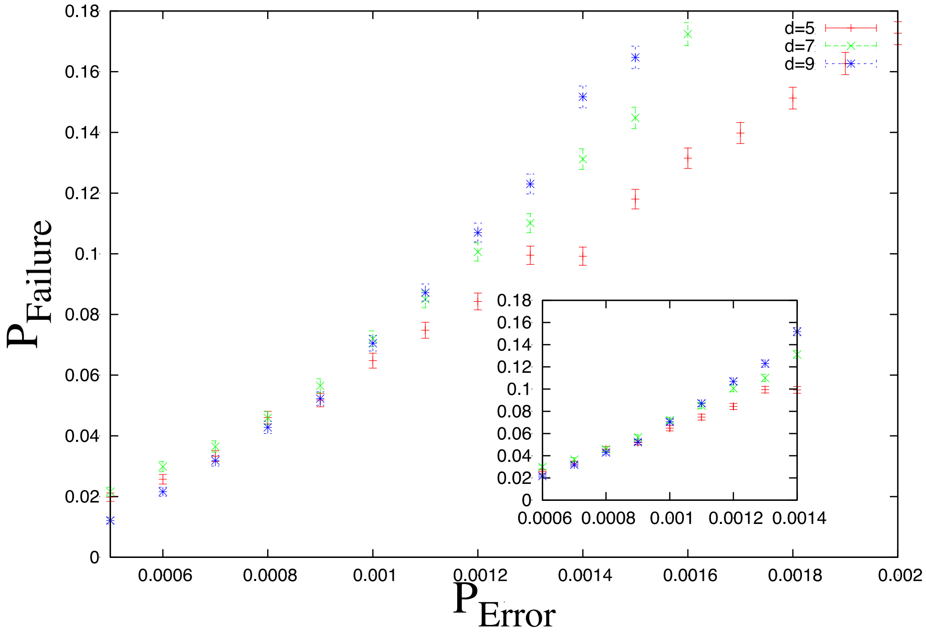
$n_{\text{faces}} \leftarrow \frac{1}{4}(d+1)^2 - 1.$
for $i = 1$ **to** N **do**
  // *Generate data and syndrome errors for d time slices.*
  for $t = 1$ **to** d **do**
    for $j = 1$ **to** n **do**
      $E[t,j] \leftarrow 1$ with probability p. // *Data errors.*
    end for
    for $j = n+1$ **to** $n+1+n_{\text{faces}}$ **do**
      $E[t,j] \leftarrow 1$ with probability p. // *Synd. errors.*
    end for
  end for
  $E_{\min} \leftarrow \text{Decode}(\text{Syndrome}(E))$. // *3D error volume.*
  $E' \leftarrow \bigoplus_t E[t] \oplus E_{\min}[t]$. // *2D error plane.*
  $E'_{\min} \leftarrow \text{Decode}(\text{Syndrome}(E'))$. // *Ideal decoding.*
  if $(\bigoplus_i E'[i] \oplus E'_{\min}[i] = 1)$ **then**
    $N_{\text{fail}} \leftarrow N_{\text{fail}} + 1.$
  end if
end for
return $p_{\text{fail}}^{(\text{est})} = N_{\text{fail}}/N.$

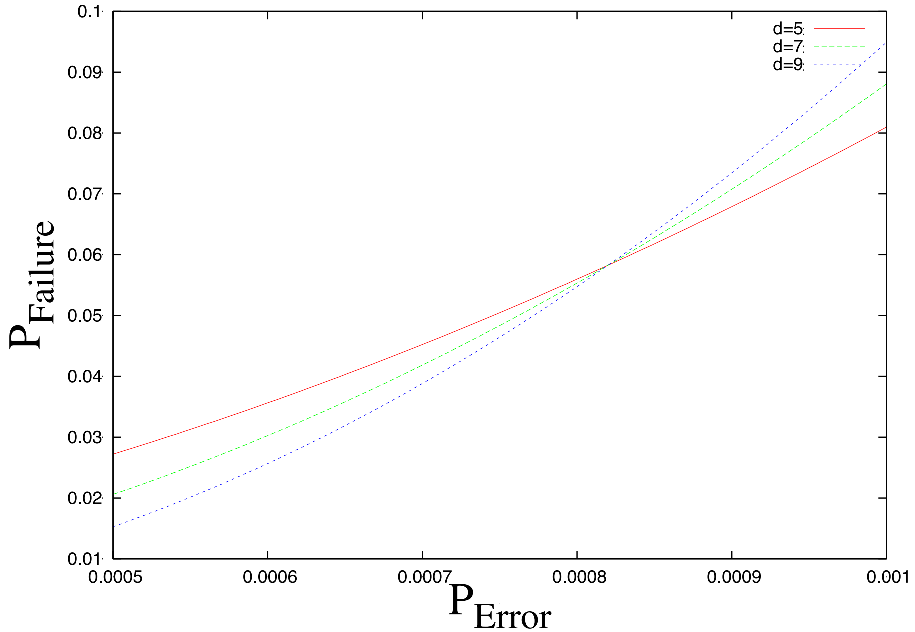
$p_{\text{th}} = 3.05(4)\%$





Circuit threshold





- **Circuit generates correlated errors up to weight four (“hooks”).**
- **Can accommodate by modifying fitting function:**

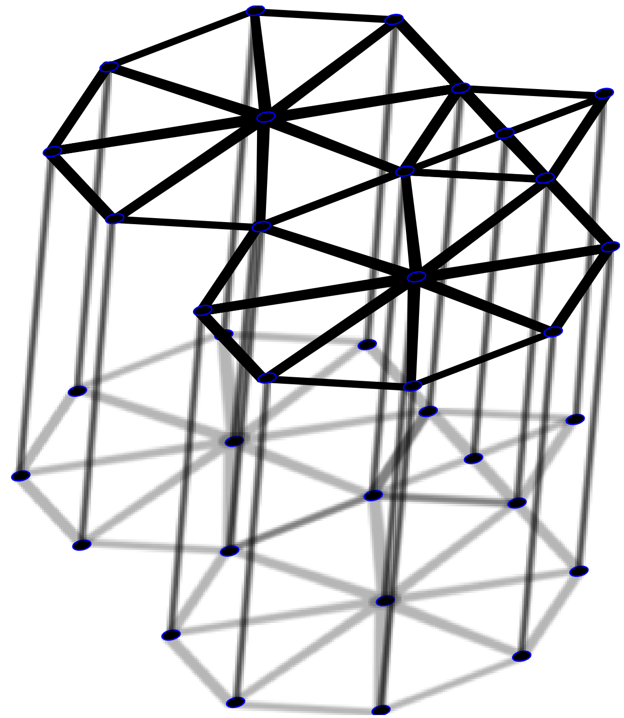
$$p_{\text{fail}} = A + B(p - p_c)d^{1/\nu_0} + C(p - p_c)^2 d^{2/\nu_0}.$$

$$p_{\text{th}}^{(XZ\ \text{interleaved})} = 8.2(3) \times 10^{-4}$$
$$p_{\text{th}}^{(XZ\ \text{sequential})} = 8.0(4) \times 10^{-4}$$

Self-avoiding walk bound

$$p_{\text{fail}} = \sum_{\mathcal{O}\in\text{log ops}} p(\mathcal{O} \in E + E_{\min})$$
$$= \sum_{\mathcal{O}\in\text{string-net}} p(\mathcal{O} \in E + E_{\min}) + \sum_{\mathcal{O}\in\text{string}} p(\mathcal{O} \in E + E_{\min})$$
$$\leq 2 \sum_{\mathcal{O}\in\text{SAW}(\geq d)} p(\mathcal{O} \in E + E_{\min})$$
$$\leq 2 \cdot \underset{\substack{\uparrow \\ \text{\# starting points for SAW}}}{n} \sum_{L\geq d} n_{\text{SAW}}(L) \cdot \underset{\substack{\uparrow \\ \text{\# ways to choose flips on } E + E_{\min}}}{2^L} \cdot \underset{\substack{\uparrow \\ \text{Prob. flips at those locations}}}{[p(1-p)]^{L/2}}$$

$$n_{\text{SAW}}(L) \leq \Delta_{\max}(\Delta_{\max}-1)^{L-1} \qquad p_{\text{th}}(1-p_{\text{th}}) \leq \frac{1}{4\mu^2}$$
$$n_{\text{SAW}}(L) \leq P(L)\,\mu^L$$



- **Code-capacity noise model**

$$\mu_{4.8.8} \approx 1.808\,830\,01(6)$$
$$1.804\,596 \leq \mu_{4.8.8} \leq 1.829\,254$$
$$\boxed{p_{\text{th}} \geq 8.335\,745(1)\%}$$

cf. 10.56(1)%

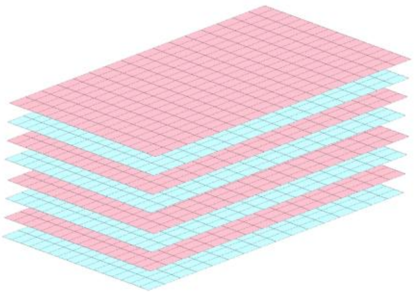
- **Phenomenological noise model**

$$\Delta_{\max} = 10$$
$$\boxed{p_{\text{th}} \geq 0.502\,5\%}$$

cf. 3.05(4)%

[Dennis, Kitaev, Landahl, & Preskill, JMP **43**, 4452 (2002)]

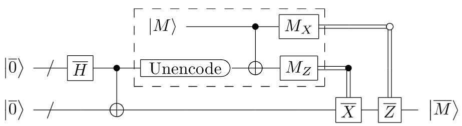
Architectures & computation



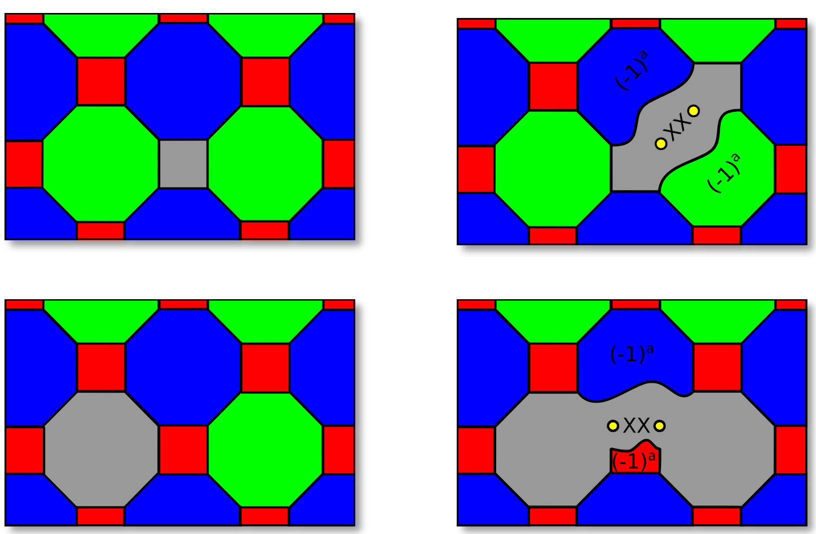
Transversal

- Push noise channels through gates to get effective noise channels for FTQEC
- Magic states for *T* gate injected by teleportation.

$$p_I = p_X = p_Z = p_{|0\rangle} = p_{|+\rangle} = \frac{3}{2} p_{CNOT} = \frac{3}{2} p_{M_X}^{(\text{nondestructive})} = \frac{3}{2} p_{M_Z}^{(\text{nondestructive})}$$
$$= 2p_H = 2p_S^{(\text{bit-flip})} = 2p_T^{(\text{bit-flip})} = 3p_S^{(\text{phase-flip})} = 3p_T^{(\text{bit-flip})}$$
$$= 8.2(3) \times 10^{-4}$$



$$p_{M_X}^{(\text{destructive})} = p_{M_Z}^{(\text{destructive})} = 10.56(1)\%$$
$$p_{|\pi/4\rangle} = (\sqrt{2} - 1)/2\sqrt{2} \approx 14.6\%$$



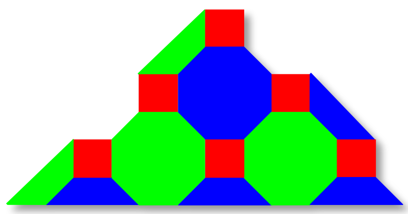
Code deformation

- Local ops to grow, shrink, & move "defects."
- *H, S, CNOT* gate by code deformation.
- Magic states for *T* gate injected by code deformation.

Summary

Color codes vs. surface codes

- Code-capacity & phenomenological thresholds comparable to surface codes. (11%, 3%)
- Circuit-model threshold about 10 times smaller than surface-code's.
- Rigorous bounds on threshold very weak at this point.
- Qubit overhead comparable to surface codes.



Open questions:

- TSCCs: Do they have better thresholds?
- Leakage errors: How well are they tolerated?
- Efficient decoders: TCC = 2 surface codes---run efficient matching on these. Threshold?
- Rigorous lower bounds: Tighter analysis techniques?
- Magic-state injection: How much of 14.6% threshold is consumed by imperfect injection?
- Topological quantum computation: How to formulate a color-code quantum double model?