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Electromagnetic modes in periodically structured cavities

A Scattering Approach

Paul S. Davids[1]

[1] Applied Photonics and Microsystems
Sandia National Labs
Albuquerque NM USA

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Outline

1. Introduction
2. Modal Solutions to Maxwell's Equations
   - General non-symmetric eigenvalue problems
   - Mode bi-orthogonality and reciprocity
   - Planewave decomposition
3. Scattering from periodic structures
   - Transfer matrix
   - Scattering matrix
4. Cavity Modes

Introduction

- Casimir Force is a vacuum fluctuation driven force in an electromagnetic cavity.
  - Macroscopic manifestation of quantum electrodynamics.
  - Classical electromagnetic modes in cavity strongly influence Casimir force.
- Focus on electromagnetic modes in periodically modulated planar cavities.
  - 1D periodic - gratings
  - 2D periodic - planar metamaterials and 2D photonic crystals
- Goals:
  - Modify fluctuation electromagnetic forces in cavities by nanostructuring.
  - Provide computational tool to predict cavity fluctuation forces from nanostructures.

Maxwell's equations Electromagnetic wave propagation in a dispersive media is typically examined by solving Maxwell's equations in the frequency domain,

$$
\begin{aligned}
\nabla \times \vec{E} &= ik\mu(\omega)\vec{H} \qquad & \nabla \cdot \vec{D} = 0 \\
\nabla \times \vec{H} &= -ik\epsilon(\omega)\vec{E} \qquad & \nabla \cdot \vec{B} = 0,
\end{aligned}
\tag{1}
$$

Microscopic Relations

$$
\begin{aligned}
\vec{D}(\omega) &= \vec{E}(\omega) + 4\pi\vec{P}(\omega) \\
\vec{B}(\omega) &= \vec{H}(\omega) + 4\pi\vec{M}(\omega),
\end{aligned}
\tag{2}
$$

Constitutent Relations

$$
\begin{aligned}
\vec{D}(\omega) &= \epsilon(\omega)\vec{E}(\omega) \\
\vec{B}(\omega) &= \mu(\omega)\vec{H}(\omega).
\end{aligned}
\tag{3}
$$

- Decomposition of the fields into transverse and longitudinal components.

$$\vec{E} = \vec{E}_t + \hat{e}_3 E_z \quad (4)$$
$$\vec{H} = \vec{H}_t + \hat{e}_3 H_z. \quad (5)$$

- Split gradient operator into $\nabla = \nabla_t + \hat{e}_3 \partial_z$.

$$-ik\frac{\partial \vec{E}_t}{\partial z} = \nabla_{\mathbf{t}} \left[\frac{1}{\epsilon} \hat{e}_3 \cdot \nabla_{\mathbf{t}} \times \vec{H}_t \right] - k^2 \mu \hat{e}_3 \times \vec{H}_t, \quad (6)$$

$$-ik\frac{\partial \vec{H}_t}{\partial z} = -\nabla_{\mathbf{t}} \left[\frac{1}{\mu} \hat{e}_3 \cdot \nabla_{\mathbf{t}} \times \vec{E}_t \right] + k^2 \epsilon \hat{e}_3 \times \vec{E}_t \quad (7)$$

- Assume $\epsilon = \epsilon(x, y)$

Longitudinal Fields

$$E_z = \frac{i}{k\epsilon}\hat{e}_3 \cdot \left(\nabla_{\mathbf{t}} \times \vec{H}_t\right) \tag{8}$$

$$H_z = -\frac{i}{k}\hat{e}_3 \cdot \left(\nabla_{\mathbf{t}} \times \vec{E}_t\right). \tag{9}$$

$$\nabla_t \cdot (\epsilon \vec{E}_t) = -\epsilon \partial_z E_z \tag{10}$$

$$\nabla_t \cdot \vec{H}_t = -\partial_z H_z. \tag{11}$$

Eigenmodes

- Transverse Maxwell's equation Eq. (6-7)

$$-ik\partial_z\Psi = \mathbf{H}\Psi, \tag{12}$$

- $\Psi(x,y,z) = u(x,y)\exp(i\gamma z)$
- γ is the eigenmode propagation constant.

$$\Psi = \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} \exp(i\gamma) \tag{13}$$

and the operator **H** is defined as

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & -\partial_x(\frac{1}{\epsilon}\partial_y) & \partial_x(\frac{1}{\epsilon}\partial_x) + k^2 \\ 0 & 0 & -\partial_y(\frac{1}{\epsilon}\partial_y) - k^2 & \partial_y(\frac{1}{\epsilon}\partial_x) \\ \partial_x\partial_y & -\partial_x^2 - k^2\epsilon & 0 & 0 \\ \partial_y^2 + k^2\epsilon & -\partial_y\partial_x & 0 & 0 \end{pmatrix}. \tag{14}$$

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Non-symmetric eigenvalue problem

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- Eigenvalue for propagation constant γ.

$$k\gamma u = \mathbf{H}u. \tag{15}$$

- Secular determinant

$$\det(\mathbf{H} - k\gamma\mathbf{1}) = 0 \tag{16}$$

- $\mathbf{H} \neq \mathbf{H}^T$ is not symmetric and in general not self-adjoint ($\mathbf{H} \neq \mathbf{H}^\dagger$) Hermetian.
- left and right eigenvectors

$$k\gamma u = \mathbf{H}u \tag{17}$$
$$k\gamma v = \mathbf{H}^T v \tag{18}$$
$$k\gamma^* z = \mathbf{H}^\dagger z \tag{19}$$

- Formally $z = v^*$.

Left and Right Eigenmodes

- From operator definition: right eigenvector

$$u = \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} \tag{20}$$

- Left eigenvector

$$k\gamma z^\dagger = z^\dagger \mathbf{H} \tag{21}$$

- Sometimes

$$z = \begin{pmatrix} H_y \\ -H_x \\ -E_y \\ E_x \end{pmatrix}. \tag{22}$$

Outline

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- Complex eigenvalues (γ). If γ is an eigenvalue then γ^* is also an eigenvalue.
- Nondegenerate eigenvalues

$$k\gamma_j z_j^\dagger = z_j^\dagger \cdot H \tag{23}$$

$$k\gamma_i u_i = H \cdot u_i. \tag{24}$$

- Proof: We multiply the upper equation by u_i on the right and lower equation $z_j^\dagger$ on the left.

$$k(\gamma_j - \gamma_i) z_j^\dagger \cdot u_i = z_j^\dagger \cdot H \cdot u_i - z_j^\dagger \cdot H \cdot u_i = 0. \tag{25}$$

- Bi-orthogonality

$$z_j^\dagger \cdot u_i = S_i \, \delta_{i,j} \tag{26}$$

Poynting Vector Reciprocity

- Poynting Vector and Energy conservation for eigenmode

$$\vec{S} = \frac{c}{4\pi}\vec{E}_\mu \times \vec{H}_\mu \tag{27}$$

$$\nabla \cdot \vec{S} + \frac{1}{c}\frac{\partial dU}{\partial t} = 0 \tag{28}$$

- Source free region no dissipation, and U is electromagnetic energy density.
- Reciprocity gives

$$S_z^{\mu,\nu} = \left(E_x^\nu H_y^\mu - E_y^\nu H_x^\mu - H_x^\nu E_y^\mu + H_y^\nu E_x^\mu\right), \tag{29}$$

- Bloch Modes for tranvserse fields

$$\vec{E}_t = e^{i\vec{k}\cdot\vec{r}} \sum_{nm} \vec{E}_{nm} \exp\left(i\frac{2\pi n}{L_x} x + i\frac{2\pi m}{L_y} y \right), \tag{30}$$

and

$$\vec{H}_t = e^{i\vec{k}\cdot\vec{r}} \sum_{nm} \vec{H}_{nm} \exp\left(i\frac{2\pi n}{L_x} x + i\frac{2\pi m}{L_y} y \right) \tag{31}$$

- Eigenmode equation

$$-ik\frac{\partial \Psi_{n'm'}}{\partial z} = \sum_{nm} H_{n'm':nm} \Psi_{nm} \tag{32}$$

- Implicitly ϵ is periodic in x and y.

Eigenvalue Problem

$$k^2\beta\, u_{n'm'} = \sum_{nm} H_{n'm':nm}\, u_{nm} \qquad \Psi_{nm} = \begin{pmatrix} E^x_{nm} \\ E^y_{nm} \\ H^x_{nm} \\ H^y_{nm} \end{pmatrix} \exp(i\gamma z),$$

- Eigenmodes are Fourier coefficients of transverse fields.
- Eigenvalues are mode propagation constants, $\gamma = k\beta$.
- $H_{n'm':nm}$ is

$$\begin{pmatrix} 0 & 0 & q_{n'}q_m\chi & -q_{n'}q_n\chi + k^2\mu \\ 0 & 0 & q_{m'}q_m\chi - k^2\mu & -q_nq_{m'}\chi \\ -q_{n'}q_m\zeta & q_{n'}q_n\zeta - k^2\epsilon & 0 & 0 \\ -q_{m'}q_m\zeta + k^2\epsilon & q_nq_{m'}\zeta & 0 & 0 \end{pmatrix}$$

- $\epsilon = \epsilon_{n'-n,m'-m}$ and $\chi = 1/\epsilon$ with $\chi = \chi_{n'-n,m'-m}$

Bloch Mode Orthogonality

The transverse modal fields are given by

$$X_\nu(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \sum_{nm} u^\nu_{nm} \exp\left(i\frac{2\pi n}{L_x}x + i\frac{2\pi m}{L_y}y \right), \tag{33}$$

and we have a complementary expression for the left handed modal fields, $\bar{X}^\mu(\vec{r})$,

$$\bar{X}_\nu(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \sum_{nm} z^\nu_{nm} \exp\left(i\frac{2\pi n}{L_x}x + i\frac{2\pi m}{L_y}y \right). \tag{34}$$

The transverse modal fields obey the orthogonality relationship,

$$\frac{1}{L_x L_y} \int \bar{X}^\dagger_\mu(\vec{r}) X_\nu(\vec{r}) d\vec{r} = \delta_{\mu,\nu}. \tag{35}$$

where we have used explicitly the bi-orthogonality of u and $z^\dagger$.

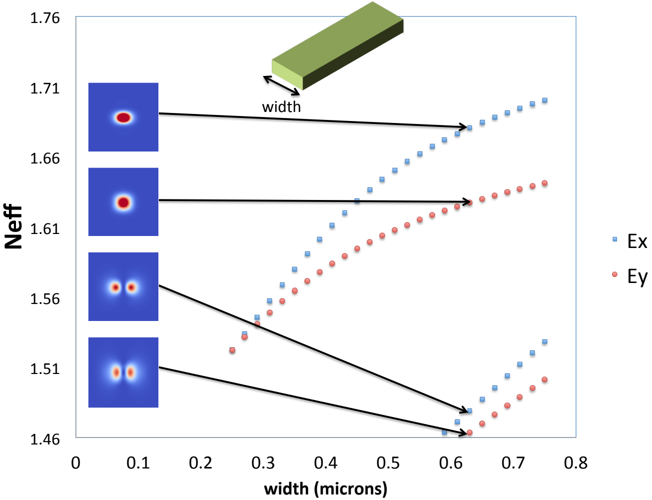
Modal Expansion

$$\Psi(\vec{r}, z) = \sum_{\nu} C_\nu X_\nu(\vec{r}) \exp(ik\beta_\nu z),$$

- C_ν are modal expansion coefficients.
- Must truncate Fourier series in X_ν.
- Define $N = N_x N_y$ where $0 \le n < N_x$ and $0 \le m < N_y$.
- H is $4N \times 4N$ matrix.
- $4N$ eigenvalues $0 \le \nu < 4N$, β_ν.
- Polarization: $2N$ $\beta^{(+)}$ and $2N$ $\beta^{(-)}$.

Note, $\epsilon_{n-n',m-m'}$ and $\chi_{n-n',m-m'}$ has twice the spatial frequency content.

Mode Examples: Rectangular dielectric



Mode Examples: Circular dielectric

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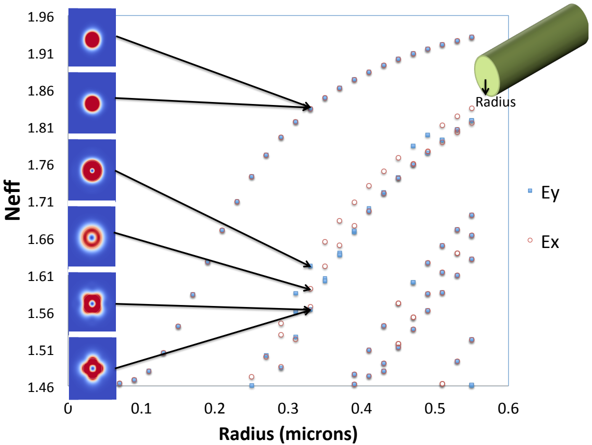
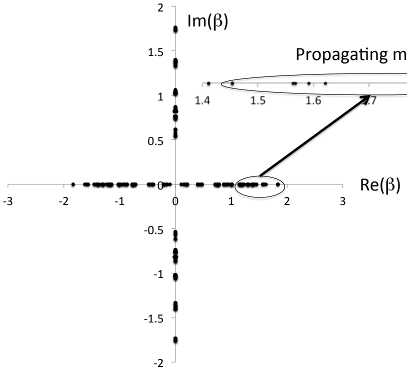


Note, modes are not uniquely labeled by Cartesian polarization.

Eigenmode Spectrum

Real and Imaginary effective indices of modes



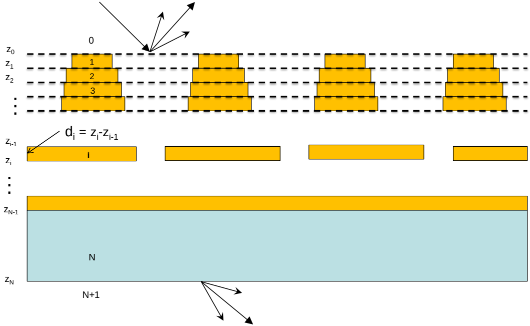
Propagating modes bound between index of core and cladding.

Modal RCWA

- Periodic in x and y and consisting of slices along z axis.
- Modal expansion in ith layer

$$\Psi^i(\vec{r}, z) = \sum_{\nu} A^i_{\nu} X^{(+)}_{i,\nu}(\vec{r}) \exp(ik\beta^{(+)}_{i,\nu} z) + B^i_{\nu} X^{(-)}_{i,\nu}(\vec{r}) \exp(-ik\beta^{(-)}_{i,\nu} z)$$

- *A* is forward propagating mode, *B* is backward propagating mode.



Outline

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Boundary Conditions

- Continuity of transverse fields $\Psi_i(z_i) = \Psi_{i+1}(z_i)$
- Transfer matrix relates modal expansion coefficients

$$\begin{pmatrix} A_{i+1} \\ B_{i+1} \end{pmatrix} = \begin{pmatrix} t_{11}\exp(i\gamma^+) & t_{12}\exp(i\gamma^-) \\ t_{21}\exp(i\gamma^+) & t_{22}\exp(i\gamma^-) \end{pmatrix} \cdot \begin{pmatrix} A_i \\ B_i \end{pmatrix} \tag{36}$$

- $\gamma^\pm = k\beta^\pm d_i$ where d_i is layer thickness.
- Transfer matrices are mode overlap between layers.

$$\begin{aligned} t_{11} &= \sum_{nm} z_{nm}^{*\nu(+)}(i+1)u_{nm}^{\mu(+)}(i) \\ t_{12} &= \sum_{nm} z_{nm}^{*\nu(+)}(i+1)u_{nm}^{\mu(-)}(i) \\ t_{21} &= \sum_{nm} z_{nm}^{*\nu(-)}(i+1)u_{nm}^{\mu(+)}(i) \\ t_{22} &= \sum_{nm} z_{nm}^{*\nu(-)}(i+1)u_{nm}^{\mu(-)}(i) \end{aligned} \tag{37}$$

Reflection and Transmission matrices

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- Global transfer matrix

$$\begin{pmatrix} A_{N+1} \\ B_{N+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \quad (38)$$

- Transfer matrix product.

$$T_{\nu,\mu} = \sum_{\mu_N} \ldots \sum_{\mu_1} T^{N+1,N}_{\nu,\mu_N} T^{N,N-1}_{\mu_N,\mu_{N-1}} \ldots T^{1,0}_{\mu_1,\mu}, \quad (39)$$

- Reflection and Transmission in terms of eigenmode coefficients.

$$\begin{aligned} B_0 &= -T_{22}^{-1} \cdot T_{21} \cdot A_0 \\ A_{N+1} &= (T_{11} - T_{12} \cdot T_{22}^{-1} \cdot T_{21}) \cdot A_0. \end{aligned} \quad (40)$$

- Unstable due to growing exponential in thick layers.

Outline

- Modification of Transfer matrix approach (L. Li)

$$\begin{pmatrix} A_{i+1} \\ B_i \end{pmatrix} = \begin{pmatrix} s^i_{11} & s^i_{12} \\ s^i_{21} & s^i_{22} \end{pmatrix} \cdot \begin{pmatrix} A_i \\ B_{i+1} \end{pmatrix}.$$

- Layer S-matrix $\gamma^{\pm}$

$$\begin{pmatrix} s^i_{11} & s^i_{12} \\ s^i_{21} & s^i_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \exp(-i\gamma^-) \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \exp(i\gamma^+) & 0 \\ 0 & 1 \end{pmatrix}$$

- In terms of transfer matrices

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} t_{11} - t_{12} t_{22}^{-1} t_{21} & t_{12} t_{22}^{-1} \\ -t_{22}^{-1} t_{21} & t_{22}^{-1} \end{pmatrix}.$$

- Define total S-Matrix

$$\begin{pmatrix} A_{N+1} \\ B_0 \end{pmatrix} = \begin{pmatrix} S_{11}^{(N)} & S_{12}^{(N)} \\ S_{21}^{(N)} & S_{22}^{(N)} \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ B_{N+1} \end{pmatrix}$$

- S-Matrix recursion (iterate to find Reflection and Transmission)

$$\begin{aligned} S_{11}^{(i)} &= s_{11}^{i}\left(1 - S_{12}^{(i-1)} s_{21}^{i}\right)^{-1} S_{11}^{(i-1)} \\ S_{12}^{(i)} &= s_{12}^{i} + s_{11}^{i}\left(1 - S_{12}^{(i-1)} s_{21}^{i}\right)^{-1} S_{12}^{(i-1)} s_{22}^{i} \\ S_{21}^{(i)} &= S_{21}^{(i-1)} + S_{22}^{(i-1)}\left(1 - s_{21}^{i} S_{12}^{(i-1)}\right)^{-1} s_{21}^{i} S_{11}^{(i-1)} \\ S_{22}^{(i)} &= S_{22}^{(i-1)}\left(1 - s_{21}^{i} S_{12}^{(i-1)}\right)^{-1} s_{22}^{i} \end{aligned}$$

- Initial is incident layer S-Matrix : $S^{(0)} = s^{(0)}$

R and T Matrices

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- The reflection and transmission matrices are

$$\begin{aligned} R &= S_{21} \\ T &= S_{11} \end{aligned} \tag{41}$$

- R and T are $2N \times 2N$ matrices.
- Numerical considerations
  - S-matrix is stable for thick stacks due to $\exp(-i\gamma^-)$ and $\exp(i\gamma^+)$ terms in def.
  - k_x and k_y are fixed in first Brillouin zone, $-\pi/L_x \leq k_x \leq \pi/L_x$ and $-\pi/L_y \leq k_y \leq \pi/L_y$.
  - Fourier series converge to mean value at field discontinuity.
  - What is proper cutoff for converged S-matrix?

RCWA Example

- This is item

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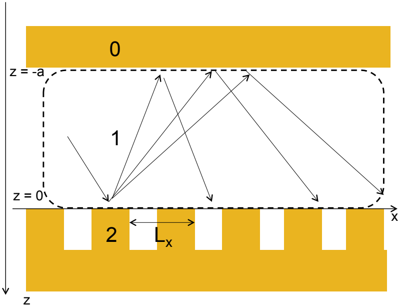
Putting it all together

Vacuum cavity (region 1) between 2 plates.



Periodically patterned (region 2) Upper plate (region 0) is uniform.

- Eigenmode expansions in 3 regions of cavity.

$$\Psi_0 = \sum_{\mu} C_\mu X^{(-)}_{\mu,0} e^{-iq_z^{(0)} z} \quad (42)$$

$$\Psi_1 = \sum_{\mu} A_\mu X^{(+)}_{\mu,1} e^{iq_z^{(1)} z} + B_\mu X^{(-)}_{\mu,1} e^{-iq_z^{(1)} z} \quad (43)$$

$$\Psi_2 = \sum_{\mu} D_\mu X^{(+)}_{\mu,2} e^{iq_z^{(2)} z}, \quad (44)$$

- Propagation constants in uniform media, $q_z^{(i)} = \sqrt{k^2 \epsilon_i - q_n^2 - q_m^2}$.
- Eigenvalue indexed by $\mu = (n, m, \sigma)$ where $\sigma =$ s or p polarization.

Boundary Conditions

- Cavity split into 3 regions:
  - Region 1 is vacuum space between periodic surface and planar mirror.

$$\Psi_1 = \sum_\mu \left(A_\mu X^{(+)}_{\mu,1} e^{iq_z^{(1)} z} + \sum_\nu R_{\mu,\nu} A_\nu X^{(-)}_{\mu,1} e^{-iq_z^{(1)} z} \right), \quad (45)$$

  - $R = S_{21}$ has been determined at $z = 0$, $B = R \cdot A$ in matrix notation.
- Boundary conditions on tranvserse fields
  - $\Psi_0(z = -a) = \Psi_1(z = -a)$
  - $\Psi_1(z = 0) = \Psi_2(z = 0)$.
- Use bi-orthogonality to project and determine eigenmode expansion coefficients.

Coefficient projections

$$C_\mu e^{iq_z^0 a} = i\alpha_\mu A_\mu e^{-iq_z^1 a} + \beta_\mu B_\mu e^{iq_z^1 a}, \tag{46}$$

where

$$\alpha_\mu = \begin{cases} \frac{1}{2\sqrt{q_z^{(0)} q_z^{(1)}}}\left(q_z^{(1)} - q_z^{(0)}\right) & \text{for s polarization} \\ \frac{1}{2\sqrt{q_z^{(0)} q_z^{(1)}}}\left(\sqrt{\frac{\epsilon_0}{\epsilon_1}} q_z^{(1)} - \sqrt{\frac{\epsilon_1}{\epsilon_0}} q_z^{(0)}\right) & \text{for p polarization,} \end{cases}$$

and

$$\beta_\mu = \begin{cases} \frac{1}{2\sqrt{q_z^{(0)} q_z^{(1)}}}\left(q_z^{(1)} + q_z^{(0)}\right) & \text{for s polarization} \\ \frac{1}{2\sqrt{q_z^{(0)} q_z^{(1)}}}\left(\sqrt{\frac{\epsilon_0}{\epsilon_1}} q_z^{(1)} + \sqrt{\frac{\epsilon_1}{\epsilon_0}} q_z^{(0)}\right) & \text{for p polarization.} \end{cases}$$

$$C_\mu = -\frac{i}{\alpha_\mu} A_\mu e^{-i(q_z^1+q_z^0)a}. \tag{47}$$

The secular equation is

$$\sum_\nu \left(-i\delta_{\mu,\nu} \left(\frac{1}{\alpha_\mu} + \alpha_\mu \right) e^{-iq_z^1 a} - e^{iq_z^1 a} \beta_\mu R_{\mu,\nu} \right) A_\nu = 0. \tag{48}$$

$$\sum_\nu \left(\delta_{\mu,\nu} - e^{2iq_{z,\mu}^{(1)} a} \rho_\mu R_{\mu,\nu} \right) A_\nu = 0, \tag{49}$$

$$\rho_\mu = \begin{cases} ir_s^{nm}(1,0) & \text{for s polarization} \\ ir_p^{nm}(1,0) & \text{for p polarization,} \end{cases}$$

and r_s and r_p are the Fresnel reflection coefficients.

Cavity Eigenmodes

- The generalized cavity secular equation

$$\det(\mathbf{I} - e^{iq_z a} \cdot \rho \cdot e^{iq_z a} \cdot R) = 0, \tag{50}$$

- Solve for cavity eigenfrequencies, $\omega(k_x, k_y)$.
- Condition for non-trivial coefficients
- Define Cavity matrix D

$$D_{\mu,\nu} = \delta_{\mu,\nu} - (e^{iq_z a} \cdot \rho \cdot e^{iq_z a} \cdot R)_{\mu,\nu}. \tag{51}$$

- Here ρ is a diagonal reflection matrix.
- Propagation matrices are $e^{iq_z a}$.

Cavity spectrum

- Cavity eigenmodes spectrum

$$\sum_{\nu} D_{\mu,\nu} x_{\nu} = \Lambda_{\mu} x_{\mu}, \tag{52}$$

- Secular equation

$$\det(D) = \prod_{n=1}^{2N} \Lambda_n = f(\omega, k_x, k_y, a), \tag{53}$$

- Exact Eigenfrequencies correspond to zero eigenvalue.
- Approximate Eigenfrequencies correspond to large condition number, $\kappa = |\Lambda_{max}|/|\Lambda_{min}|$

Theorem

Given a closed rectifiable Jordan curve L, suppose $g(z)$ is analytic on $\overline{I(L)}$, while $f(z)$ is is analytic on $\overline{I(L)}$ except for poles in $I(L)$, $b_0, b_1 ... b_n$. Moreover, suppose $f(z)$ has A points, $a_0, a_1, ..., a_m$ in $I(L)$ but none on L itself. Then

$$\frac{1}{2\pi i}\int_L g(z)\frac{f'(z)}{f(z)-A}dz = \sum_{k=1}^{m}\alpha_k g(a_k) - \sum_{k=1}^{n}\beta_k g(b_k), \tag{54}$$

it where α_k is the order of A-points and b_k is the order of the pole. The A-points refer to solutions of the equations $f(a_k) = A$ and if A=0, correspond to the zeroes of the function f.

[R. Silverman, Introductory Complex Analysis,]

- Cavity secular determinant

$$\frac{f'(\omega)}{f(\omega)} = \frac{d\log(f(\omega))}{d\omega} \to \frac{d\log(\det(D(\omega)))}{d\omega}, \tag{55}$$

- Hurwitz Roche

$$\frac{1}{2\pi i}\int_L \frac{f'(\omega)}{f(\omega)} d\omega = \sum_{k=1}^{m} \alpha_k - \sum_{k=1}^{n} \beta_k, \tag{56}$$

and

$$\frac{1}{2\pi i}\int_L \omega \frac{f'(\omega)}{f(\omega)} d\omega = \sum_{k=1}^{m} \alpha_k \omega_k - \sum_{k=1}^{n} \beta_k \zeta_k, \tag{57}$$

- Cavity eigenfrequencies ω_k depend implicitly on (k_x, k_y) and the separation a.
- ζ_k are the poles of the secular determinant in the region bounded by the closed contour.

Distinct Eigenvalues

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Recall

$$f(\omega, k_x, k_y, a) = \det(D) = \det(\mathbf{1} - e^{iq_z a} \cdot \rho \cdot e^{iq_z a} \cdot R) \qquad (58)$$

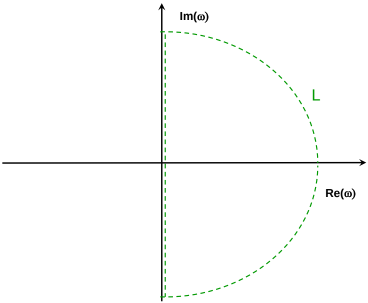
- Distinct non-degenerate cavity eigenvalues with no poles in L. $\alpha_k = 1$, $\beta_k = 0$, and $\zeta_k = 0$.

$$N(k_x, k_y, a) = \frac{1}{2\pi i} \int_L d\omega \frac{d \log(\det(D(\omega)))}{d\omega} \qquad (59)$$

$$E(k_x, k_y, a) = \sum_{k=1}^{m} \hbar\omega_k = \frac{\hbar}{2\pi i} \int_L d\omega\omega \frac{d \log(\det(D(\omega)))}{d\omega} \qquad (60)$$

- $N(k_x, k_y, a)$ is the number of cavity eigenmodes. $E(k_x, k_y, a)$ cavity energy.

- Choose contour L in complex plane.
- Integration contribution along $\omega = i\zeta$
- Integrate over first Brillioun zone.



Cavity energy per unit area

$$\mathcal{E} = \frac{\hbar}{2\pi} \int_{-\frac{\pi}{L_x}}^{\frac{\pi}{L_x}} \frac{dk_x}{2\pi} \int_{-\frac{\pi}{L_y}}^{\frac{\pi}{L_y}} \frac{dk_y}{2\pi} \int_0^\infty d\zeta \log(\det(D(i\zeta))) \qquad (61)$$

Force per unit area

$$\mathcal{P} = -\frac{\partial \mathcal{E}}{\partial a} \qquad (62)$$

Casimir Free Energy

- Starting point for Casimir Force calculation

$$\mathcal{F} = \frac{1}{\beta} \int_{-\frac{\pi}{L_x}}^{\frac{\pi}{L_x}} \frac{dk_x}{2\pi} \int_{-\frac{\pi}{L_y}}^{\frac{\pi}{L_y}} \frac{dk_y}{2\pi} {\sum_{n=0}^{\infty}}' \log(\det(\mathbf{1} - e^{-2q_{zn}a} \cdot \rho(\zeta_n) \cdot R(\zeta_n)))$$

- Finite Temperature Free-energy $\mathcal{F}$
- Imaginary Matsubara frequencies $\zeta_n = 2\pi n/\hbar\beta$ with n a positive integer, and $\beta = 1/kT$.
- Matsubara wavevectors are $\kappa_n = 2\pi n/\hbar c\beta$.

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - Something you haven't solved.
  - Something else you haven't solved.

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Best title

- This is item

TE-TM Decomposition

$$\vec{E}_t = \frac{i}{k^2\epsilon - \gamma^2}\left(\gamma\nabla_t E_z - k\hat{e}_3 \times \nabla H_z\right) \quad (63)$$

$$\vec{H}_t = \frac{i}{k^2\epsilon - \gamma^2}\left(\gamma\nabla_t H_z + k\epsilon\hat{e}_3 \times \nabla E_z\right). \quad (64)$$

TE Decomposition

The transverse fields can be expressed in terms of a scalar field variable, $\psi_e = H_z$,

$$\vec{E}_{TE} = -\frac{ik}{k^2\epsilon - \gamma^2}(\hat{e}_3 \times \nabla_t \psi_e) \tag{65}$$

$$\vec{H}_{TE} = \frac{i\gamma}{k^2\epsilon - \gamma^2}\nabla_t \psi_e. \tag{66}$$

TM Decomposition

The tranverse magnetic field modes require that the z component of the magnetic field vanish in the entire cross-section and we have

$$\vec{E}_{TM} = \frac{i\gamma}{k^2\epsilon - \gamma^2}(\nabla_t \psi_m) \qquad (67)$$

$$\vec{H}_{TM} = \frac{ik\epsilon}{k^2\epsilon - \gamma^2}(\hat{e}_3 \times \nabla_t \psi_m), \qquad (68)$$

where $\psi_m = E_z$.