SAND2013-9422C

Eigenfunction Expansion of the Space-Time Dependent Neutron Survival Probability

RYAN J KAMM
SANDIA NATIONAL LABORATORIES

ANIL K PRINJA
THE UNIVERSITY OF NEW MEXICO

2013 ANS WINTER MEETING
REACTOR PHYSICS: GENERAL
2013.11.13

OVERVIEW

- Deterministic v. stochastic neutron transport
  - The Pál-Bell equation
- The survival probability equation
  - The diffusion approximation
- 0-D equation
- Steady-state space-dependent solutions
- Eigenfunction expansion technique
- Conclusions and future work

Classes of Transport Problems

Deterministic	Stochastic
• Deals with the mean behavior of large neutron populations	• Deals with the probability of having populations of arbitrary size
• Only meaningful for large populations	• Necessary for describing small populations

Stochastic Transport Problems

- For small populations, the actual number of particles may depart significantly from its expected/average value

- Behavior is unpredictable, necessitating stochastic models
  - Criticality accidents
  - Pulsed reactors (e.g. – Godiva)
  - Weapon safety

The Probability Distribution Generating Function

The Generating Function and its Compliment

$$\mathrm{G}(z;\mathrm{R},t_f;\vec{r},t,\vec{\Omega}) = \sum_{n=0}^{\infty} z^n p_n(\mathrm{R},t_f;\vec{r},t,\vec{\Omega})$$

$$\mathcal{G} = 1 - \mathrm{G}$$

$$\mathcal{G}_0 = 1 - \int_{4\pi} \mathrm{G}(z;\mathrm{R},t_f;\vec{r},t,\vec{\Omega}')\frac{d\vec{\Omega}'}{4\pi}$$

Pál-Bell Equation*

$$-\hat{\Omega}\cdot\nabla\mathcal{G} - \frac{1}{v}\frac{\partial\mathcal{G}}{\partial t} = -\Sigma_T\mathcal{G} + \Sigma_S\mathcal{G}_0 + \Sigma_F\bar{\nu}\mathcal{G}_0 - \Sigma_F\sum_{j=2}^{J}\frac{(-1)^j\chi_j}{j!}\mathcal{G}_0{}^j$$

*One-speed, isotropic, delayed neutron precursor-free

Neutron Survival Probability

$$G(z; \mathrm{R}, t_f; \vec{r}, t, \vec{\Omega}) = \sum_{n=0}^{\infty} z^n p_n(\mathrm{R}, t_f; \vec{r}, t, \vec{\Omega})$$

⇓

$$G(0; \mathrm{R}, t_f; \vec{r}, t, \vec{\Omega}) = p_0(\mathrm{R}, t_f; \vec{r}, t, \vec{\Omega})$$

⇓

$$P \equiv \mathcal{G}(0; \mathrm{R}, t_f; \vec{r}, t, \vec{\Omega})$$

- The probability that a neutron will lead to a nonzero number of neutrons in R at t_f.
- As $t_f \to \infty$, it is the probability that of a divergent chain having formed; the Probability of Initiation (POI)

The Survival Probability Equation

$$-\hat{\Omega}\cdot\nabla P - \frac{1}{v}\frac{\partial P}{\partial t} = -\Sigma_T P + \Sigma_S P_0 + \Sigma_F\left[\bar{\nu}P_0 - \sum_{j=2}^{J}\frac{(-1)^j\chi_j}{j!}{P_0}^j\right]$$

<u>Final and Boundary Conditions</u>

$$P\left(\vec{r},\hat{\Omega},t_f\right) = 1 \qquad \vec{r},\hat{\Omega}\in \mathrm{R}$$

$$= 0 \qquad \vec{r},\hat{\Omega}\notin \mathrm{R}$$

$$P\left(\vec{r}_b,\hat{\Omega},t\right) = 0 \qquad \hat{\Omega}\cdot\vec{n} > 0$$

The Survival Probability Diffusion Equation

$$P\left(\vec{r},\hat{\Omega},\tau\right) \cong P_0\left(\vec{r},\tau\right) + 3\hat{\Omega}\cdot\vec{P}\left(\vec{r},\tau\right)$$

$$\vec{P}\left(\vec{r},\tau\right) = \int_{4\pi} \hat{\Omega} P\left(\vec{r},\hat{\Omega},\tau\right)\frac{d\hat{\Omega}}{4\pi} = D\nabla P_0$$

$$\tau = \frac{(t_f - t)}{t_l}$$

$$-\frac{\partial P_0}{\partial \tau} + L^2\nabla^2 P_0 = \left(1-k_\infty\right)P_0 + \frac{\Sigma_F}{\Sigma_A}\sum_{j=2}^{J}\frac{(-1)^j\chi_j}{j!}P_0{}^j$$

“Initial” and Boundary Conditions

$$P_0\left(\vec{r},0\right) = 1 \qquad \vec{r}\in \mathrm{R}$$

$$P_0\left(\vec{r},\tau\right) = 0 \qquad \vec{r}\notin \mathrm{R}$$

General Observations and Expectations

$$-\frac{\partial P}{\partial \tau} + L^2 \nabla^2 P = (1 - k_\infty) P + \frac{\Sigma_F}{\Sigma_A} \sum_{j=2}^{J} \frac{(-1)^j \chi_j}{j!} P^j$$

$$
\begin{aligned}
t_l &= 8.0297 \text{ ns} \\
L^2 &= 15.082 \text{ cm}^2 \\
k_\infty &= 2.6407 \\
\Sigma_F &= 0.0594 \text{ cm}^{-1} \\
\Sigma_A &= 0.0618 \text{ cm}^{-1} \\
\bar{\nu} &= 2.7500 \\
\chi_2/2! &= 3.0760 \\
\chi_3/3! &= 1.7720 \\
\chi_4/4! &= 0.5640 \\
\chi_5/5! &= 0.0950
\end{aligned}
$$

- Small values of the survival probability diminish the import of nonlinear terms
- For small POI (steady-state), the shape should be well-characterized by the fundamental mode

0-D Survival Probability

Time-Dependent Equation

$$\frac{dQ}{d\tau} = \left(k_{eff} - 1\right) Q\left(\tau\right) - \frac{k_{eff}}{\bar{\nu}} \sum_{j=2}^{J} \frac{(-1)^j \chi_j}{j!} Q\left(\tau\right)^j$$

<u>"Initial" Condition</u>

$$Q\left(0\right) = 1$$

Quadratic Approximation for the 0-D POI

$$Q\left(\tau\right) = \frac{1}{\frac{1}{\rho}\frac{\chi_2/2!}{\bar{\nu}}\left(1 - e^{-(k_{eff}-1)\tau}\right) + e^{-(k_{eff}-1)\tau}}$$

- The POI is linearly proportional to the excess reactivity

$$Q_\infty \equiv Q(\tau \to \infty) = \rho \frac{\bar{\nu}}{\chi_2/2!}$$

Space-Dependent POI

Equation	Examinations
$\nabla^2 P_\infty = -B_m^2 P_\infty(\vec{r}) + \frac{\Sigma_F}{D}\sum_{j=2}^{J}\frac{(-1)^j \chi_j}{j!} P_\infty(\vec{r})^j$ <u>Boundary Conditions</u> $P_\infty(\vec{r}_b) = 0$	• Shape and magnitude as function of ○ Method of solution ○ System reactivity ○ Degree of nonlinearity ○ System geometry

The Fundamental Mode Approximation

$${}_{N=1}P_{\infty}(\vec{r}) = A_1 R_1(\vec{r})$$

$$\nabla^2 R_1 = -B_g^2 R_1$$

$$\varrho = \int_{\vec{r}} R_1^2(\vec{r}) d\vec{r}$$

$$0 = \rho A_1 - \frac{1}{\varrho \bar{\nu}} \sum_{j=2}^{J} \frac{(-1)^j \chi_j}{j!} A_1^j \int_{\vec{r}} R_1^{j+1}(\vec{r}) d\vec{r}$$

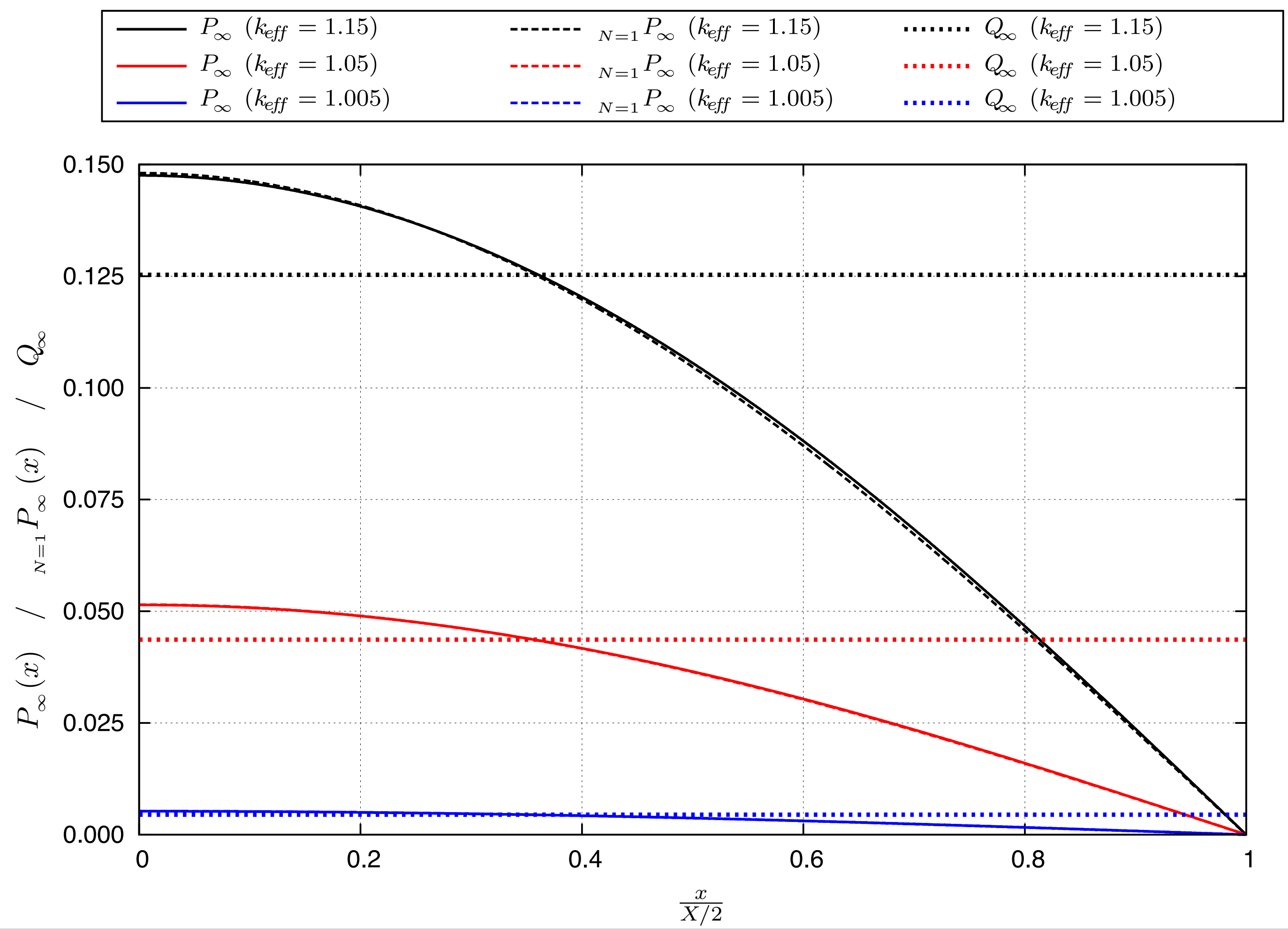
- As with 0-D model, the POI is linearly proportional to the excess reactivity

Semi-Analytic Solution for 1-D Slab

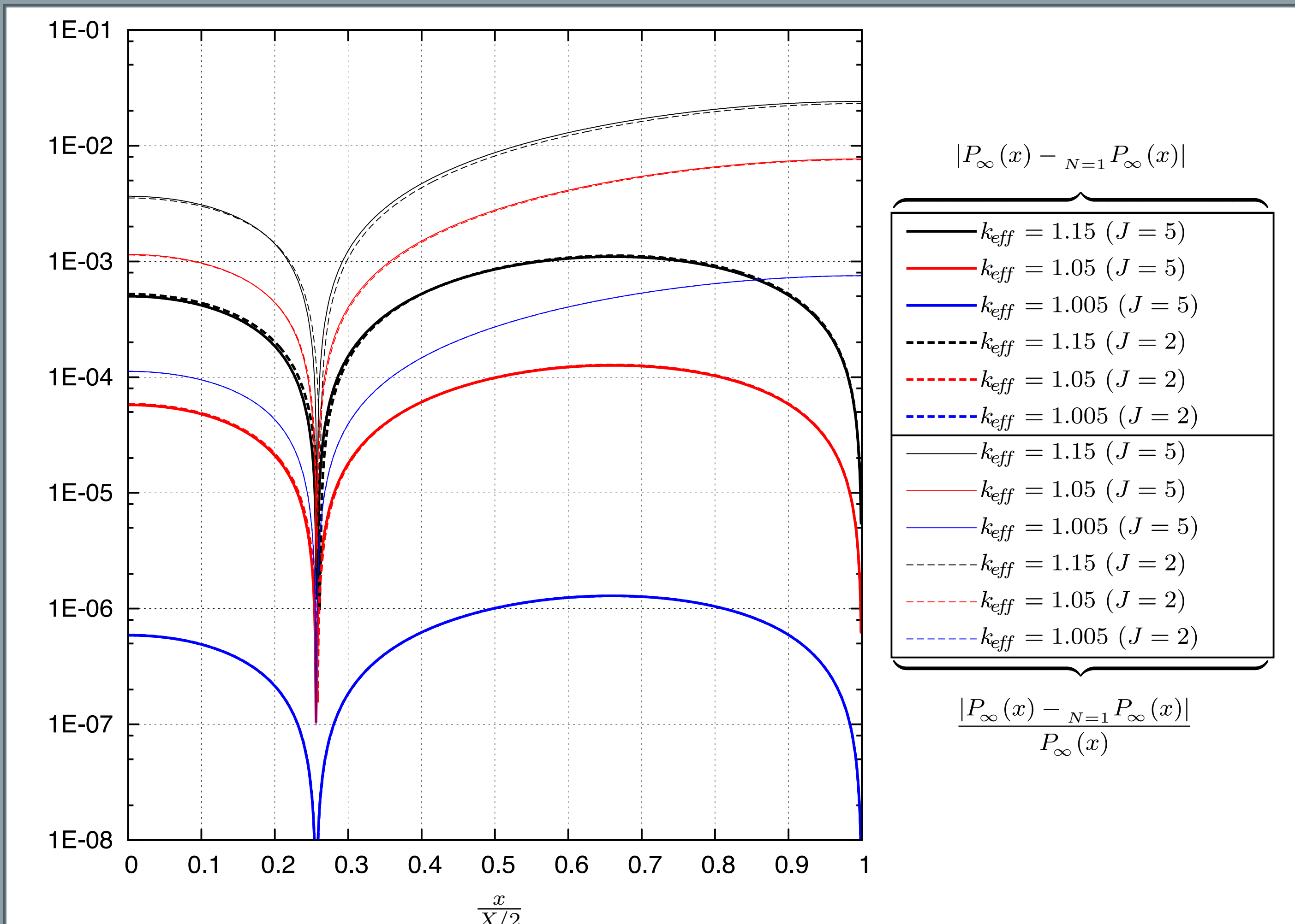
$$\int_0^{P_\infty(x)} \frac{dP_\infty{}'}{\sqrt{-B_m^2\left(P_\infty{}'^2 - P_\infty(0)^2\right) + 2\frac{\Sigma_F}{D}\sum_{j=2}^{J}\frac{(-1)^j\chi_j}{(j+1)!}\left(P_\infty{}'^{j+1} - P_\infty(0)^{j+1}\right)}} = \frac{X}{2} - x$$

- No closed-form solution exists
- Newton-Raphson iteration produces values at specified intervals
- Because it is semi-analytic, profile can serve as a benchmark

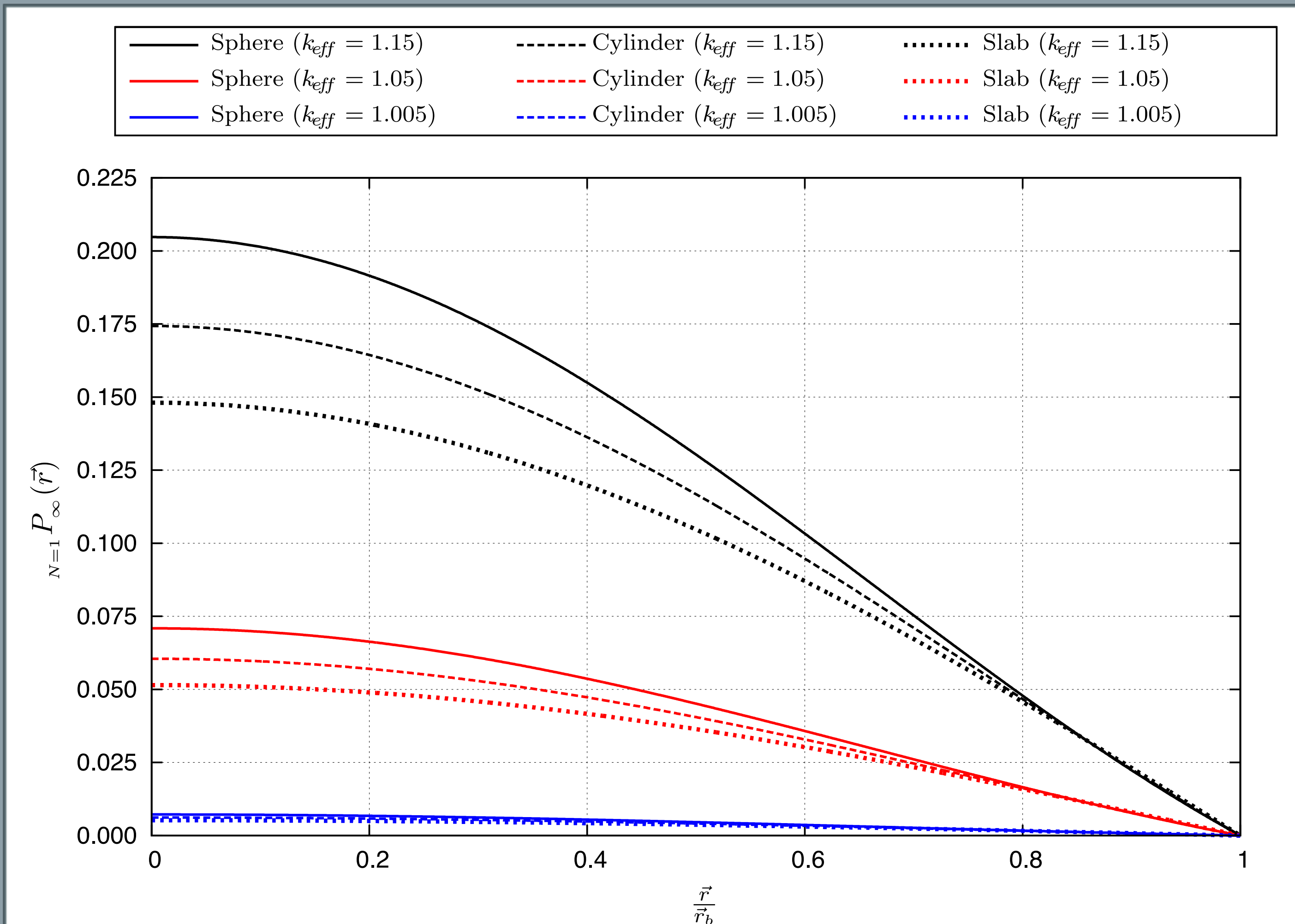
POI in a 1-D Slab (Full Nonlinearity)



Error in Approximation in a 1-D Slab



POI in 1-D Geometries (Full Nonlinearity)



Time and Space-Dependent Survival Probability

Equation	Examinations

$$-\frac{\partial P}{\partial \tau} + L^2 \nabla^2 P = (1 - k_\infty) P + \frac{\Sigma_F}{\Sigma_A} \sum_{j=2}^{J} \frac{(-1)^j \chi_j}{j!} P^j$$

<u>"Initial" and Boundary Conditions</u>

$$P(\vec{r}, 0) = 1 \qquad \vec{r} \in \mathrm{R}$$

$$P(\vec{r}, \tau) = 0 \qquad \vec{r} \notin \mathrm{R}$$

- Efficacy of Eigenfunction Expansion (EFE) technique throughout parameter-space
- Shape and magnitude as function of
  - Survival time
  - System reactivity
  - Degree of nonlinearity
  - System geometry

EFE Technique (1/4)

Assumptions

- Separable in space and time
- Always piecewise smooth

$$P(\vec{r},\tau) = \sum_{n=1}^{\infty} T_n(\tau) R_n(\vec{r})$$

$$L^2 \sum_{n=1}^{\infty} T_n \nabla^2 R_n - \sum_{n=1}^{\infty} \frac{\partial T_n}{\partial \tau} R_n = (1-k_\infty) \sum_{n=1}^{\infty} T_n R_n + \frac{\Sigma_F}{\Sigma_A} \sum_{j=2}^{J} \frac{(-1)^j \chi_j}{j!} \left[\sum_{n=1}^{\infty} T_n R_n \right]^j$$

$$\sum_{n=1}^{\infty} T_n(0) R_n(\vec{r}) = 1 \qquad \vec{r} \in \mathrm{R}$$

$$\sum_{n=1}^{\infty} T_n(\tau) R_n(\vec{r}) = 0 \qquad \vec{r} \notin \mathrm{R}$$

EFE Technique (2/4)

Assumptions

- Eigenfunction expansion of linear portion provides spatial eigenfunctions

$$\nabla^2 R_n = -\lambda_n^2 R_n$$

$$-\int_{\vec{r}} \sum_{n=1}^{\infty} \frac{dT_n}{d\tau} R_n R_m d\vec{r} = \int_{\vec{r}} \sum_{n=1}^{\infty} \left(1 - k_\infty + L^2 \lambda_n^2\right) T_n R_n R_m d\vec{r} + \frac{\Sigma_F}{\Sigma_A} \sum_{j=2}^{J} \frac{(-1)^j \chi_j}{j!} \int_{\vec{r}} \left[\sum_{n=1}^{\infty} T_n R_n\right]^j R_m d\vec{r}$$

$$\int_{\vec{r}} \sum_{n=1}^{\infty} T_n(0) R_n R_m d\vec{r} = \int_{\vec{r}} R_m d\vec{r}$$

EFE Technique (3/4)
Geometry Specific Eigenfunctions

Geometry	Eigenfunction (R_n)	Geometric Buckling (B_g^2)
1-D Slab	$\cos\left(\frac{(2n-1)\pi}{X}x\right)$	$\left(\frac{\pi}{X}\right)^2$
1-D Cylinder	$J_0\left(\frac{j_n}{R}r\right)$	$\left(\frac{j_1}{R}\right)^2$
Sphere	$\frac{1}{r}\sin\left(\frac{n\pi}{R}r\right)$	$\left(\frac{\pi}{R}\right)^2$
2-D Slab	$\cos\left(\frac{(2a-1)\pi}{X}x\right)\cos\left(\frac{(2b-1)\pi}{Y}y\right)$	$\left(\frac{\pi}{X}\right)^2+\left(\frac{\pi}{Y}\right)^2$

EFE Technique (4/4)

Assumptions

- N modes accurately represent the survival probability

$$P(\vec{r},\tau) = \sum_{n=1}^{N} T_n(\tau) R_n(\vec{r})$$

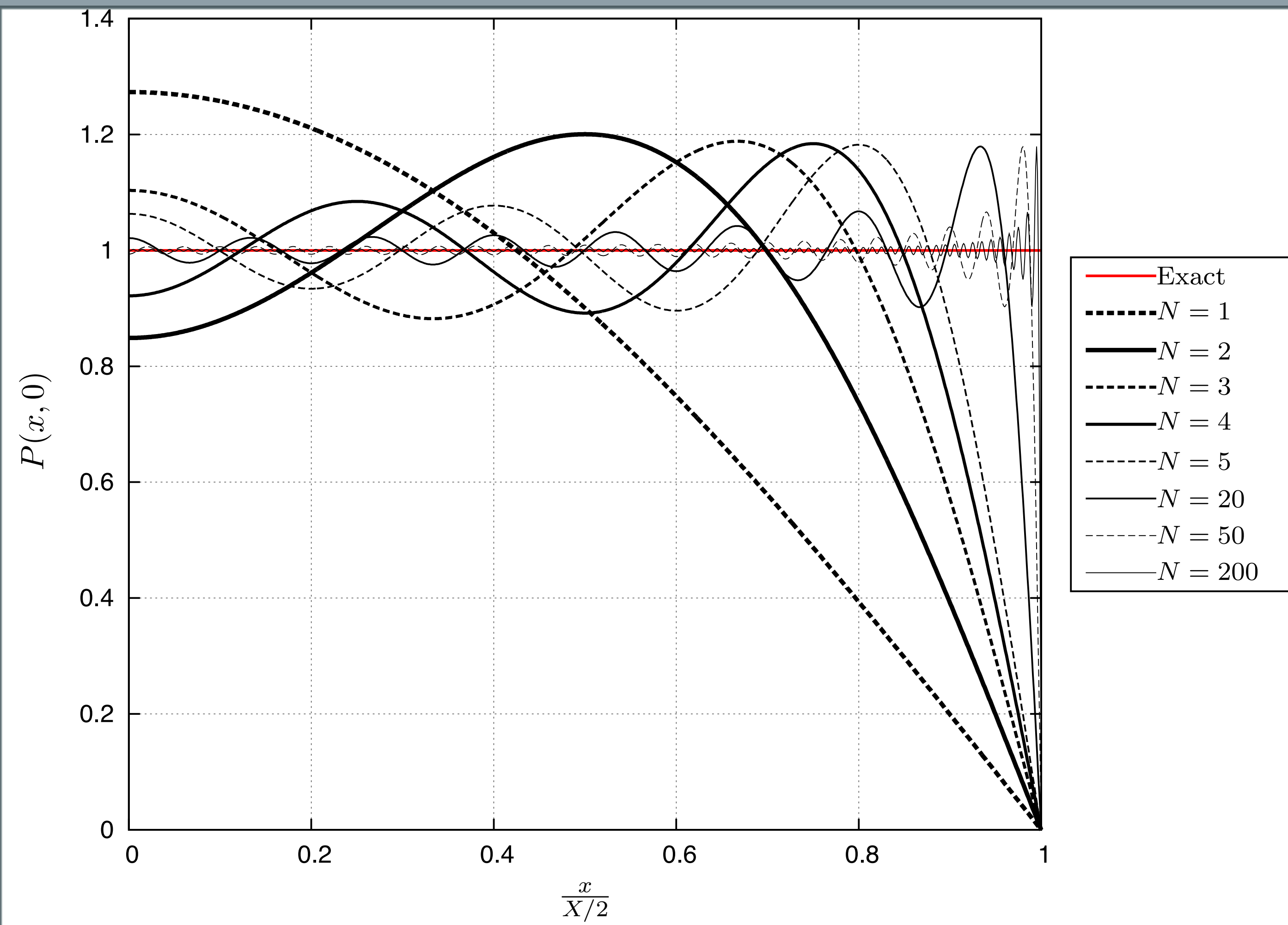
$$\frac{dT_m}{d\tau} = \left(k_\infty - 1 - L^2\lambda_m^2\right) T_m -$$

$$\frac{\Sigma_F}{\Sigma_A \varrho_m} \sum_{j=2}^{J} \frac{(-1)^j \chi_j}{j!} \int_{\vec{r}} R_m \left[\sum_{n=1}^{N} T_n R_n\right]^j d\vec{r}$$

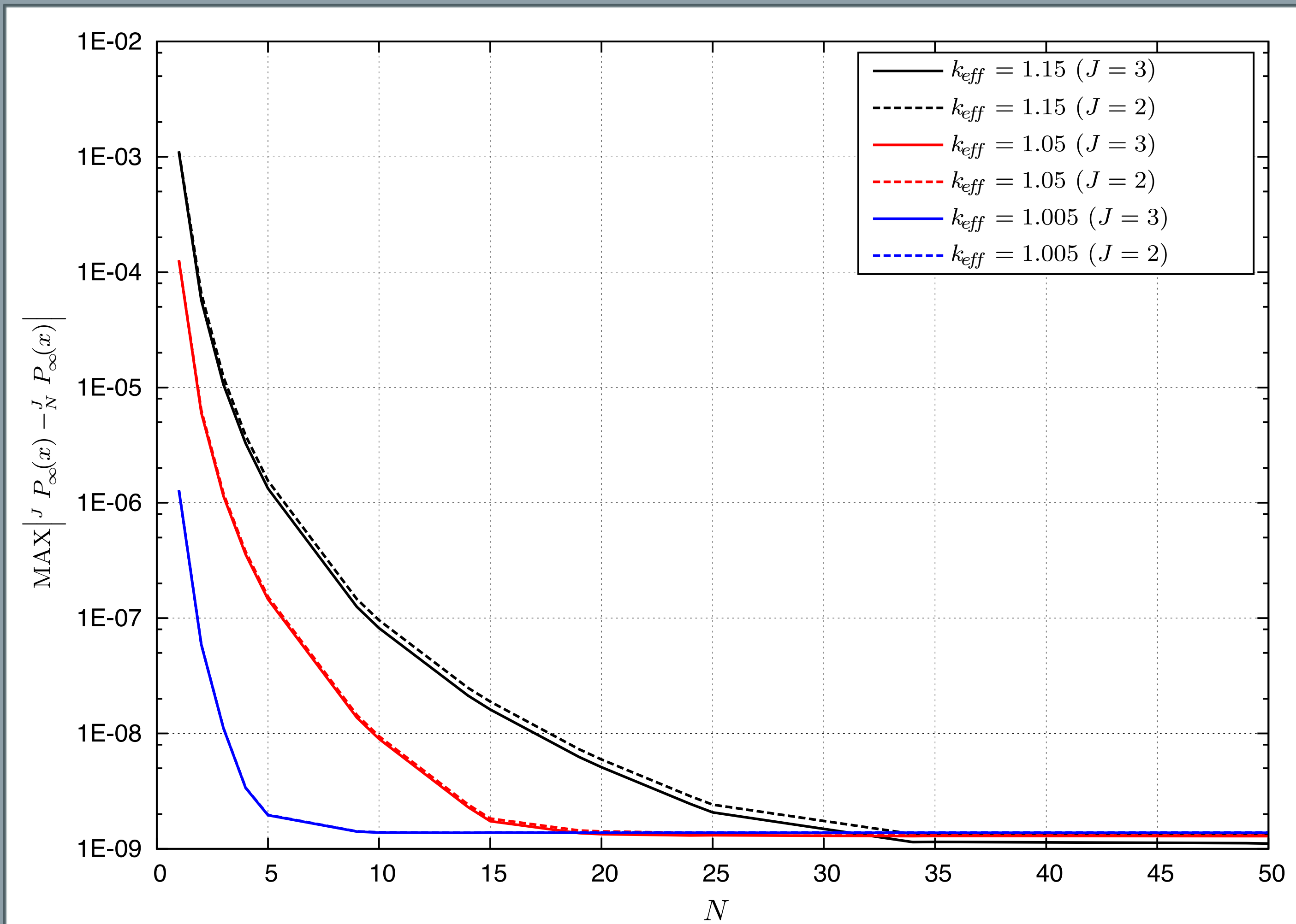
$$T_m(0) = \frac{1}{\varrho_m} \int_{\vec{r}} R_m d\vec{r}$$

$$\varrho_m = \int_{\vec{r}} R_m^2 d\vec{r}$$

“Initial” Condition in 1-D Slab



Convergence on Semi-Analytic POI

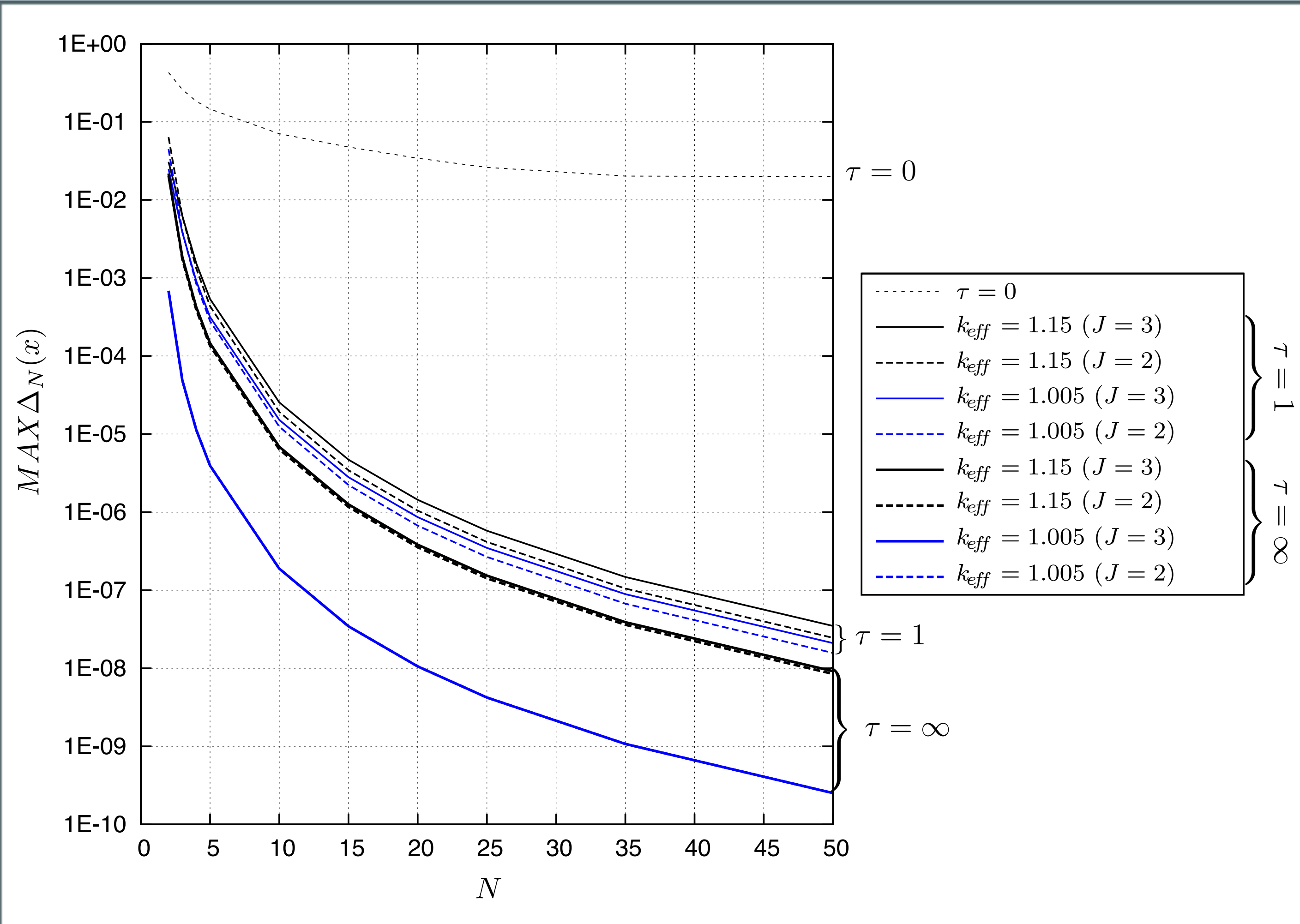


Measures of Efficacy

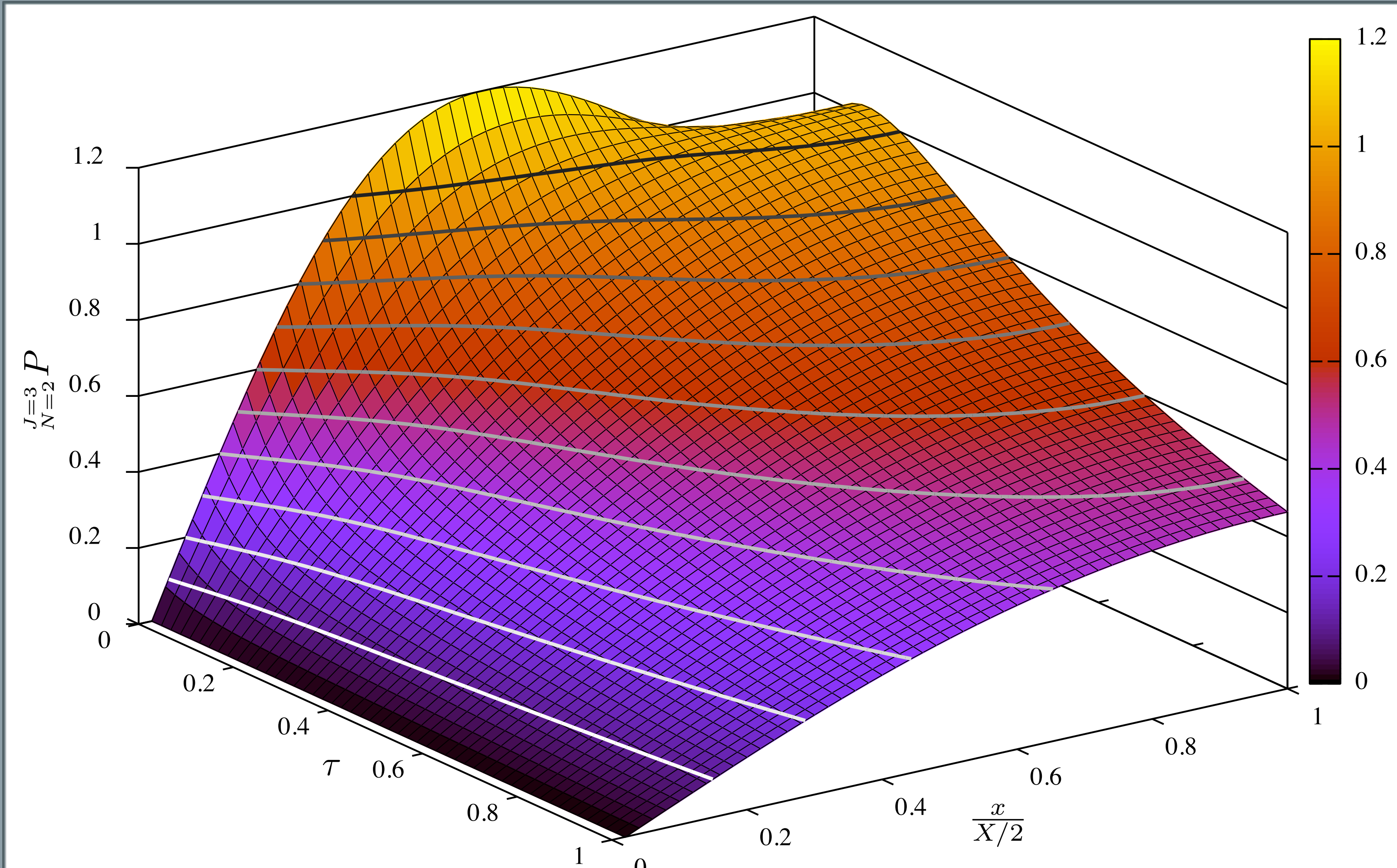
- Available benchmarks
  - "Initial" condition
  - Convergence on semi-analytical POI
  - Temporal evolution identical to analytical expression available with fundamental mode and quadratic approximations
  - Value of equilibrium eigenfunction coefficients identical to steady state coefficients
- Relative modal error

$$\Delta_N(\vec{r},\tau) = \frac{\left| {}_{N-1}P(\vec{r},\tau) - {}_{N}P(\vec{r},\tau)\right|}{{}_{N_{MAX}}P(\vec{r},\tau)}$$

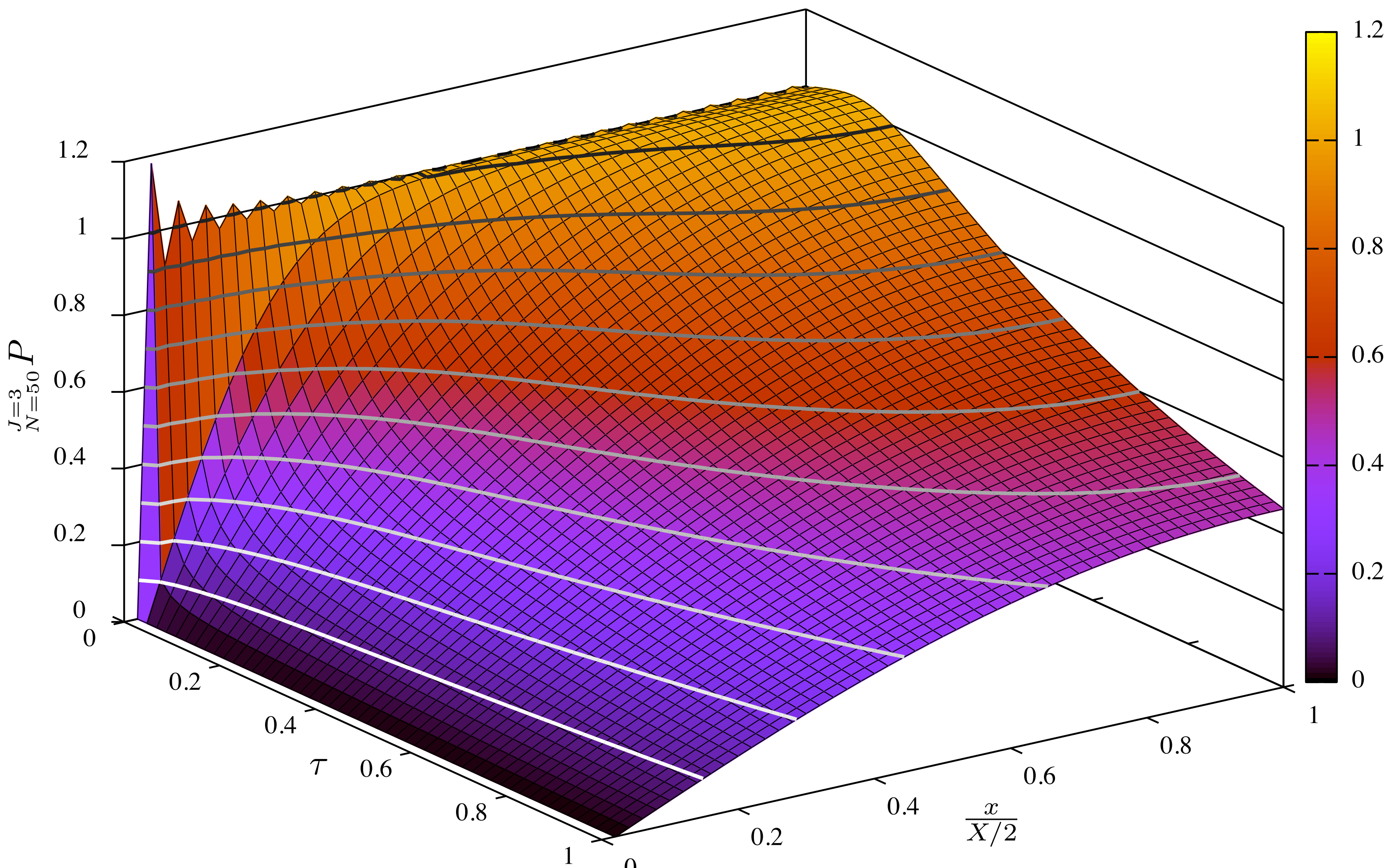
Modal Error for Various Survival Times



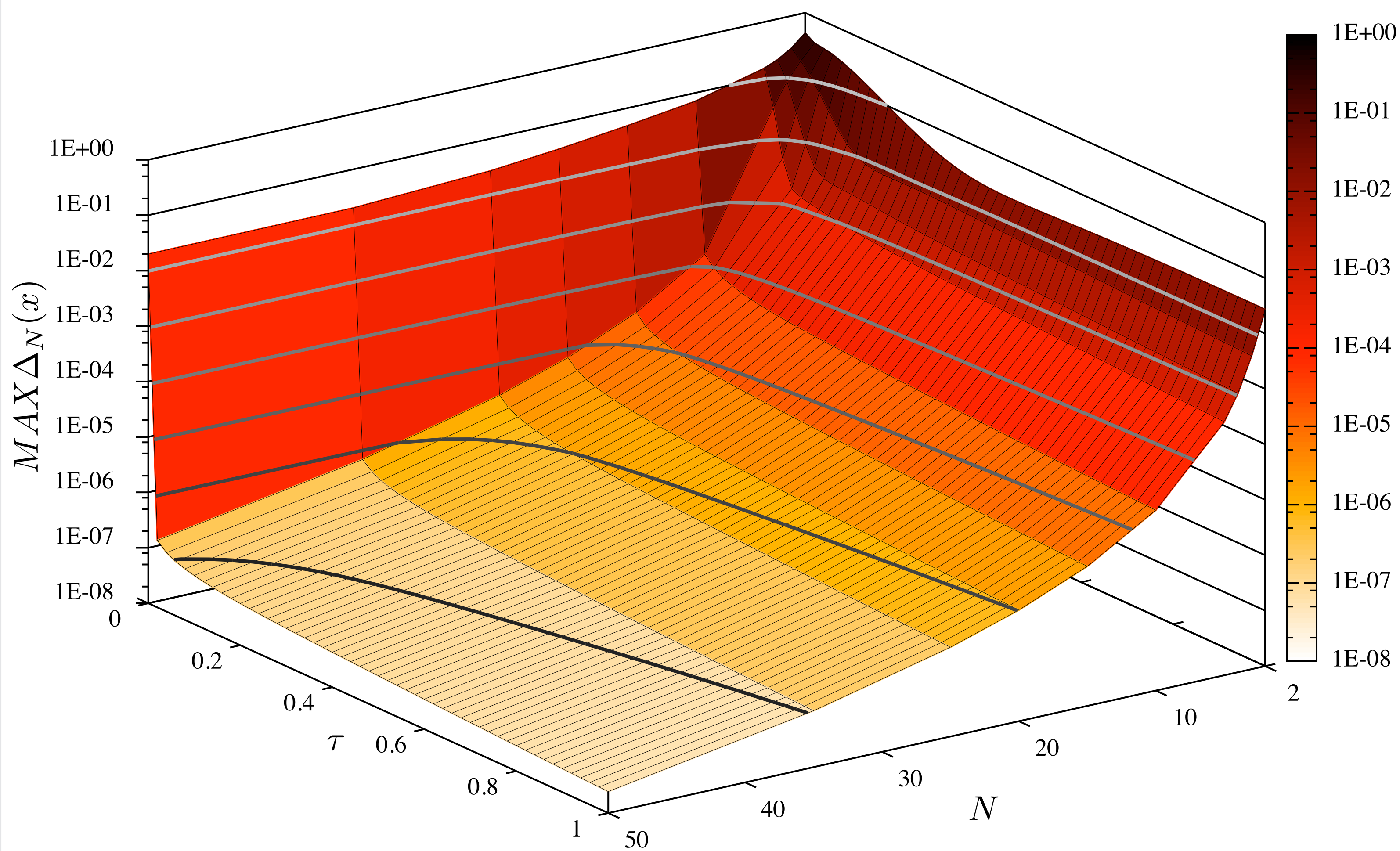
Behavior for Very Small Survival Times



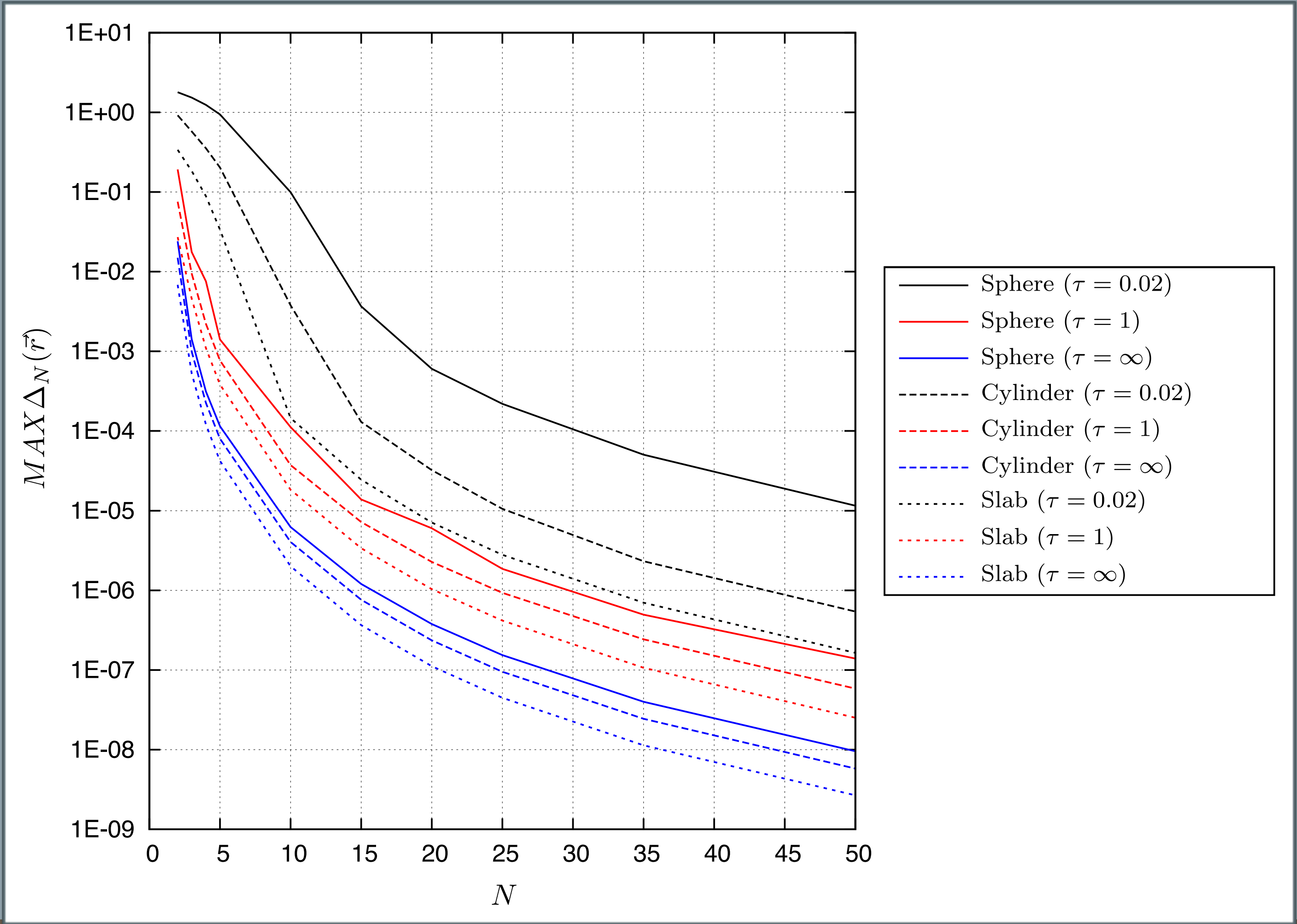
Behavior for Very Small Survival Times



Modal Error for Very Small Survival Times



Modal Error for Various Geometries

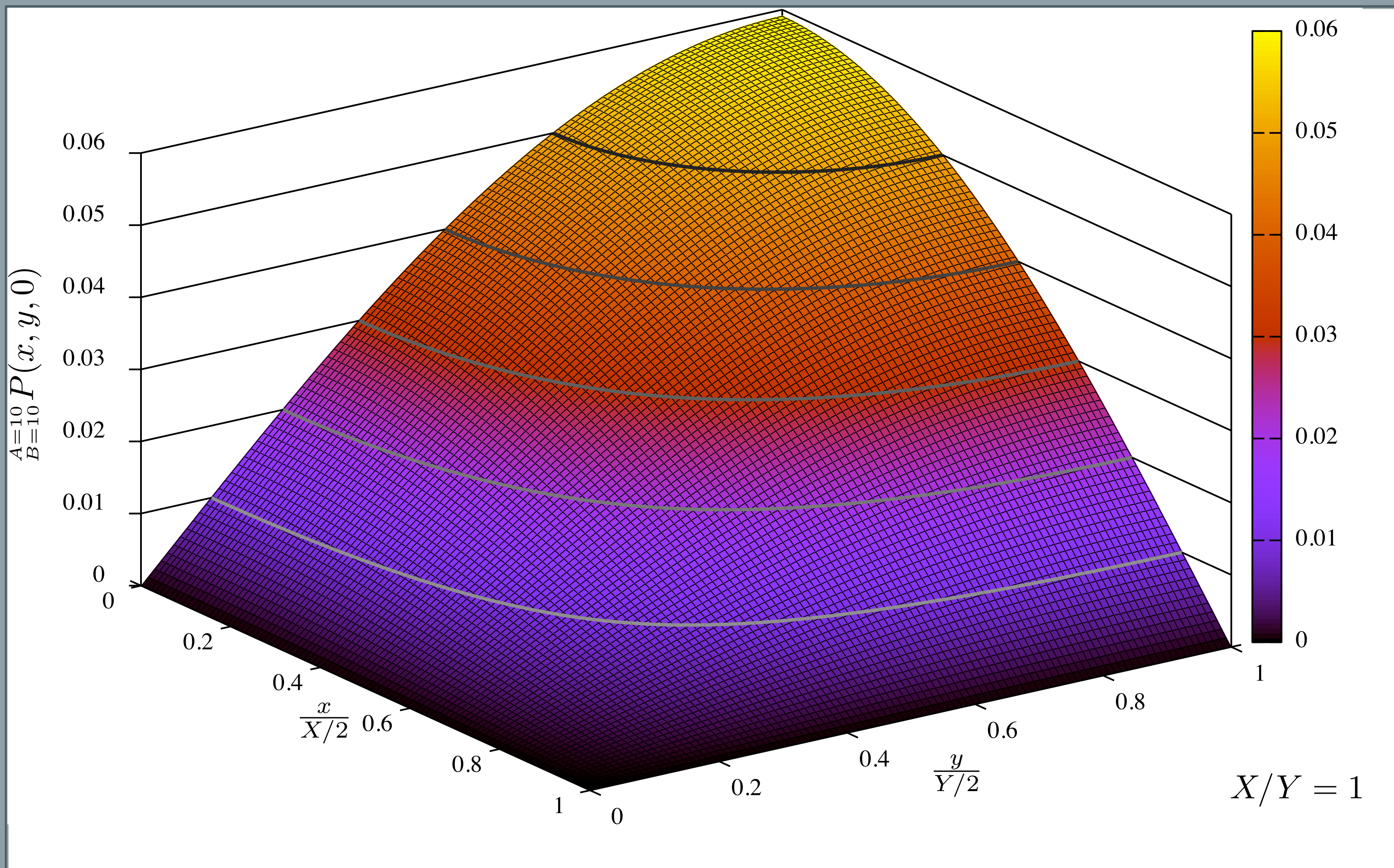


EFE in Multi-Dimensional Geometries

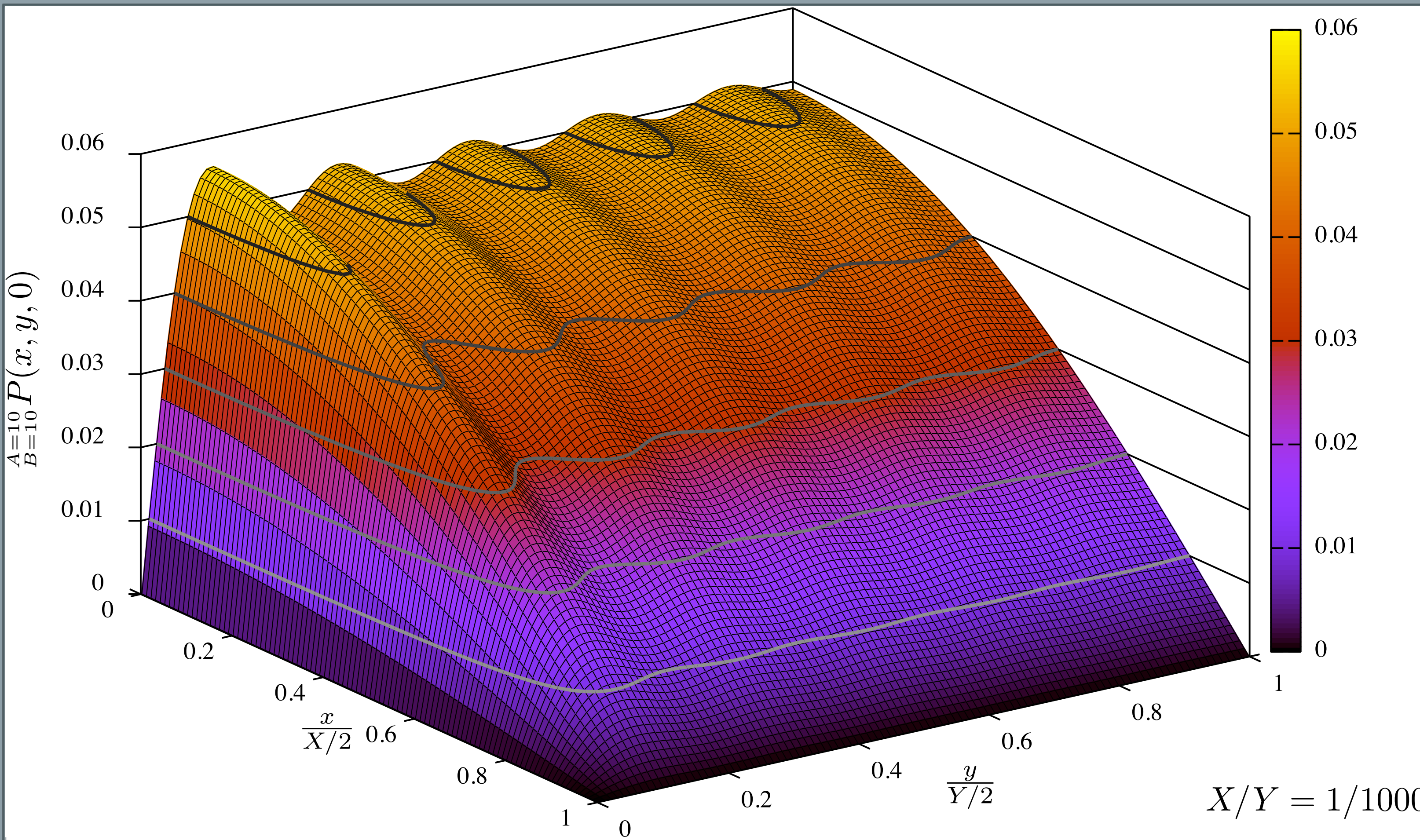
$$P\left(x,y,\tau\right)=\sum_{a=1}^{A}T_a\left(\tau\right)R_a\left(x\right)\sum_{b=1}^{B}T_b\left(\tau\right)R_b\left(y\right)$$

$$\frac{dT_{cd}}{d\tau}=\left(k_\infty-1-L^2\left[\lambda_c^2+\lambda_d^2\right]\right)T_{cd}$$
$$-\frac{\Sigma_F}{\Sigma_A\varrho_{cd}}\sum_{j=2}^{J}\frac{(-1)^j\chi_j}{j!}\int_{-X/2}^{X/2}\int_{-Y/2}^{Y/2}\left[\sum_{a=1}^{A}\sum_{b=1}^{B}T_{ab}R_aR_b\right]^j R_cR_d\,dy\,dx$$

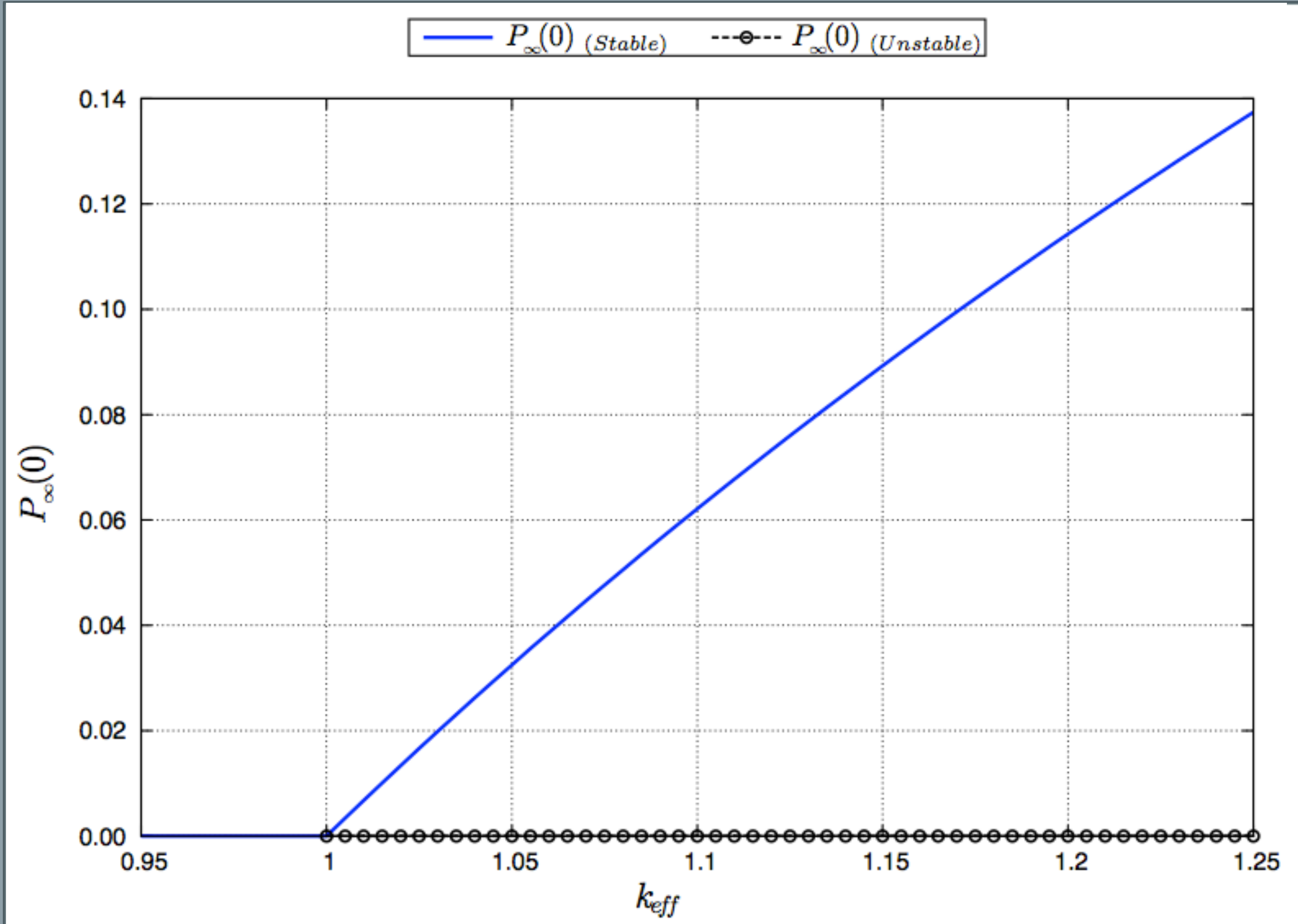
POI for Equal Side Lengths



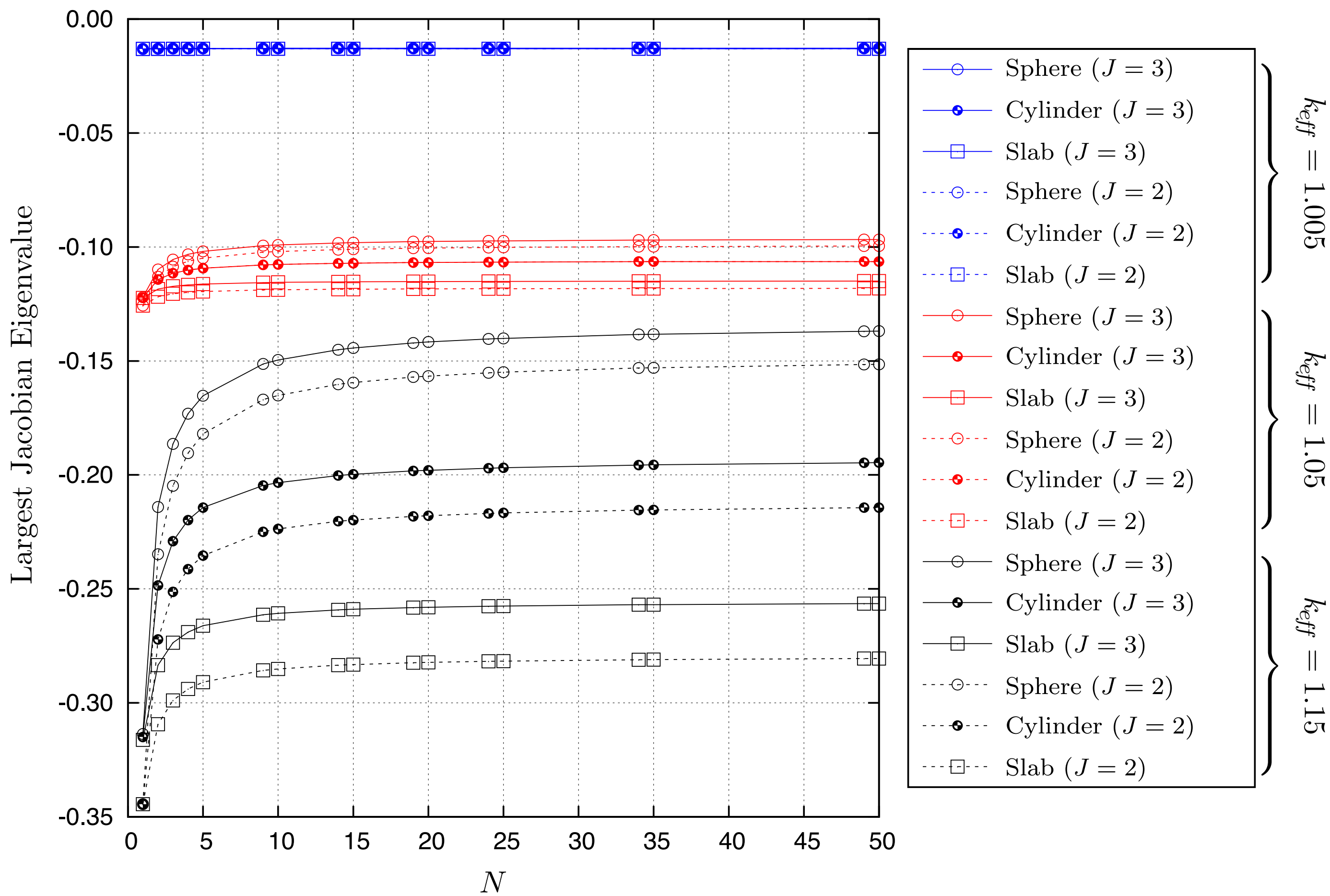
POI for Unequal Side Lengths



Linear Stability of POI



Results of Linear Stability Analysis



Summary of EFE

- Benchmarked perfectly against available solutions
- Offers extremely broad problem parameter space
- By tailoring the nonlinearity and size of expansion to suit problem, computational effort can be minimized to achieve desired accuracy
- Linear Stability Analyses shows the equilibrium solutions to be stable throughout examined parameter space

Future Work

- Transport as opposed to diffusion
- Multi-group
- Multi-region
- Time-dependent reactivity
- Intrinsic source
- "Throttling" capability (adaptivity) for factorial moments and number of modes