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Parameterized Reduced Order Models Constructed Using Hyper Dual Numbers

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Abstract

In order to assess the predicted performance of a manufactured system, analysts must typically consider random variations (both geometric and material) in the development of a finite element model, instead of a single deterministic model of an idealized geometry. The incorporation of random variations, however, could potentially require the development of thousands of nearly identical solid geometries that must be meshed and separately analyzed, which would require an impractical number of man-hours to complete. This paper proposes a new approach to uncertainty quantification by developing parameterized reduced order models. These parameterizations are based upon Taylor series expansions of the system's matrices about the ideal geometry, and a component mode synthesis representation for each linear substructure is used to form an efficient basis with which to study the system. The numerical derivatives required for the Taylor series expansions are obtained efficiently using hyper dual numbers, which enable the derivatives to be calculated precisely to within machine precision. The theory is applied to a stepped beam system in order to demonstrate proof of concept. The accuracy and efficiency of the method, as well as the level at which the parameterization is introduced, are discussed. Hyper dual numbers can be used to construct parameterized models both efficiently and accurately and constitute an appropriate methodology to account for perturbations in a structural system.

1 Introduction

Modern engineering analysis must take into account the effects of aleatoric (parametric) uncertainty in the analysis of a system. As a real system is manufactured, part-to-part variations are introduced that can have significant ramifications on the functionality of the system. Thus, in order to account

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for these variations at the design stage, a methodology is needed to assess the performance of many (often thousands) of permutations of a design to qualify the performance of a manufactured system.

One method to simulate the performance of a system is via high fidelity modeling, such as using the finite element (FE) method. High fidelity computational simulations can often provide very accurate predictions; however, they have a very high computational cost. In order to develop simulations that are both efficient and accurate enough, reduced order models (ROMs) often are used as surrogates for a full order model in order to decrease the computational expense of analysis.

To model the perturbations that are found in manufactured systems without a systematic, efficient reduced order approach would be prohibitively difficult. For example, consider a scenario where it takes several weeks to develop a high quality mesh for one relatively simple component. To quantify the aleatoric uncertainty associated with manufacturing, thousands of perturbations of the ideal geometry are necessary, and each likely requires a new mesh. Even with factoring in time saved from some automation of the process, the number of man hours required to construct these meshes is on the order of 20 years. In addition, the computational time to analyze all of these models is on the order of several years assuming that an entire super computer can be dedicated to the analysis. Clearly, decades of time are infeasible constraints to be incorporated into a design cycle. One method of accounting for these perturbations is to create a parameterized reduced order model (PROM) of the system [1–4]. This allows the behavior of the system to be inexpensively predicted over a range of perturbations based on a few simulations distributed about the nominal design.

A standard approach to constructing ROMs for structural dynamics is Craig-Bampton (C-B) Component Mode Synthesis (CMS) [5]. Figure 1 shows two approaches that can be taken to analyze a system composed of several components. The left branch of the diagram shows the traditional approach of forming a full order FE model for the system, which often is prohibitively expensive for assessing aleatoric uncertainty. The right branch shows the steps when C-B CMS is used. To account for aleatoric uncertainty, PROMs cab be utilized. Simple PROMs can be constructed from a finite Taylor series expansion; for instance, in computing some scalar quantity of interest, $f(x)$, as a function of some perturbation to the nominal design, Δx, $f(x)$ can be approximated as $\bar{f}(x)$, which is based on a Taylor series expansion. For the FE and CMS cases outlined in Fig. 1, $f(x)$ could be elements of the mass and stiffness matrices or it could be the result of the system analysis, such as displacements or eigenvalues. The perturbations to the nominal design, Δx, could be changes in material properties or geometric variations. A parameterized model can then be created as

$$\bar{f}(x + \Delta x) = f(x) + (\Delta x) f'(x) + \frac{(\Delta x)^2}{2} f''(x) + \frac{(\Delta x)^3}{6} f'''(x) + \dots . \tag{1}$$

In practice, this infinite sum is truncated to be finite valued. In what follows, a second order and a third order expansion are considered.

The use of these parameterized models with FE analysis requires the calculation of derivatives at the nominal design. These derivatives can be computed in many ways, such as using finite difference approximations; however, finite difference approximations can require multiple meshes for each dimension of a perturbation, which can result in an impractically large amount of man-hours. For instance, a fourth order accurate cubic finite difference expansion for just two variables could require 49 separate meshes, three variables could require 343 meshes, and N variables could require as many as 7^N. The approach proposed here is to compute the derivatives using hyper dual numbers [6]. Hyper dual numbers produce exact values of the derivatives, and only require a single evaluation at the nominal design, in contrast to finite difference approximations. The use of

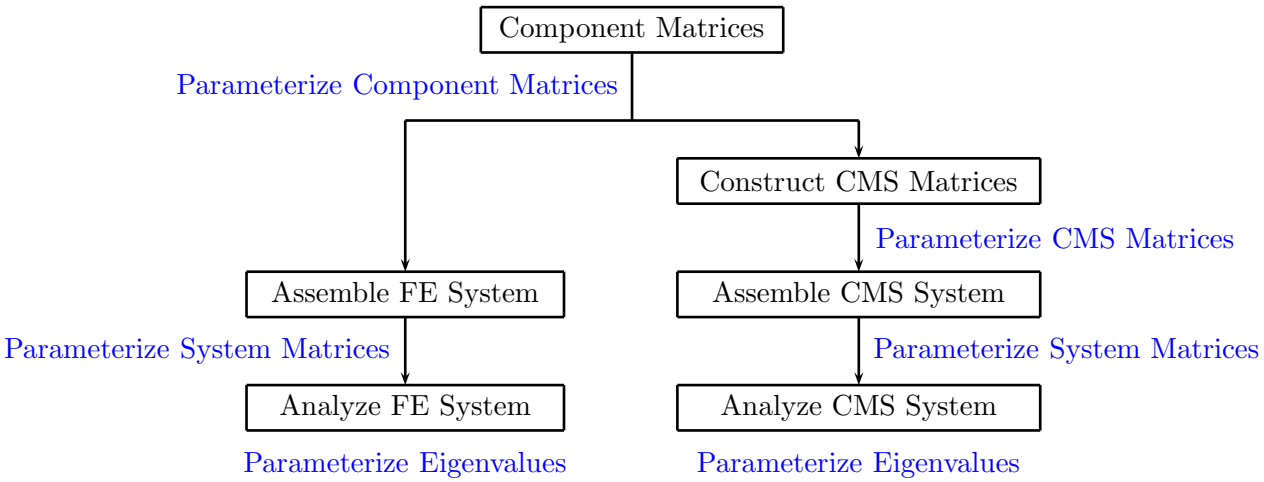


Figure 1: Two possible approaches for analyzing a system composed of several components, and the levels at which parameterized models can be constructed.

hyper dual numbers only requires the nominal mesh, but the information on how the mesh would change due to the perturbations needs to be known or calculated. In practice, this can require the introduction of hyper dual numbers into the meshing procedure.

Parameterized models can be created at several levels in the analysis procedure outlined in Figure 1. One possibility is to construct parameterized models for the component mass and stiffness matrices. Another possibility is to parameterize the output of the system analysis, i.e. displacements or eigenvalues. When using CMS it is also possible to parameterize the reduced order mass and stiffness matrices. One goal of this research is to assess the efficacy of developing PROMs at each level identified in Fig. 1.

This paper presents a brief overview of C-B CMS in §2, and then in §3 introduces hyper dual numbers and discusses the development of PROMs, where the required derivatives are computed using hyper dual numbers [6]. Next, §4 assesses the efficacy of each potential level of developing a PROM, as indicated in Fig. 1, and conclusions are presented in §5.

2 Craig-Bampton Component Mode Synthesis

The equation of motion for an unforced and undamped structure with mass matrix $[M]$, stiffness matrix $[K]$, and displacement $\{u\}$ is

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\}\,. \tag{2}$$

The C-B CMS [5] method is based on a substructuring of the component's degrees of freedom (DOFs) into boundary (u_b) DOFs and internal (u_i) DOFs as $\{u\} = \{u_b, u_i\}^T$. Boundary DOFs typically are defined as DOFs where excitations are applied or where output quantities are desired (such as the displacement of a particular flange), whereas internal DOFs are all non-boundary DOFs, which generally are inside of the component and thus have no applied loads. This substruc-

turing is used to recast Eq. 2 as

$$\begin{bmatrix} M_{bb} & M_{bi} \\ M_{ib} & M_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{u}_b \\ \ddot{u}_i \end{Bmatrix} + \begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix} \begin{Bmatrix} u_b \\ u_i \end{Bmatrix} = \{0\}. \tag{3}$$

Modal analysis is performed on the internal degrees of freedom,

$$(K_{ii} - \lambda_\ell M_{ii})\, \phi_\ell = 0, \tag{4}$$

where λ_ℓ are individual eigenvalues and ϕ_ℓ are the corresponding eigenvectors of the fixed interface normal modes. Constraint modes, ϕ_C, are a static deflection shape and are computed by imposing a unit displacement at each individual boundary DOF while holding the other boundary DOFs fixed at zero displacement. The constraint modes can therefore be computed as

$$\phi_C = -K_{ii}^{-1} K_{ib}. \tag{5}$$

Subsequently, the physical DOFs (u_b and u_i) are related to the hybrid set of physical and modal DOFs (u_b and q_m) by

$$\begin{Bmatrix} u_b \\ u_i \end{Bmatrix} = \begin{bmatrix} I & 0 \\ \phi_C & \phi_N \end{bmatrix} \begin{Bmatrix} u_b \\ q_m \end{Bmatrix}, \tag{6}$$

where I is the identity matrix and ϕ_N is a matrix of the eigenvectors ϕ_ℓ. This allows the equation of motion for each component to be written as

$$\begin{bmatrix} M_{CC} & M_{CN} \\ M_{NC} & I \end{bmatrix} \begin{Bmatrix} \ddot{u}_b \\ \ddot{q}_m \end{Bmatrix} + \begin{bmatrix} K_{CC} & 0 \\ 0 & \Lambda^2 \end{bmatrix} \begin{Bmatrix} u_b \\ q_m \end{Bmatrix} = \{0\}, \tag{7}$$

assuming the eigenvectors are normalized with respect to the mass matrix M_{ii}, and with

$$M_{CC} = M_{bb} + M_{bi}\phi_C + \phi_C^T + \phi_C^T M_{ii} \phi_C, \tag{8}$$

$$M_{CN} = M_{NC}^T = \left(M_{bi} + \phi_C^T M_{ii}\right) \phi_N, \tag{9}$$

$$K_{CC} = K_{bb} + M_{bi}\phi_C, \tag{10}$$

where Λ is a diagonal matrix of the eigenvalues λ_ℓ.

The number of degrees of freedom can be reduced by retaining only enough fixed interface normal modes to capture the behavior of interest. In general, keeping more modes results in a more accurate approximation of the true behavior at the expense of an increased computational cost. The reduced mass and stiffness matrices for each component can be combined to form a ROM for the system. This ROM is less expensive to analyze and provides accurate results given that enough modes are kept to capture the behavior of interest.

3 Parameterization Using Hyper Dual Numbers

The PROM method proposed in the present work is derived using hyper dual numbers because hyper dual numbers allow for exact calculations of derivatives without needing multiple points at which the derivatives are evaluated. This is achievable due to the definition of a dual number. A dual number is defined as a class of generalized complex numbers where the non-real part is defined by the non-zero root of the number zero, as described in [6]. A hyper dual number is a dual

number defined in more than one dimension. For example, two dimensional hyper dual numbers are defined to consist of one real part and three non-real parts, where the three non-real units ϵ_1, ϵ_2, and $\epsilon_1\epsilon_2$ have the properties that $\epsilon_1^2 = \epsilon_2^2 = (\epsilon_1\epsilon_2)^2 = 0$ but $\epsilon_1 \neq \epsilon_2 \neq \epsilon_1\epsilon_2 \neq 0$. Higher dimensional hyper dual numbers can also be considered, such as described later in the text. The Taylor series for a real-valued function subjected to a hyper dual perturbation truncates exactly at the second derivative term

$$f(x + h_1\epsilon_1 + h_2\epsilon_2 + 0\epsilon_1\epsilon_2) = f(x) + h_1 f'(x)\epsilon_1 + h_2 f'(x)\epsilon_2 + h_1 h_2 f''(x)\epsilon_1\epsilon_2 \tag{11}$$

for arbitrary perturbations h_1 and h_2. There is no truncation error because all the higher order terms contain ϵ_1^2 or ϵ_2^2 or higher powers and are zero by definition. The first and second derivatives are the leading terms of the non-real parts, and these terms can be found by taking the individual non-real parts and dividing by the step size. There is no required difference operation, as in finite difference approximations, which would lead to subtractive cancelation error. The first and second derivatives can thus be computed exactly, regardless of the step size.

For the present research, third order parameterizations are needed since many geometry variables enter into the stiffness matrix as cubic terms. A hyper dual implementation that produces exact third derivatives is created by including ϵ_3 terms. This yields a Taylor series that truncates exactly at the third derivative term (with an arbitrary perturbation h_3)

$$\begin{aligned} f(x + h_1\epsilon_1 + h_2\epsilon_2 + h_3\epsilon_3 + 0\epsilon_1\epsilon_2 + 0\epsilon_1\epsilon_3 + 0\epsilon_2\epsilon_3 + 0\epsilon_1\epsilon_2\epsilon_3) \\ = f(x) + h_1 f'(x)\epsilon_1 + h_2 f'(x)\epsilon_2 + h_3 f'(x)\epsilon_3 + h_1 h_2 f''(x)\epsilon_1\epsilon_2 \\ + h_1 h_3 f''(x)\epsilon_1\epsilon_3 + h_2 h_3 f''(x)\epsilon_2\epsilon_3 + h_1 h_2 h_3 f'''(x)\epsilon_1\epsilon_2\epsilon_3. \end{aligned} \tag{12}$$

The use of hyper dual numbers requires overloading all of the functions in the analysis code to operate on hyper dual numbers instead of on real numbers (that is, creating a new method for a function that operates on hyper dual numbers instead of the originally intended data structure). However, there are often cases where functions are used for which the code is not available and therefore cannot be modified. In these situations it may still be possible to use hyper dual numbers, if the effect of computing the derivatives can be achieved. One example of this is the solution of a linear system, $\mathbf{A}\mathbf{y} = \mathbf{b}$, where derivatives can be computed by several calls to the real-valued routine [7]. First derivatives of the solution of a linear system, $\mathbf{A}\mathbf{y} = \mathbf{b}$, can be computed by solving

$$\mathbf{A}\frac{\partial \mathbf{y}}{\partial x_i} = \frac{\partial \mathbf{b}}{\partial x_i} - \frac{\partial \mathbf{A}}{\partial x_i}\mathbf{y}. \tag{13}$$

Second derivatives can then be found by solving

$$\mathbf{A}\frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j} = \frac{\partial^2 \mathbf{b}}{\partial x_i \partial x_j} - \frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_j}\mathbf{y} - \frac{\partial \mathbf{A}}{\partial x_i}\frac{\partial \mathbf{y}}{\partial x_j} - \frac{\partial \mathbf{A}}{\partial x_j}\frac{\partial \mathbf{y}}{\partial x_i}, \tag{14}$$

and third derivatives by solving

$$\begin{aligned} \mathbf{A}\frac{\partial^3 \mathbf{y}}{\partial x_i \partial x_j \partial x_k} = \frac{\partial^3 \mathbf{b}}{\partial x_i \partial x_j \partial x_k} - \frac{\partial^3 \mathbf{A}}{\partial x_i \partial x_j \partial x_k}\mathbf{y} - \frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_j}\frac{\partial \mathbf{y}}{\partial x_k} - \frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_k}\frac{\partial \mathbf{y}}{\partial x_j} \\ - \frac{\partial^2 \mathbf{A}}{\partial x_j \partial x_k}\frac{\partial \mathbf{y}}{\partial x_i} - \frac{\partial \mathbf{A}}{\partial x_i}\frac{\partial^2 \mathbf{y}}{\partial x_j \partial x_k} - \frac{\partial \mathbf{A}}{\partial x_j}\frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_k} - \frac{\partial \mathbf{A}}{\partial x_k}\frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j}. \end{aligned} \tag{15}$$

Using hyper dual numbers to compute derivatives for the parameterization of eigenvalues or CMS matrices requires a hyper dual version of the eigenvalue calculation routine. As with the solution of a linear system, the effect of using hyper dual numbers can be recreated without modifying the real-valued eigenvalue solver.

3.1 Derivatives of Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are solutions of

$$(K - \lambda_\ell M)\,\phi_\ell = F_\ell \phi_\ell = 0, \tag{16}$$

with $F_\ell = (K - \lambda_\ell M)$. This equation can be differentiated to give

$$\frac{\partial F_\ell}{\partial x_i}\phi_\ell + F_\ell \frac{\partial \phi_\ell}{\partial x_i} = 0. \tag{17}$$

Pre-ultiplying this equation by the transpose of the eigenvector, and making use of the fact that $F_\ell \phi_\ell = 0$, yields

$$\phi_\ell^T \frac{\partial F_\ell}{\partial x_i}\phi_\ell = \phi_\ell^T \left(\frac{\partial K}{\partial x_i} - \lambda_\ell \frac{\partial M}{\partial x_i} - \frac{\partial \lambda_\ell}{\partial x_i} M \right) \phi_\ell. \tag{18}$$

The eigenvectors are orthonormal with respect to M, so $\phi_\ell^T M \phi_\ell = 1$, and (18) can be rearranged to give the first derivative of the eigenvalue

$$\frac{\partial \lambda_\ell}{\partial x_i} = \phi_\ell^T \left(\frac{\partial K}{\partial x_i} - \lambda_\ell \frac{\partial M}{\partial x_i} \right) \phi_\ell. \tag{19}$$

There are several methods for computing the first derivatives of eigenvectors, as summarized by Alvin [8]. The method of Nelson [9] is exact but can be computationally expensive since it involves solving a linear system for each derivative. The modal superposition method [10] represents the derivative of an eigenvector as a superposition of the other eigenvectors and is less computationally intensive than Neslon's method.

Figure 2 shows the first derivative of the second eigenvector with respect to the cross-sectional height computed using finite differences, Nelson's method, and modal superposition. Nelson's method and the finite difference calculation are in good agreement. The first derivative of the eigenvector has sharp corners where the individual components join together. Modal superposition requires the use of all eigenvectors in order to be exact. Using a smaller subset results in an approximation. Figure 2 shows the result of modal superposition with 20 modes. The modal superposition method does not capture the sharp corners, and exhibits Gibb's phenomena, as would be expected of a method relying on a finite summation of modes. When the application is CMS, only a few eigenvectors are kept, so the computational cost of Nelson's method is not expected to be an issue and is preferred over modal superposition.

Nelson's method for computing the first derivative of an eigenvector is to represent it as a sum of two terms

$$\frac{\partial \phi_\ell}{\partial x_i} = z_i + c_i \phi_\ell. \tag{20}$$

The quantity z_i is found by solving (17) with $\frac{\partial \phi_\ell}{\partial x_i}$ replaced by z_i,

$$F_\ell z_i = -\frac{\partial F_\ell}{\partial x_i}\phi_\ell. \tag{21}$$

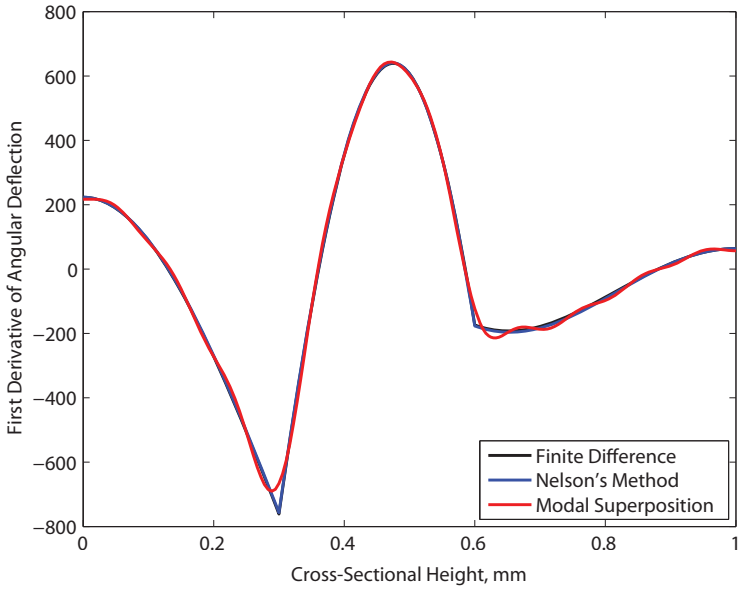


Figure 2: A comparison of three methods for computing derivatives of eigenvectors (the system and its parameters are described in §4.

The contribution of ϕ_ℓ is added back in by multiplying by c_i, where c_i is found by differentiation the orthonormalization equation

$$\phi_j^T M \phi_i = \delta_{ij}, \tag{22}$$

where δ_{ij} is the Kronecker delta function. The first derivative of the orthonormalization equation is

$$\phi^T \frac{\partial M}{\partial x_i}\phi + 2\phi^T M \frac{\partial \phi_\ell}{\partial x_i} = 0. \tag{23}$$

Substituting Eq. 20 into this equation produces

$$\phi^T \frac{\partial M}{\partial x_i}\phi + 2\phi^T M \left(z_i + c_i\phi_\ell\right) = 0, \tag{24}$$

and c_i can then be computed as

$$c_i = -\frac{1}{2}\phi_\ell^T \frac{\partial M}{\partial x_i}\phi_\ell - \phi_\ell^T M z_i. \tag{25}$$

Expressions for second and third derivatives can be derived following a similar procedure. The second derivative of an eigenvalue can be computed as

$$\frac{\partial^2 \lambda_\ell}{\partial x_i \partial x_j} = \phi_\ell^T \left(\frac{\partial^2 K}{\partial x_i \partial x_j} - \frac{\partial \lambda_\ell}{\partial x_i}\frac{\partial M}{\partial x_j} - \frac{\partial \lambda_\ell}{\partial x_j}\frac{\partial M}{\partial x_i} - \lambda \frac{\partial^2 M}{\partial x_i \partial x_j} \right) \phi_\ell + \phi_\ell^T \frac{\partial F_\ell}{\partial x_i}\frac{\partial \phi_\ell}{\partial x_j} + \phi_\ell^T \frac{\partial F_\ell}{\partial x_j}\frac{\partial \phi_\ell}{\partial x_i}, \tag{26}$$

and the second derivative of the corresponding eigenvector can then be computed as

$$\frac{\partial^2 \phi_\ell}{\partial x_i \partial x_j} = z_{ij} + c_{ij}\phi_\ell, \tag{27}$$

where

$$c_{ij} = -\frac{1}{2}\phi_\ell^T \frac{\partial^2 M}{\partial x_i \partial x_j}\phi_\ell - \phi_\ell^T \frac{\partial M}{\partial x_i}\frac{\partial \phi_\ell}{\partial x_j} - \phi_\ell^T \frac{\partial M}{\partial x_j}\frac{\partial \phi_\ell}{\partial x_i} - \frac{\partial \phi_\ell^T}{\partial x_j} M \frac{\partial \phi_\ell}{\partial x_i} - \phi_\ell^T M z_{ij}. \tag{28}$$

The term z_{ij} is found by solving

$$F_\ell z_{ij} = -\frac{\partial^2 F_\ell}{\partial x_i \partial x_j}\phi_\ell - \frac{\partial F_\ell}{\partial x_i}\frac{\partial \phi_\ell}{\partial x_j} - \frac{\partial F_\ell}{\partial x_j}\frac{\partial \phi_\ell}{\partial x_i}. \tag{29}$$

The expression for the third derivative of an eigenvalue is

$$\begin{aligned}\frac{\partial^3 \lambda_\ell}{\partial x_i \partial x_j \partial x_k} = \phi_\ell^T \Bigg(&\frac{\partial^3 K}{\partial x_i \partial x_j \partial x_k} - \frac{\partial^2 \lambda_\ell}{\partial x_j \partial x_k}\frac{\partial M}{\partial x_i} - \frac{\partial^2 \lambda_\ell}{\partial x_i \partial x_k}\frac{\partial M}{\partial x_j} - \frac{\partial^2 \lambda_\ell}{\partial x_i \partial x_j}\frac{\partial M}{\partial x_k} - \frac{\partial \lambda_\ell}{\partial x_i}\frac{\partial^2 M}{\partial x_j \partial x_k} \\ &- \frac{\partial \lambda_\ell}{\partial x_j}\frac{\partial^2 M}{\partial x_i \partial x_k} - \frac{\partial \lambda_\ell}{\partial x_k}\frac{\partial^2 M}{\partial x_i \partial x_j} - \lambda \frac{\partial^3 M}{\partial x_i \partial x_j \partial x_k}\Bigg)\phi_\ell + \phi_\ell^T \frac{\partial^2 F_\ell}{\partial x_j \partial x_k}\frac{\partial \phi_\ell}{\partial x_i} + \phi_\ell^T \frac{\partial^2 F_\ell}{\partial x_i \partial x_k}\frac{\partial \phi_\ell}{\partial x_j} \\ &+ \phi_\ell^T \frac{\partial^2 F_\ell}{\partial x_i \partial x_j}\frac{\partial \phi_\ell}{\partial x_k} + \phi_\ell^T \frac{\partial F_\ell}{\partial x_i}\frac{\partial^2 \phi_\ell}{\partial x_j \partial x_k} + \phi_\ell^T \frac{\partial F_\ell}{\partial x_j}\frac{\partial^2 \phi_\ell}{\partial x_i \partial x_k} + \phi_\ell^T \frac{\partial F_\ell}{\partial x_k}\frac{\partial^2 \phi_\ell}{\partial x_i \partial x_j}.\end{aligned} \tag{30}$$

The third derivative of the corresponding eigenvector can then be computed as

$$\frac{\partial^3 \phi_\ell}{\partial x_i \partial x_j \partial x_k} = z_{ijk} + c_{ijk}\phi_\ell, \tag{31}$$

where

$$\begin{aligned}c_{ijk} = -\frac{1}{2}\phi_\ell^T \frac{\partial^3 M}{\partial x_i \partial x_j \partial x_k}\phi_\ell &- \phi_\ell^T \frac{\partial^2 M}{\partial x_i \partial x_j}\frac{\partial \phi_\ell}{\partial x_k} - \phi_\ell^T \frac{\partial^2 M}{\partial x_i \partial x_k}\frac{\partial \phi_\ell}{\partial x_j} - \phi_\ell^T \frac{\partial^2 M}{\partial x_j \partial x_k}\frac{\partial \phi_\ell}{\partial x_i} - \frac{\partial \phi_\ell^T}{\partial x_k}\frac{\partial M}{\partial x_i}\frac{\partial \phi_\ell}{\partial x_j} \\ &- \frac{\partial \phi_\ell^T}{\partial x_k}\frac{\partial M}{\partial x_j}\frac{\partial \phi_\ell}{\partial x_i} - \frac{\partial \phi_\ell^T}{\partial x_k} M \frac{\partial^2 \phi_\ell}{\partial x_i \partial x_j} - \frac{\partial \phi_\ell^T}{\partial x_j}\frac{\partial M}{\partial x_k}\frac{\partial \phi_\ell}{\partial x_i} - \frac{\partial^2 \phi_\ell^T}{\partial x_j \partial x_k} M \frac{\partial \phi_\ell}{\partial x_i} - \frac{\partial \phi_\ell^T}{\partial x_j} M \frac{\partial^2 \phi_\ell}{\partial x_i \partial x_k} \\ &- \phi_\ell^T \frac{\partial M}{\partial x_i}\frac{\partial^2 \phi_\ell}{\partial x_j \partial x_k} - \phi_\ell^T \frac{\partial M}{\partial x_j}\frac{\partial^2 \phi_\ell}{\partial x_i \partial x_k} - \phi_\ell^T \frac{\partial M}{\partial x_k}\frac{\partial^2 \phi_\ell}{\partial x_i \partial x_j} - \phi_\ell^T M z_{ijk},\end{aligned} \tag{32}$$

and z_{ijk} is found by solving

$$\begin{aligned}F_\ell z_{ijk} = -\frac{\partial^3 F_\ell}{\partial x_i \partial x_j \partial x_k}\phi_\ell &- \frac{\partial^2 F_\ell}{\partial x_j \partial x_k}\frac{\partial \phi_\ell}{\partial x_i} - \frac{\partial^2 F_\ell}{\partial x_i \partial x_k}\frac{\partial \phi_\ell}{\partial x_j} - \frac{\partial^2 F_\ell}{\partial x_i \partial x_j}\frac{\partial \phi_\ell}{\partial x_k} \\ &- \frac{\partial F_\ell}{\partial x_i}\frac{\partial^2 \phi_\ell}{\partial x_j \partial x_k} - \frac{\partial F_\ell}{\partial x_j}\frac{\partial^2 \phi_\ell}{\partial x_i \partial x_k} - \frac{\partial F_\ell}{\partial x_k}\frac{\partial^2 \phi_\ell}{\partial x_i \partial x_j}.\end{aligned} \tag{33}$$

4 Application to a Stepped Beam

The example problem being considered is a simply supported beam composed of three components, as shown in Figure 3. The beam has elastic modulus E, density ρ, width B, cross-sectional height H, location of the center of the defect ℓ, and length of the defect W, with nominal values given in Table 1. The material properties and geometry of the center section are allowed to be perturbed from their nominal values.

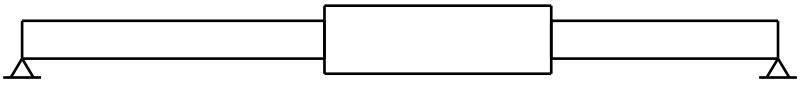


Figure 3: A simply supported beam composed of three components.

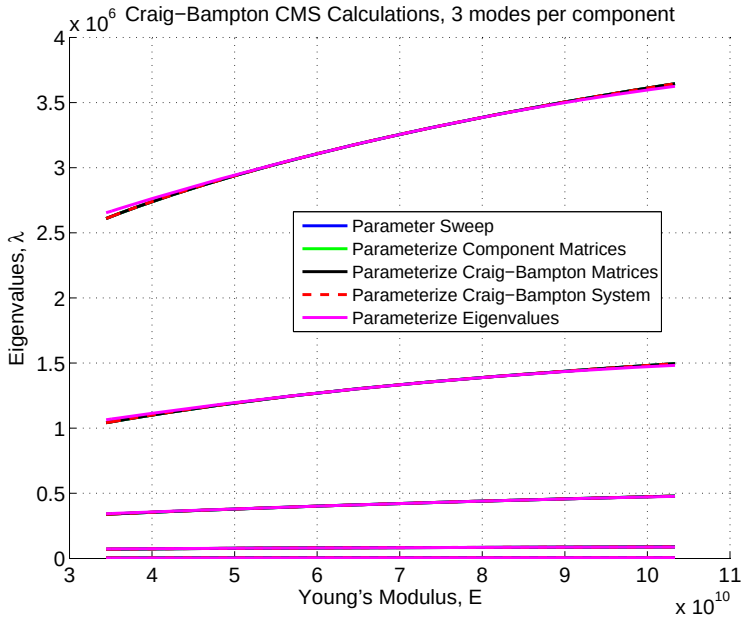
Property	Value
Property	Value
Density, ρ	2700 kg/m^3
Elastic modulus, E	68.9 GPa
Cross-sectional width, B	20 cm
Cross-sectional height, H	5 mm
Location of defect's center, ℓ	45 cm
Length of the defect, W	30 cm
Length of the beam, L	1 m

Table 1: Material and geometric properties for the beam.

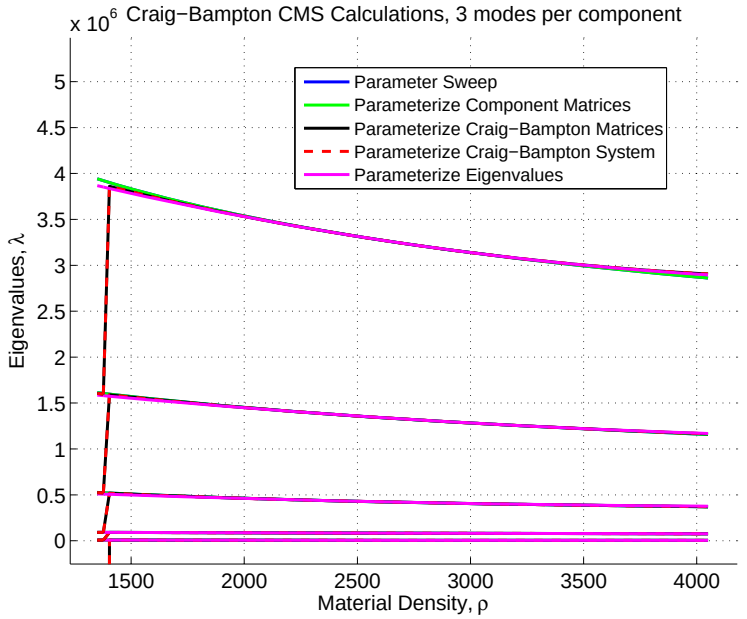
Two types of parameterizations are considered: quadratic and cubic. These parameterizations are applied at all the levels shown in Figure 1, although only results from the CMS branch are presented here. A parameter sweep was run for each case to determine the true behavior in order to assess the accuracy of the parameterized models. The CMS calculations are performed by keeping only three modes per component. This seems to produce accurate results although more testing is needed on the effect of varying the number of modes kept. Parameter variations are considered for each of the variables in Table 1, except for L.

A quadratic parameterization (the first three terms of Eq. 1) is used to produce the comparisons shown in Figs. 4-6. Quadratic parameterization produces fairly accurate results for variations in Young's modulus, material density and cross-sectional width and height. There are some issues with the CMS parameterizations at the extremes of the parameter space. For the quadratic parameterization, variations in Young's modulus, material density and cross-sectional width are exactly represented to machine precision by parameterizing the component matrices. Parameterizing the CMS matrices produces slightly less accurate results, and parameterizing the eigenvalues produces fairly accurate results. Quadratic parameterization of the full system matrices and CMS matrices is not able to accurately capture the effects of variations in cross-sectional height. Here, the system matrices are composed to represent bending stiffness (as opposed to the cross-sectional height). Bending stiffness relates to the cube of cross-sectional height, whereas the other parameters factor into the system matrices in a linear manner. For variations in location and length of the center component the parameterizations are only accurate in a small region around the nominal design. For these variations, parameterizing the eigenvalues is the most accurate, followed by parameterizing the CMS matrices, with parameterizing the component matrices the least accurate.

Cubic parameterizations, as given in Eq. 1, produce similar trends but are in general more accurate. The accuracy of the cubic parameterization is shown in Figs. 7-9. The cubic parameterization applied to the component matrices is able to accurately represent the effect of varying the



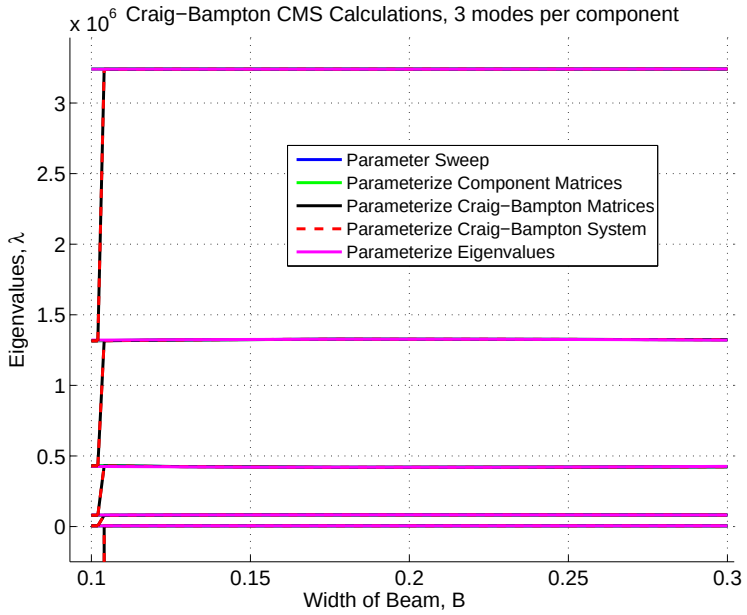
(a) Variations in Young's modulus



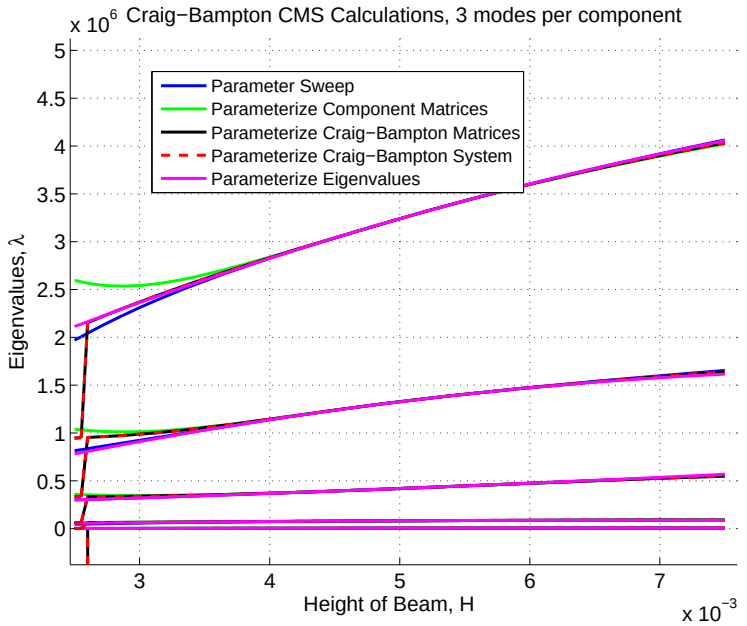
(b) Variations in material density

Figure 4: The effects on the first five eigenvalues of quadratic parameterizations for E and ρ.

cross-sectional height, and also improves the accuracy of parameterization of the CMS matrices. The geometric variations, location and length of the center component, are accurate only in a small region around the nominal design but the accuracy is better than the quadratic parameterization. For the geometric variations, parameterizing the eigenvalues is most accurate, followed by parameterizing the CMS matrices. Parameterizing the component matrices is the least accurate approach



(a) Variations in cross-sectional width

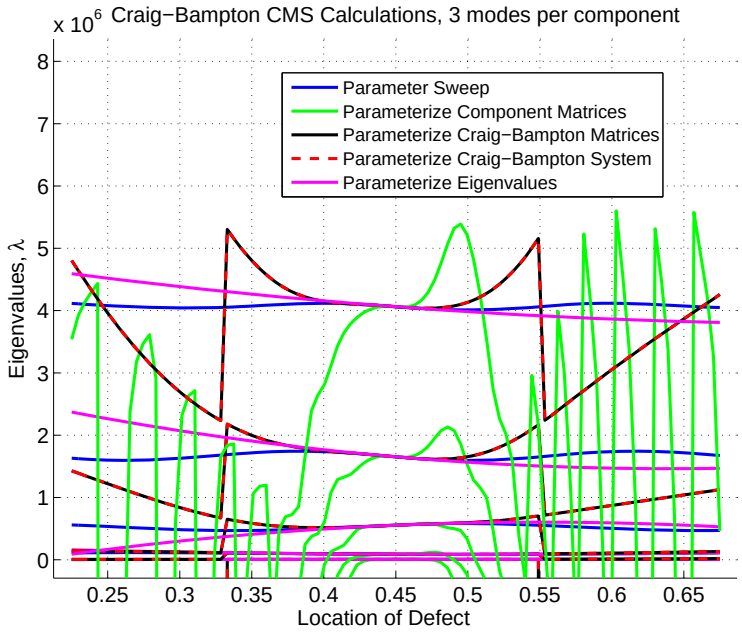


(b) Variations in cross-sectional height

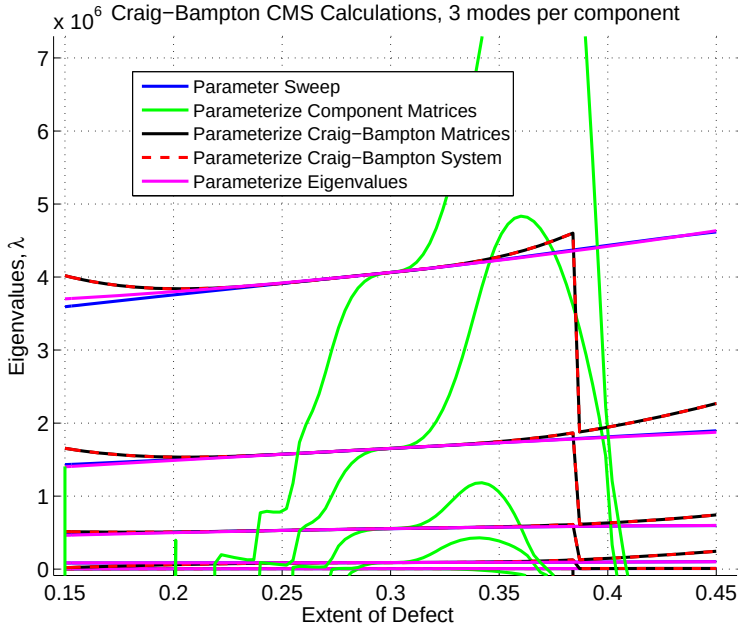
Figure 5: The effects on the first five eigenvalues of quadratic parameterizations for B and H.

for the geometric variations. These trends are consistent with those for using real-valued finite difference methods to construct the parameterizations instead of hyper dual numbers.

Parameters related to the geometric changes of the system, specifically ℓ and W, are expected to be more difficult to model due to their nonlinear effects on the stiffness matrix as they are varied. Other parameters, that affect the system in only a bulk sense such as B or H, or that are material



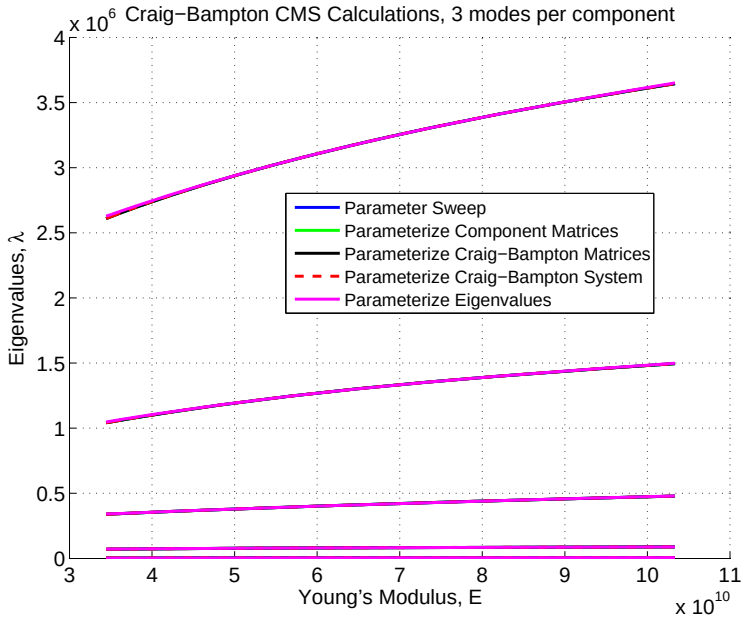
(a) Variations in location of center section



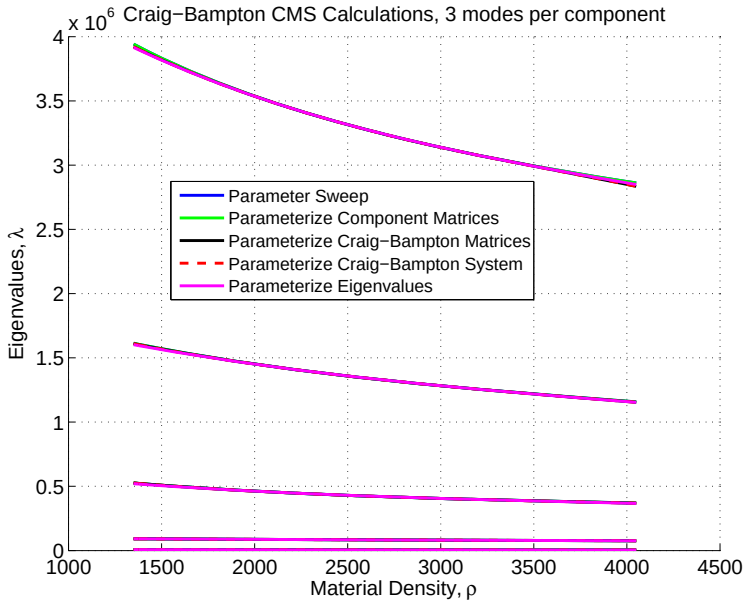
(b) Variations in length of center section

Figure 6: The effects on the first five eigenvalues of quadratic parameterizations for ℓ and W.

properties such as E or ρ, are expected to be easier to parameterize as varying them linearly varies the system matrices linearly as well.



(a) Variations in Young's modulus

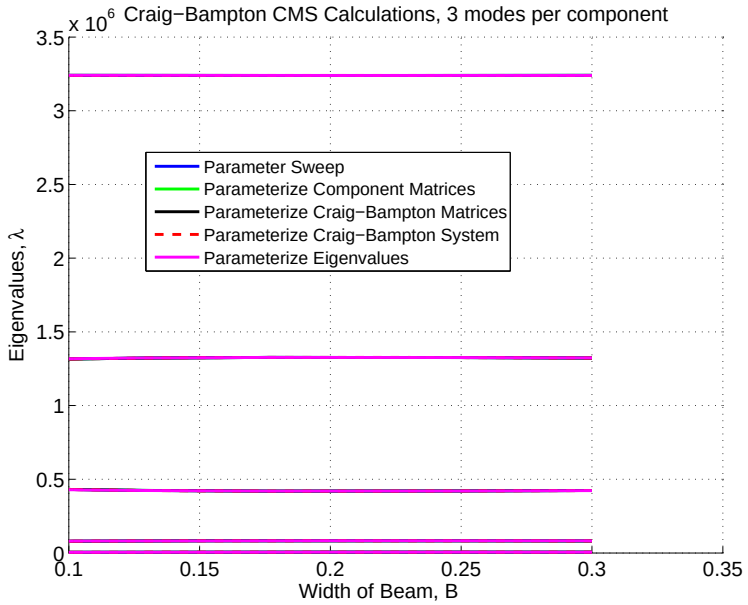


(b) Variations in material density

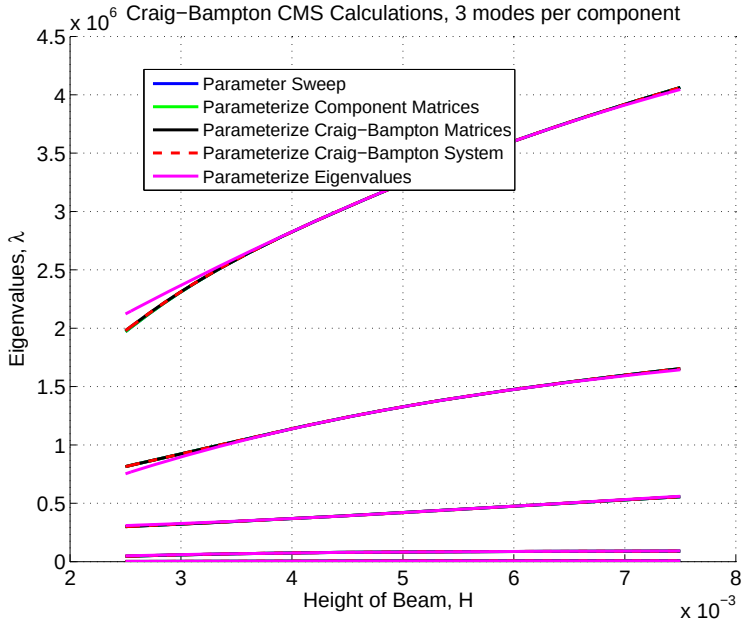
Figure 7: The effects on the first five eigenvalues of cubic parameterizations for E and ρ.

5 Conclusions and Future Work

This work demonstrates that hyper dual numbers can be used to construct parameterized reduced order models (PROMs) both efficiently (since only one mesh is needed) and accurately. With these PROMs, the task of assessing the response of a real system with aleatoric uncertainty due to



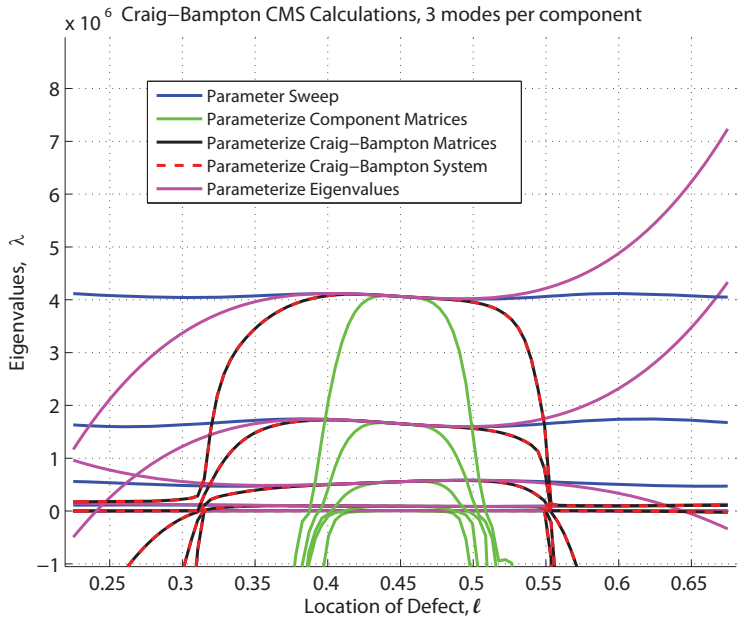
(a) Variations in cross-sectional width



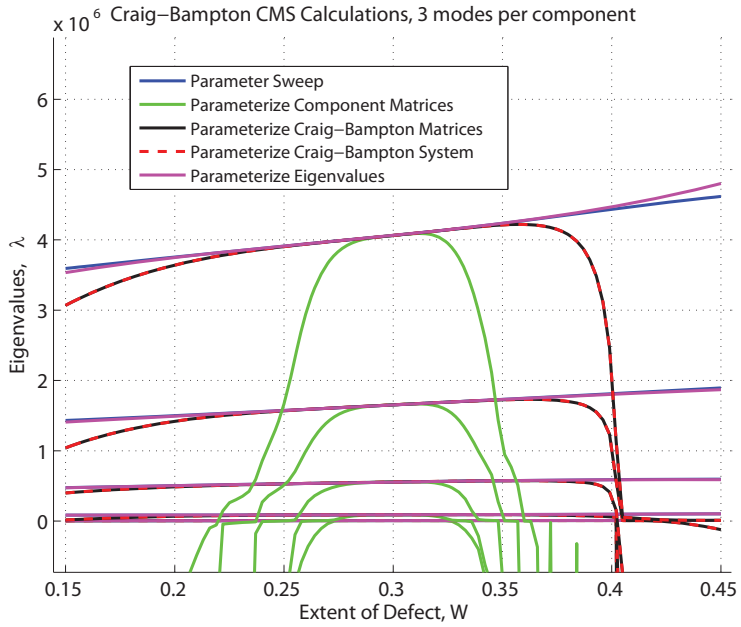
(b) Variations in cross-sectional height

Figure 8: The effects on the first five eigenvalues of cubic parameterizations for B and H.

manufacturing or other sources should now be feasible. Without these PROMs, the effort required to analyze or model aleatoric uncertainty in a real system would be prohibitively expensive. The derivatives necessary for constructing the PROMs are computed using only the nominal design, eliminating the need to construct more than one mesh as would be required by finite differencing. However, in order for this approach to be applied it is necessary to know how the mesh would



(a) Variations in location of center section



(b) Variations in length of center section

Figure 9: The effects on the first five eigenvalues of cubic parameterizations for ℓ and W.

change as the parameters of the design are varied. One approach would be to incorporate hyper dual numbers into the mesh generation routine.

This research also sought to determine the appropriate levels to develop PROMs. Results show that parameterization of the eigenvalues and eigenvectors is the most accurate way to reproduce a

perturbation in a linear system. As well, parameterization at the component matrix level and at the system level for a CMS model produced nearly identical results. In all three cases, significant savings in both computational time and man-hours are achieved by using PROMs. Future work, though, must seek to improve the accuracy of the PROMs: currently, they are well suited to studying small perturbations in system parameters (up to 5%).

In order for this approach to be useful for optimizations, the accuracy of the method must be extended to significantly larger variations. Higher order parameterizations should better capture the behavior for the geometric variations. This would require hyper dual numbers capable of producing exact fourth (or higher) derivatives. It is fairly straightforward to extend the existing hyper dual number formulation to higher derivatives, as evidenced by the extension to third derivatives. A more general approach, though, in which dual numbers are defined recursively would allow for easier extensions to arbitrary derivatives. This approach would define a hyper dual number as a dual number with dual number components. Third derivatives would then be produced by going one level further.

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