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Advances in Stochastic Peridynamic Theory

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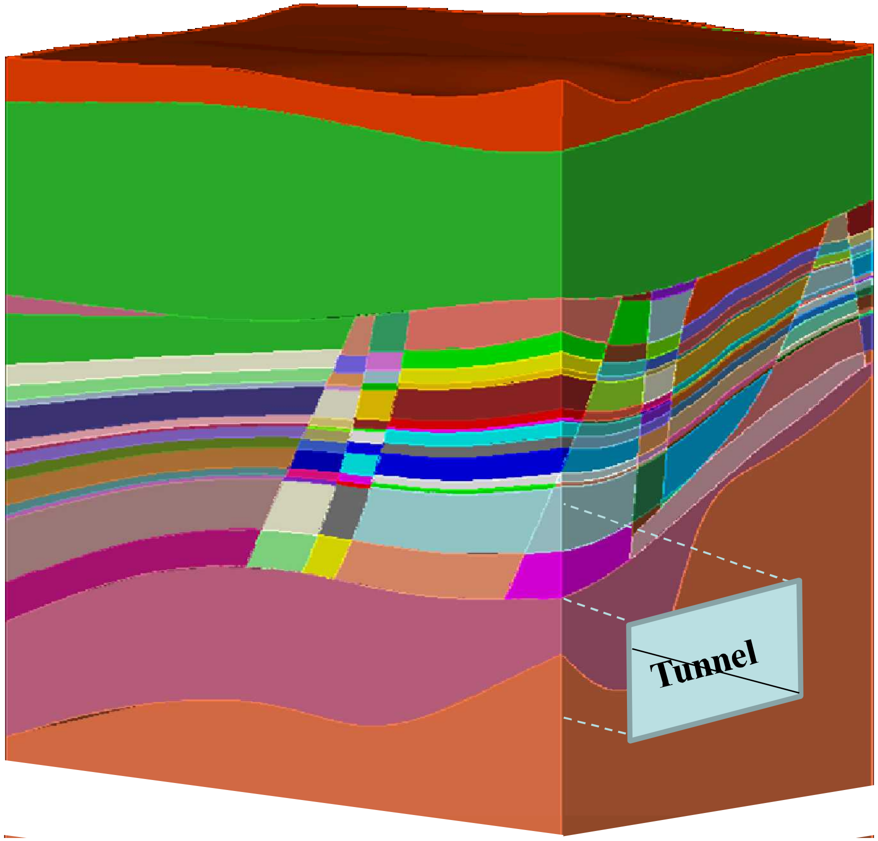
Outline of Presentation

- **Background and Significance**
- **Peridynamic Theory**
- **Stochastic Peridynamic Theory**
- **Geomaterials in Stochastic Peridynamic Theory**
- **Concluding Remarks**

Background and Significance

- **The stability of tunnels in hard rock geologies under ground shock loading is of direct consequence to studies of vulnerability or survivability of deeply-buried hard targets.**



- ***To meet these challenges, peridynamic theory, the mechanics of random media, and the mechanics of fractal media will be combined.***

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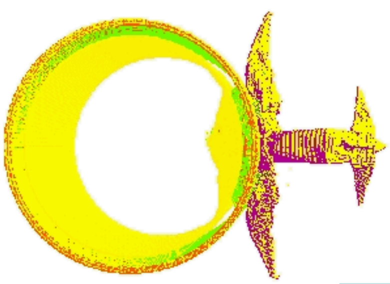
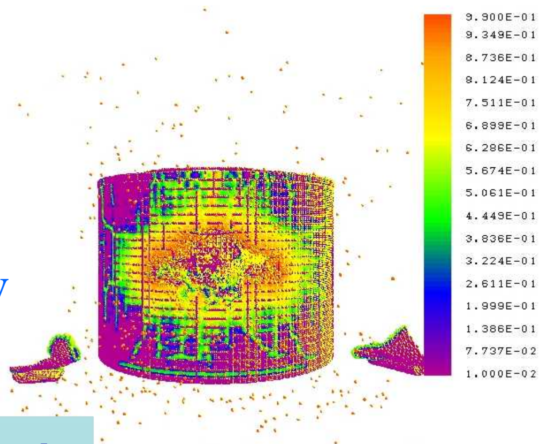
Peridynamic Theory

- ***Peridynamic theory*** **is a theory of continuum mechanics that uses integro-differential equations without spatial derivatives rather than partial differential equations.**
  - Bond-Based Peridynamics[1]
  - State-Based Peridynamics[2]
- **Peridynamic means "near force".**

Why use peridynamic theory?



The fundamental partial differential equations used in conventional finite element or particle codes do not apply at discontinuities.



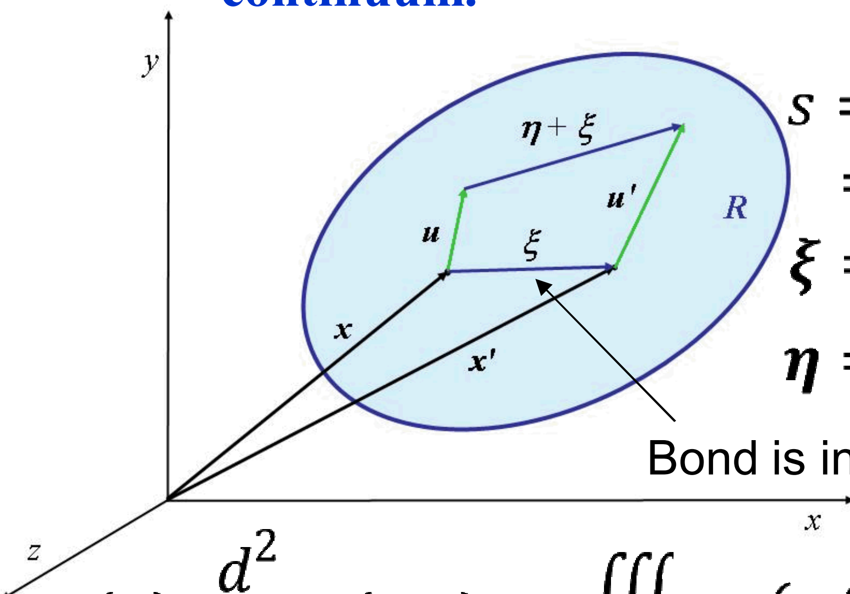
With peridynamics, cracks are part of the solution, not part of the problem.

[1]S.A. Silling, "Reformulation of elasticity theory for discontinuities and long-range forces," *J. Mech. Phys. Solids*, **48** (2000), 175-209.

[2] S.A. Silling *et al.*. "Peridynamic States and Constitutive Modeling," *J Elasticity* **88** (2007), 151–184.

The Fundamental Equation of Peridynamic Theory

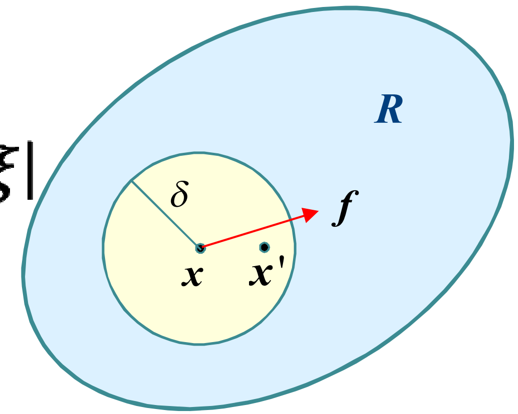
- **In bond-based peridynamics, the force state at a point is given by a functional over the pairwise interactions with all other points in the continuum.**



$$s = \text{stretch} = (|\boldsymbol{\eta} - \boldsymbol{\xi}| - |\boldsymbol{\xi}|)/|\boldsymbol{\xi}|$$

$$\boldsymbol{\xi} = \boldsymbol{x}' - \boldsymbol{x}$$

$$\boldsymbol{\eta} = \boldsymbol{u}(\boldsymbol{x}', t) - \boldsymbol{u}(\boldsymbol{x}, t)$$



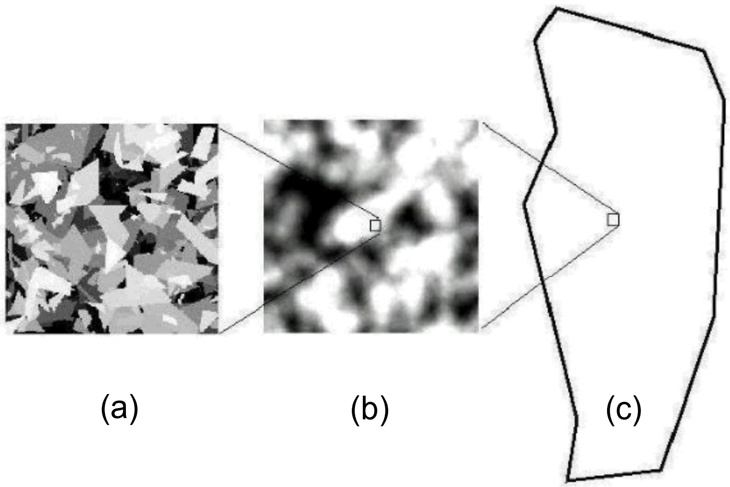
$$\rho(\boldsymbol{x}) \frac{d^2}{dt^2} \boldsymbol{u}(\boldsymbol{x}, t) = \oiiint_R \boldsymbol{f}\left(\boldsymbol{u}(\boldsymbol{x}', t) - \boldsymbol{u}(\boldsymbol{x}, t), \boldsymbol{x}' - \boldsymbol{x}\right) dV' + \boldsymbol{b}(\boldsymbol{x}, t)$$

ρ *is the density,*	$\boldsymbol{x}$ *is the position vector,*
t *is the time,*	$\boldsymbol{u}$ *is the displacement vector,*
R *is the computation domain,*	$\boldsymbol{f}$ *is the pairwise force function,*
$\boldsymbol{b}$ *is the body force,*	δ *is the horizon.*

Formulating Stochastic Peridynamics

- **General Approach**
  - Develop a stochastic peridynamic theory that combines peridynamic theory with random or fractal material characterization of (geo)materials.
  - Verify implementation by studying shock propagation in random or fractal media with joints and faults and effects at boundaries.

- **Three Scales**
  - *microscale*: average grain size d (microstructure)
  - *mesoscale*: L
    - if not representative volume element (RVE), then inhomogeneous continuum
  - *macroscale*: L_{macro}



Separation of scales $d \ll L \ll L_{macro}$ does not hold on wavefronts!

Formulating Stochastic Peridynamics

- **Field equation:** $$\rho(\boldsymbol{x},\omega)\,\frac{d^2}{dt^2}\boldsymbol{u}(\boldsymbol{x},t) = \oiiint_R \boldsymbol{f}(\boldsymbol{\eta},\boldsymbol{\xi},\omega)\,dV'$$
- **Randomness enters through the random fields:**

density $\rho(x,\omega), \omega \,\varepsilon\, \Omega$ *(sample space)*

pairwise force function $\boldsymbol{f}(\boldsymbol{\eta},\boldsymbol{\xi},\omega)$ *which for microelastic or microplastic materials depends on*

bulk modulus $K(\boldsymbol{x},\omega)$ *and*

yield strength $Y(\boldsymbol{x},\omega)$

critical stretch $s_0(\boldsymbol{x},\omega)$ *(maximum value of stretch* s*)*

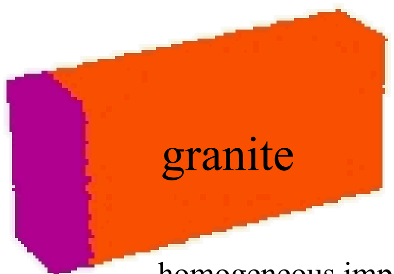
- **Thus, the random medium B is characterized by:**

$$\mathbf{B} = \{\rho(\boldsymbol{x},\omega), K(\boldsymbol{x},\omega), Y(\boldsymbol{x},\omega), s_0(\boldsymbol{x},\omega) : \boldsymbol{x} \,\varepsilon\, \mathrm{E}^3, \omega \,\varepsilon\, \Omega\}$$

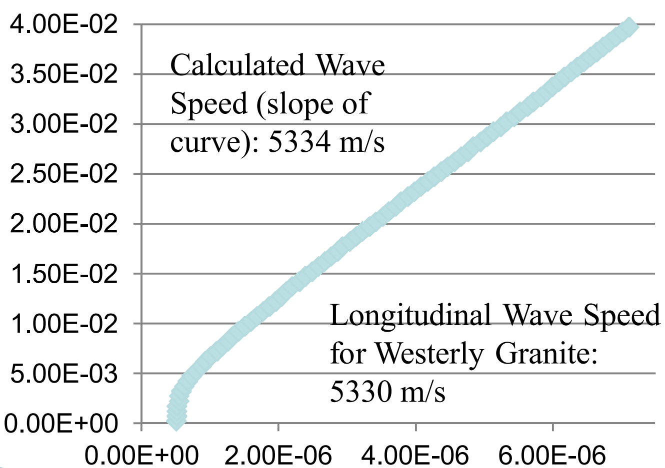
Wave Propagation in Granite

- **Impact Simulations:**
  - 0.5-cm x 2-cm x 2-cm steel object impacting a 4-cm x 2-cm x 2-cm Westerly Granite column at 100 m/s (spall) and 4000 m/s (strong shock)

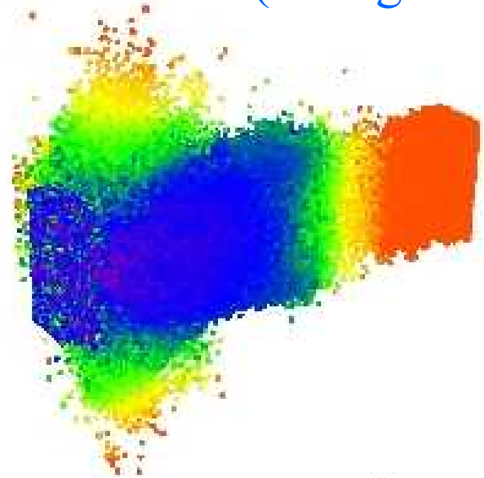
steel



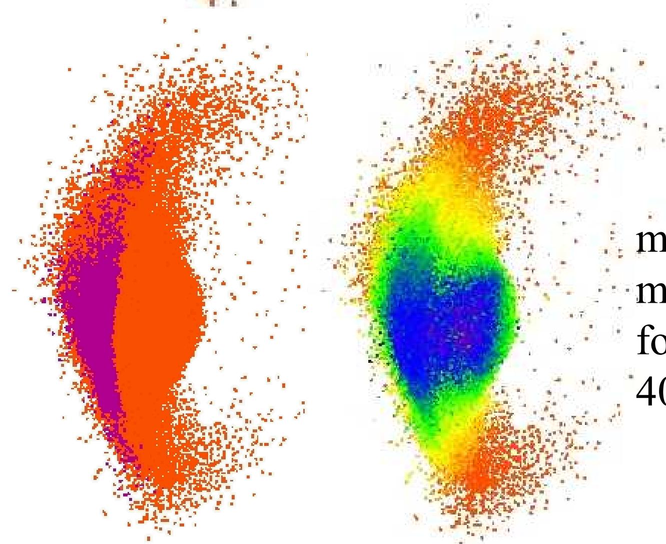
homogeneous impactor and target



Times at which Tracers First Move



velocity magnitude at 250 μs for 100 m/s impact



materials and velocity magnitude at 15.1 μs for 4000 m/s impact

Wave Propagation in Homogeneous and Heterogeneous Geologic Materials

- **We first permitted these material properties to vary spatially with the sample space via some distribution.**
  - Next 5 slides review work reported for perturbing location of nodes, random fields for some material properties, and jointed media.
- **Then we generated realizations of correlated, random-field properties using the R software.**
  - In reality, each material property is spatially correlated and we must generate the material-property field having some spatial correlation.
  - We can create different random field models depending on the assumed correlation structure and probability distribution.
  - We examine the response of the material domain to the same impact conditions in each model and make comparisons of the reference homogeneous medium case (zero noise) with random-field models, white noise and non-white noise using different spatial correlations.

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Heterogeneous Material Simulations

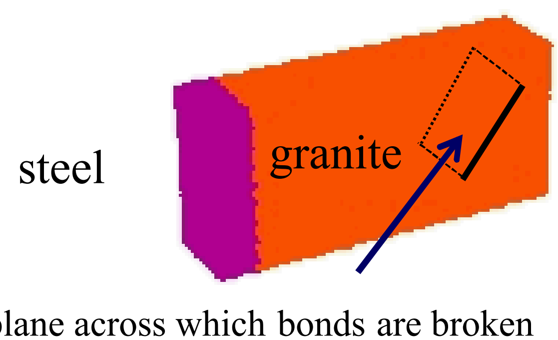
1. **Locations of nodes perturbed**
2. **Bulk modulus from a Weibull population**

$$\Psi = \frac{m}{K_0}\left(\frac{K}{K_0}\right)^{m-1} exp\left[-\left(\frac{K}{K_0}\right)^{m}\right]$$

K_0 is scale parameter
m is homogeneity index or shape parameter (larger more homogeneous)

mean $\mu = K_0\,\Gamma\left(1+\frac{1}{m}\right)$ (Γ is gamma function)

3. **Critical stretch from a Weibull population**
4. **Jointed homogeneous granite**



Summary of Results for Wave Speeds
(100 m/s impact)

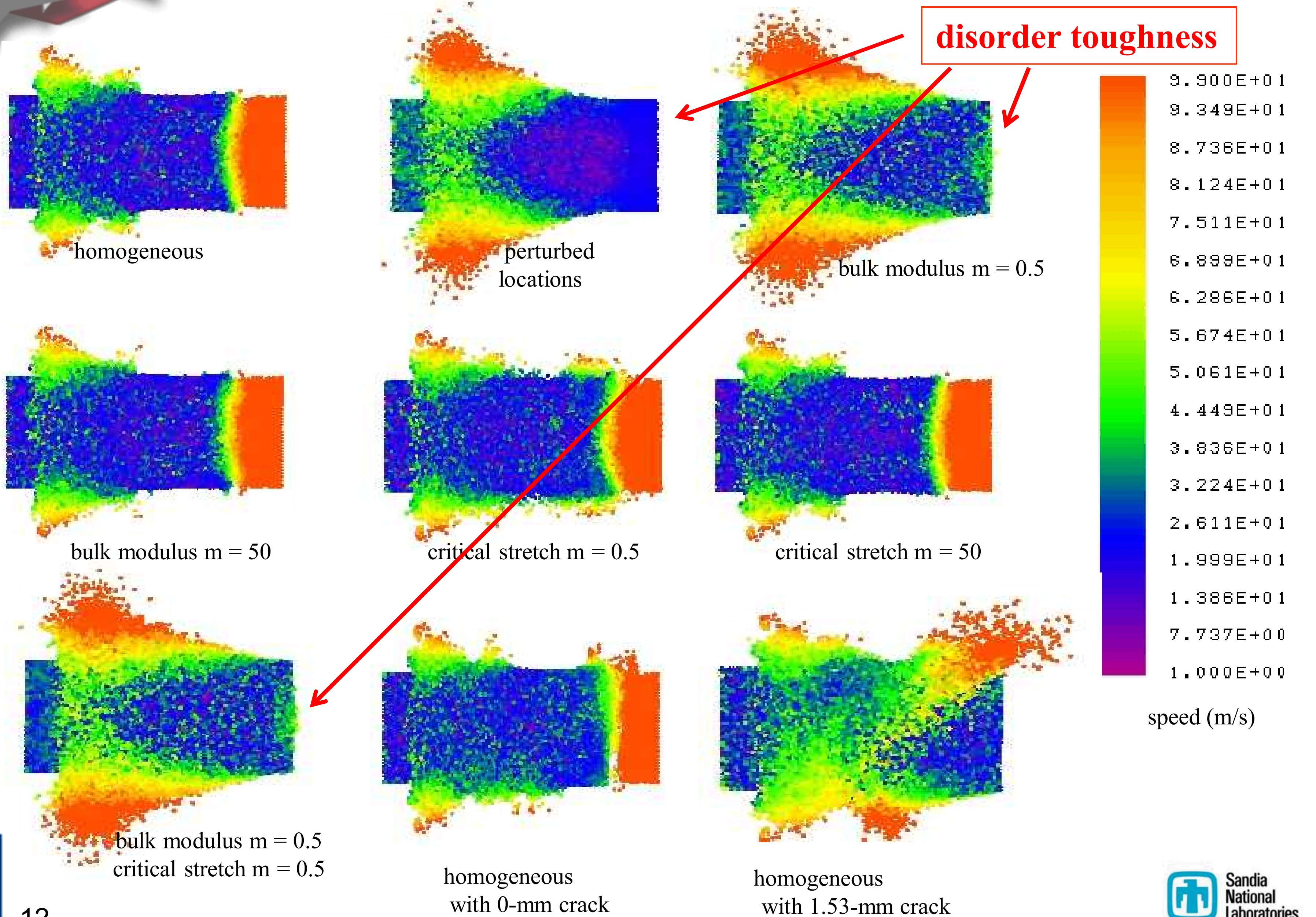
Heterogeneity	Bulk Modulus m	Critical Stretch m	Wave Speed (m/s)
Homogeneous	---	---	5334
Node Locations	---	---	5431
Weibull K	0.5	---	3371
Weibull K	50	---	5317
Weibull s_0	---	0.5	5334
Weibull s_0	---	50	5334
Weibull K and s_0	0.5	0.5	3371
Joint (oblique)*	---	---	5334
Joint (oblique)**	---	---	---

Longitudinal wave speed for Westerly granite is 5.33 km/s.
Shock wave speed for 100 m/s impact of steel into Westerly granite is 2241 m/s.
Therefore, not in strong shock regime.

*Crack thickness is zero.

**Crack thickness is 1.53 mm.

Speeds at 100 μs (100 m/s impact)



Summary of Results for Wave Speeds (4000 m/s impact)

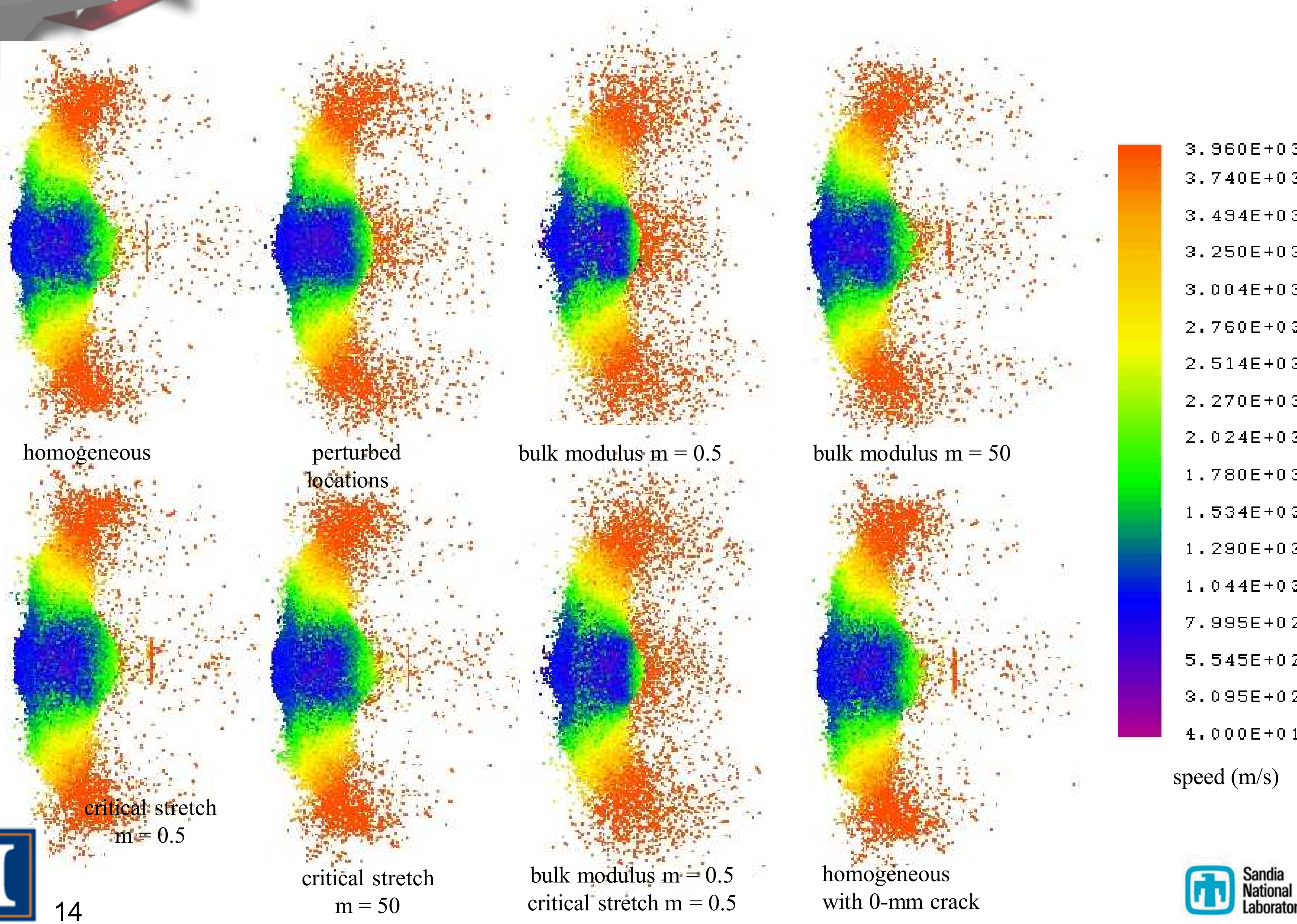
Heterogeneity	Bulk Modulus m	Critical Stretch m	Wave Speed (m/s)
Homogeneous	---	---	6909
Node Locations	---	---	6707
Weibull K	0.5	---	6330
Weibull K	50	---	6937
Weibull s_0	---	0.5	6910
Weibull s_0	---	50	6910
Weibull K and s_0	0.5	0.5	6275
Joint (oblique)*	---	---	6909
Joint (oblique)**	---	---	---

Shock wave speed for 4000 m/s impact of steel into Westerly granite is 6866 m/s. Obtained from linear Hugoniot U_{shock} = 2.10 km/s + 1.63 $u_{particle}$ (U and u velocities).

*Crack thickness is zero.

**Crack thickness is 1.53 mm.

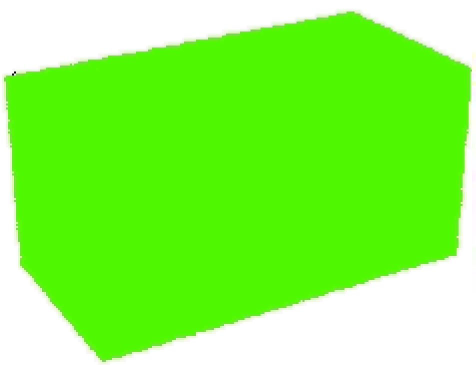
Speeds at 15 μs (4000 m/s impact)



Density Random Field

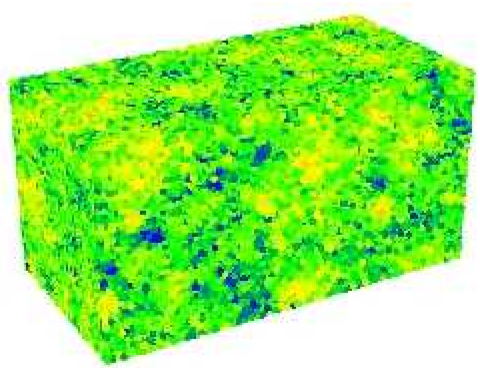
- **In the following slides, we show results for impacts at 100 m/s and 4000 m/s for the following cases:**

homogeneous (constant density)



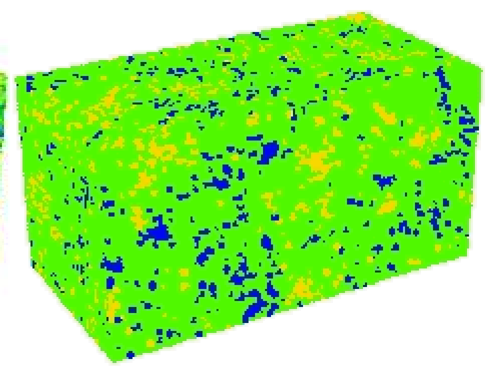
$\rho_{min} = 2627\ \mathrm{kg/m^3}$
$\rho_{max} = 2627\ \mathrm{kg/m^3}$

correlated Gaussian random variates (RVs)



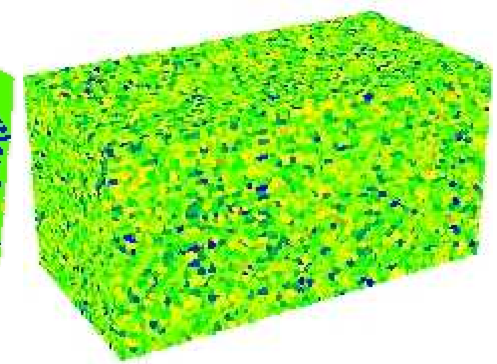
$\rho_{min} = 857\ \mathrm{kg/m^3}$
$\rho_{max} = 4398\ \mathrm{kg/m^3}$

reduction of correlated Gaussian RVs by 1/1000



$\rho_{min} = 2625\ \mathrm{kg/m^3}$
$\rho_{max} = 2629\ \mathrm{kg/m^3}$

un-correlated Gaussian RVs



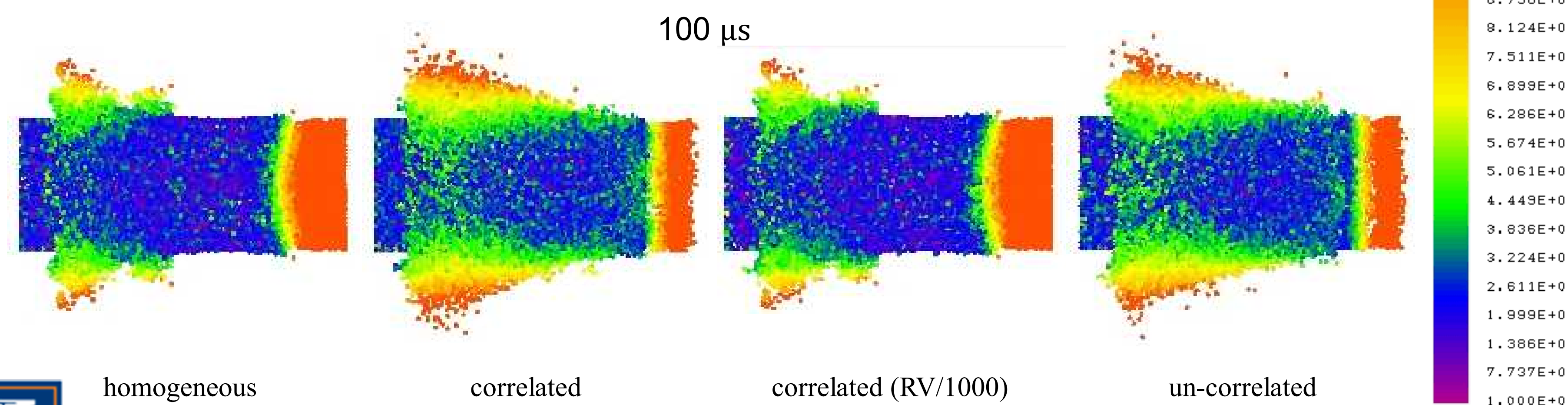
$\rho_{min} = 1422\ \mathrm{kg/m^3}$
$\rho_{max} = 3733\ \mathrm{kg/m^3}$

Summary of Results for Wave Speeds
(100 m/s impact)

Case	Average Density (kg/m^3)	Standard Deviation (kg/m^3)	Wave Speed (m/s)
Homogeneous	2627	---	5333
Correlated	2627	262.7	5198
Correlated (RV/1000)	2627	262.7	5377
Uncorrelated	2627	262.7	5269

Longitudinal wave speed for Westerly granite is 5.33 km/s. Shock wave speed for 100 m/s impact of steel into Westerly granite is 2241 m/s. Therefore, not in strong shock regime.

3% spread in wave speeds.



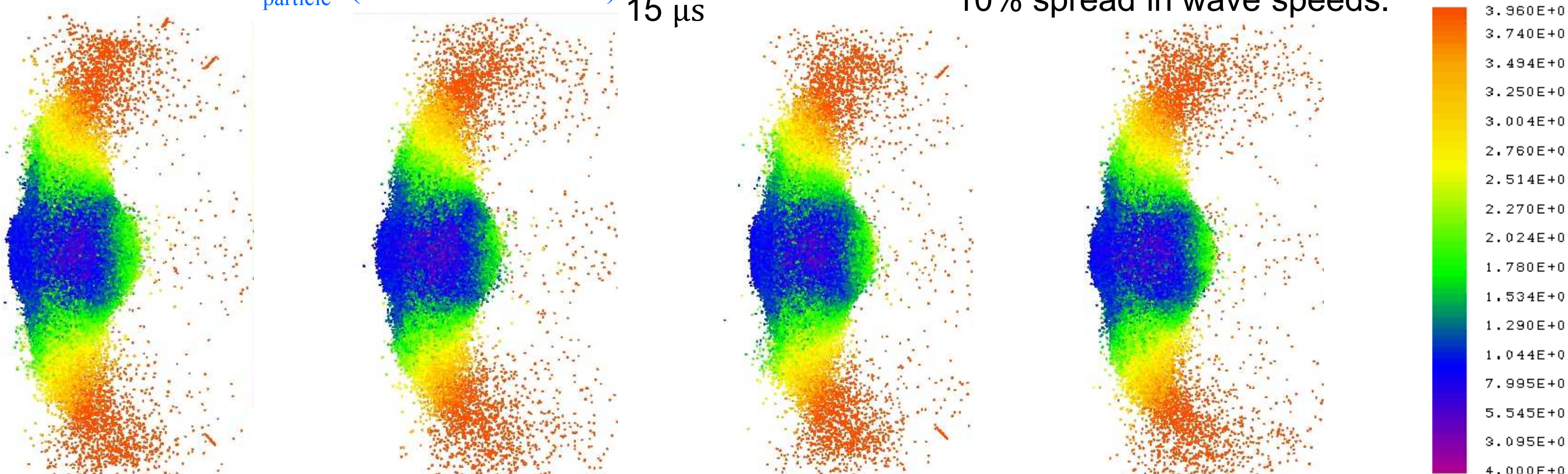
Summary of Results for Wave Speeds
(4000 m/s impact)

Case	Average Density (kg/m^3)	Standard Deviation (kg/m^3)	Wave Speed (m/s)
Homogeneous	2627	---	6866
Correlated	2627	262.7	6340
Correlated (RV/1000)	2627	262.7	6973
Uncorrelated	2627	262.7	7050

Shock wave speed for 4000 m/s impact of steel into Westerly granite is 6866 m/s. Obtained from linear Hugoniot $U_{shock} = 2.10\ \text{km/s} + 1.63\ u_{particle}$ (U and u velocities).

15 µs

10% spread in wave speeds.



homogeneous correlated correlated (RV/1000) un-correlated

Concluding Remarks

- **Principle Conclusion:**
  - Peridynamic theory is a physically reasonable and viable approach to high-impulse loading and modeling fracture and fragmentation phenomena involving geomaterials.
- **The Present:**
  - Results agree well with data and show differences with type and degree of heterogeneity.
  - 100-m/s impacts show phenomenon of disorder toughness with random perturbation of nodes or heterogeneities in bulk modulus. Disorder toughness is not observed with heterogeneities in density.
  - 4000-m/s impacts show significantly less sensitivity to heterogeneities considered and no indication of disorder toughness.
- **The Future:**
  - Continue to develop stochastic peridynamic theory.
  - Develop peridynamic theory of fractal media.
  - Continue studies of wave and shock propagation in random and fractal peridynamic media.
  - Investigate boundary effects from shock loading and tunnel-wall stability.
  - Validate numerical models and identify key parameters.