



A Massively Parallel Finite Element Capability for Forward and Inverse Structural Acoustics Simulations

SAND2013-8438C

Timothy Walsh

Computational Solid Mechanics and Structural Dynamics
Sandia National Laboratories, Albuquerque, NM

Sandia is a multiprogram engineering and science laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the US Department of Energy's National Nuclear Security Administration. (DE-AC04-94AL85000)



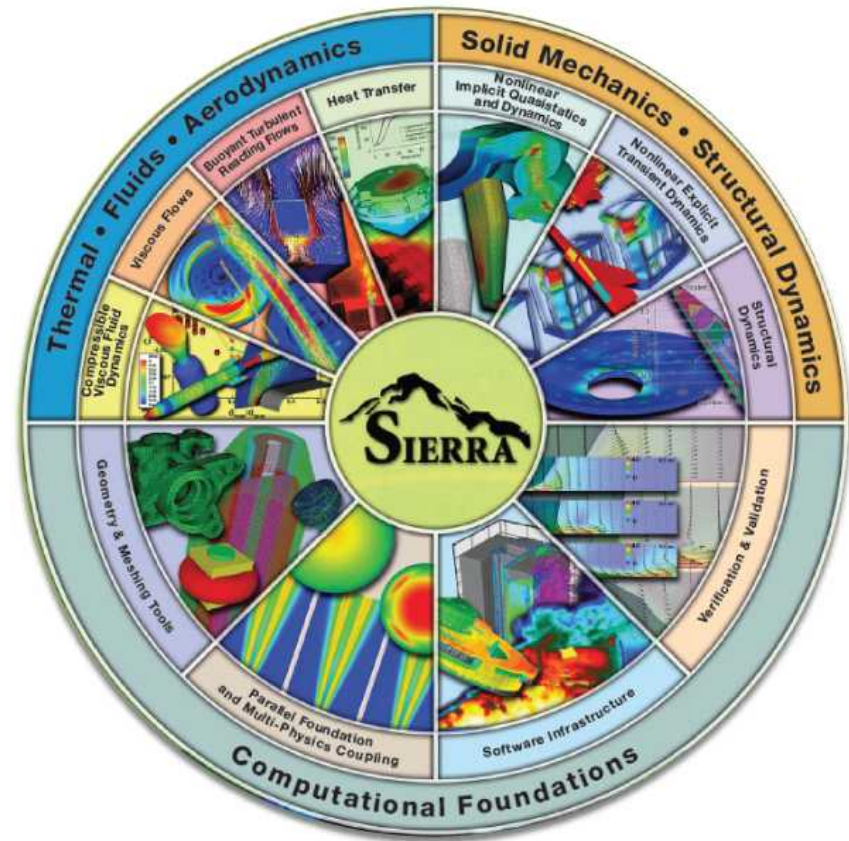


Outline

- **Quick overview of Sierra Mechanics**
- **Quick overview of Sierra-SD(Salinas)**
- **Some research areas in Sierra-SD**
 - **Nonlinear acoustics**
 - **Infinite elements**
 - **Mismatched structural/acoustic meshes**
 - **Inverse problems**
- **Example applications of Sierra-SD**

Overview of Sierra Mechanics

- **Goal:** massively parallel coupled multiphysics calculations
- **Modules for structural dynamics, solid mechanics, fluids, thermal, etc**





Overview of Sierra-SD (Salinas)

- **Massively parallel implicit finite element analysis for structural dynamics and acoustics**
- **Scalable to thousands of processors, has been run on >10,000 processors**
- **Transient, direct frequency response (Helmholtz), modal analysis capabilities**
- **Embedded fully coupled structural acoustic capability**



Sierra-SD: A Brief History

- **Sierra-SD was created in the 1990's at Sandia National Laboratories for large-scale structural analysis**
- **Intended for extremely complex structural and structural acoustics models**
 - **Routinely used to solve models with 100's of millions of degrees of freedom**
- **Scalability is the key**
 - **Sierra-SD can solve n-times larger problem using n-times many more compute processors, in nearly constant CPU time**



Sierra-SD Structural Acoustic Capabilities

- **Massively parallel**
- **Hex, wedge, tet acoustic elements**
- **Acoustic coupling with both 3D and shell (2D) structural elements**
- **Linear and nonlinear acoustics**
- **Allows for mismatched acoustic/solid meshes**
 - **Mortar or multi-point constraints (MPC)'s**
- **Solvers: GDSW/CLOP, FETI-DP, and FETI-H (for Helmholtz)**
- **Solution procedures:**
 - **Frequency response (frequency-domain)**
 - **Transient (time-domain)**
 - **Eigenvalue (modal) analysis**
 - **Linear and quadratic (complex modes)**

Structural Acoustic Equations of Motion

acoustics

$$\nabla^2 \phi = \frac{1}{c^2} \ddot{\phi}, \quad \text{in } \Omega_f \times (0, T)$$

$$\nabla \phi \cdot \mathbf{n}_f = -\rho_f \ddot{u}_n, \quad \text{on } \partial\Omega_f^N \times [0, T]$$

$$\phi = 0, \quad \text{on } \partial\Omega_f^D \times [0, T]$$

$$\phi(0, T) = 0, \quad \text{in } \Omega_f$$

$$\dot{\phi}(0, T) = 0, \quad \text{in } \Omega_f$$

solid mechanics

$$\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\mathbf{u}}, \quad \text{in } \Omega \times (0, T)$$

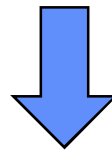
$$\boldsymbol{\sigma} \mathbf{n} = \mathbf{h}, \quad \text{on } \partial\Omega^N \times [0, T]$$

$$\boldsymbol{\sigma} = \mathbf{D} : \nabla \mathbf{u}, \quad \text{in } \Omega \times [0, T]$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega^D \times [0, T]$$

$$\mathbf{u}(0, T) = \mathbf{0}, \quad \text{in } \Omega$$

$$\dot{\mathbf{u}}(0, T) = \mathbf{0}, \quad \text{in } \Omega$$



Time domain

$$[M]\mathbf{a}(t) + [C]\mathbf{v}(t) + [K]\mathbf{u}(t) = \mathbf{f}(t)$$

Frequency domain (Helmholtz)

$$[H(\omega)]\mathbf{z}(\omega) = \mathbf{F}(\omega)$$

$$[H(\omega)] = -\omega^2[M] + i\omega[C] + [K]$$



Structural Acoustic Equations of Motion

- Fully coupled time domain formulation

$$\begin{bmatrix} M_s & 0 \\ 0 & -M_a \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} C_s & L^T \\ L & -C_a \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & -K_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} = \begin{bmatrix} f_s \\ -f_a \end{bmatrix}$$

- Fully coupled eigenanalysis formulation

$$\lambda^2 \begin{bmatrix} M_s & 0 \\ 0 & -M_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} + \lambda \begin{bmatrix} C_s & L^T \\ L & -C_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & -K_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Fully coupled frequency-domain formulation

$$-\omega^2 \begin{bmatrix} M_s & 0 \\ 0 & -M_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} + i\omega \begin{bmatrix} C_s & L^T \\ L & -C_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & -K_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} = \begin{bmatrix} f_s \\ -f_a \end{bmatrix}$$



Research Areas in Sierra-SD

- **Some research areas in Sierra-SD**
 - **Nonlinear acoustics**
 - **Infinite elements and Perfectly Matched Layers (PML)**
 - **Mismatched structural/acoustic meshes**
 - **Inverse problems**

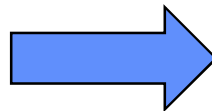
Why Nonlinear Acoustics?

Linear acoustics is inadequate for many applications

- Resonating cavities
- Large-amplitude sources
- Far-field of explosions
- Aeroacoustic noise

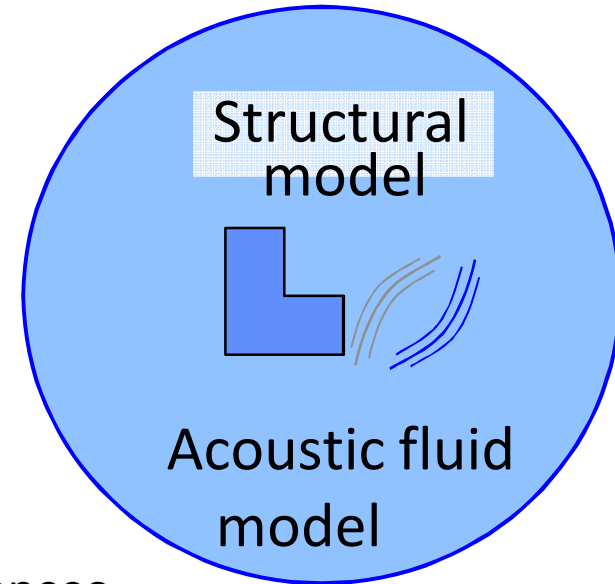
Assumptions of Linear Acoustic Theory

- Small amplitude waves
- Linear constitutive fluid model
- No fluid convection



Consequences

- Resonance leads to infinite amplitude waves
- “Sine wave remains a sine wave”
- No wave distortion
- Wavespeed independent of stress state in fluid





Eulerian Formulations for Nonlinear Acoustics

- The linear acoustic wave equation

$$\frac{1}{c^2} \phi_{tt} - \Delta \phi = 0$$

- The nonlinear Kuznetsov Equation

$$\frac{1}{c^2} \phi_{tt} - \Delta \phi + \frac{1}{c^2} \frac{\partial}{\partial t} \left[(\nabla \phi)^2 + \frac{B/A}{2c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + b \nabla^2 \phi \right] = 0$$

- Soderholm's equation

$$\frac{1}{c^2} \phi_{tt} - \Delta \phi + \frac{1}{c^2} \frac{\partial}{\partial t} [(\nabla \phi)^2 + b \nabla^2 \phi] + \frac{1}{2c^2} \nabla \psi \cdot \nabla (\nabla \phi)^2 + \frac{\gamma - 1}{c^2} \left(\frac{\partial \psi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 \right) \Delta \psi = 0$$



Nonlinear Acoustic-Structure Interaction

Equations of motion of solid

$$\rho u_{tt} - \nabla \cdot \sigma = f(x, t) \quad \Omega_e x[0, T]$$

Kuznetsov wave equation for fluid

$$\frac{1}{c^2} \phi_{tt} - \Delta \phi + \frac{1}{c^2} \frac{\partial}{\partial t} \left[(\nabla \phi)^2 + \frac{B/A}{2c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + b \nabla^2 \phi \right] = 0 \quad \Omega_f x[0, T]$$



Weak Formulation for Time Domain

Find $(u, \phi) \quad [0, T] \rightarrow H_1(\Omega_f) \times (H_1(\Omega_e))^3$

$$\rho(u_{tt}, v)_{\Omega_e} - (\sigma, \nabla v)_{\Omega_e} + (\rho \dot{\phi}, v)_{\partial\Omega} = (f, v)_{\Omega_e} \quad \forall v \in (H^1(\Omega_f))^3$$

$$\frac{1}{c^2}(\phi_{tt}, \psi)_{\Omega_f} + (\nabla \phi, \nabla \psi)_{\Omega_f} + \frac{1}{c^2}(2\nabla \phi \cdot \nabla \dot{\phi}, \psi) + \frac{1}{c^2}\left(\frac{B}{A} \phi \dot{\phi}, \psi\right) +$$

$$b(\nabla \dot{\phi}, \nabla \psi) - (\dot{u}_n, \psi)_{\partial\Omega} = 0 \quad \forall \psi \in H^1(\Omega_f)$$



Nonlinear Equations of Motion

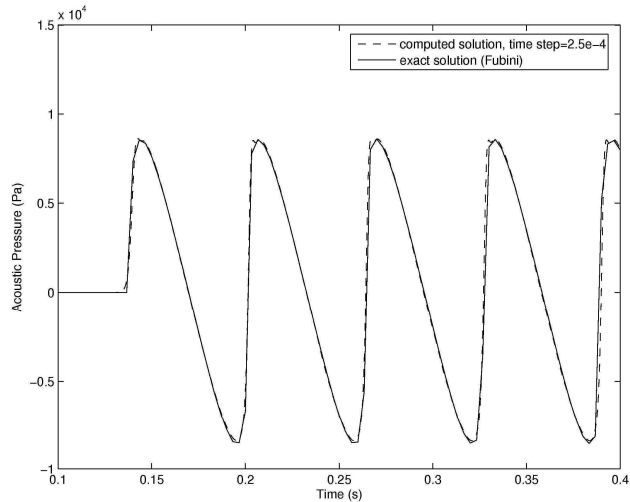
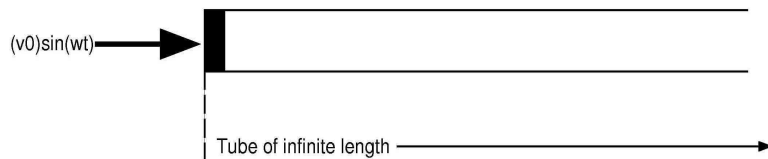
- Time domain formulation

$$\begin{bmatrix} M_s & 0 \\ 0 & -M_a \end{bmatrix} \begin{bmatrix} \ddot{\Delta u} \\ \ddot{\Delta \phi} \end{bmatrix} + \begin{bmatrix} C_s & L^T \\ L & -C_a \end{bmatrix} \begin{bmatrix} \dot{\Delta u} \\ \dot{\Delta \phi} \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & -K_a \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \phi \end{bmatrix} = \begin{bmatrix} \text{Re } s_s \\ \text{Re } s_a \end{bmatrix}$$

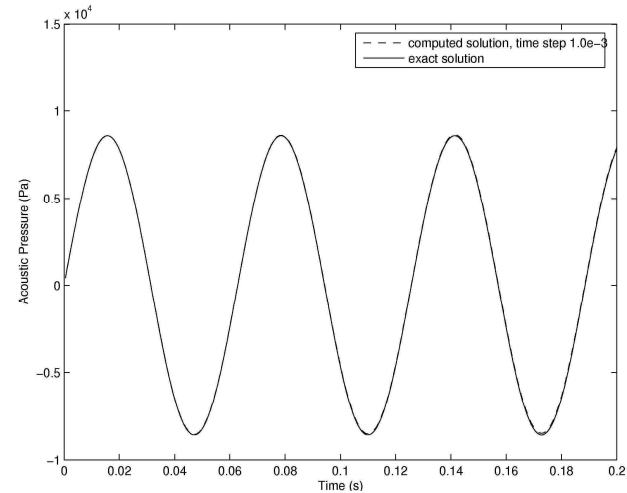
Currently we solve these equations with Newton's method

Verification of Nonlinear Acoustics

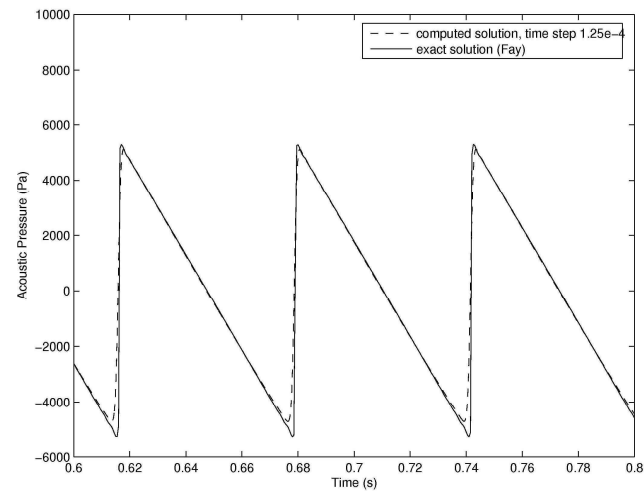
An Infinite Tube



$$\frac{x}{X} = 1$$



$$\frac{x}{X} = 0$$



$$\frac{x}{X} = 4$$

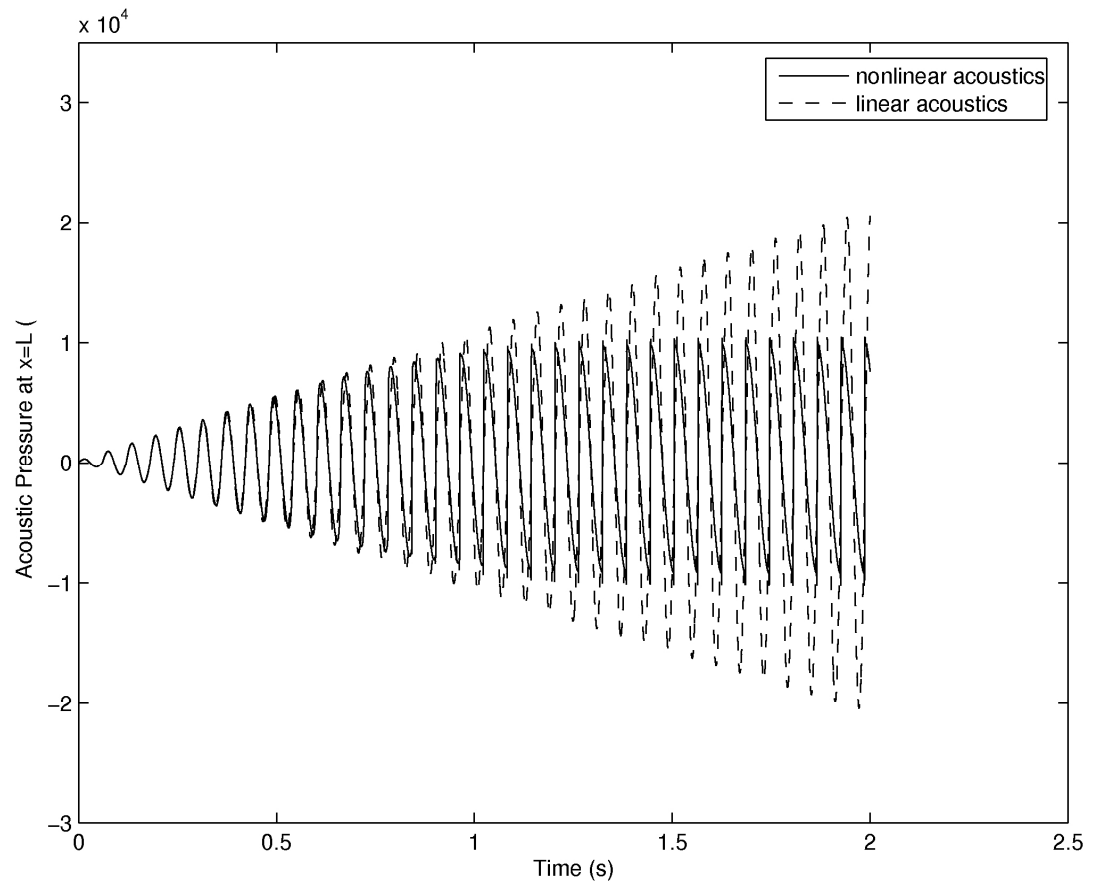
Resonating Tube Results

Source at resonant frequency:

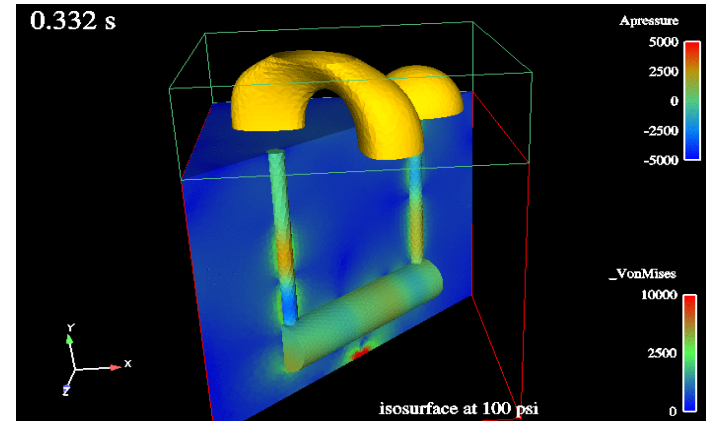
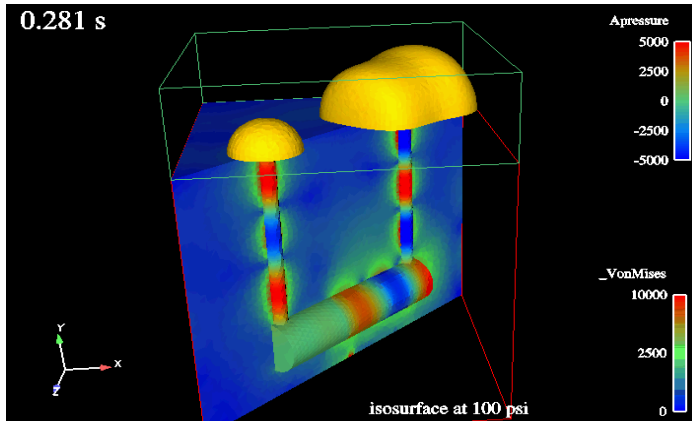
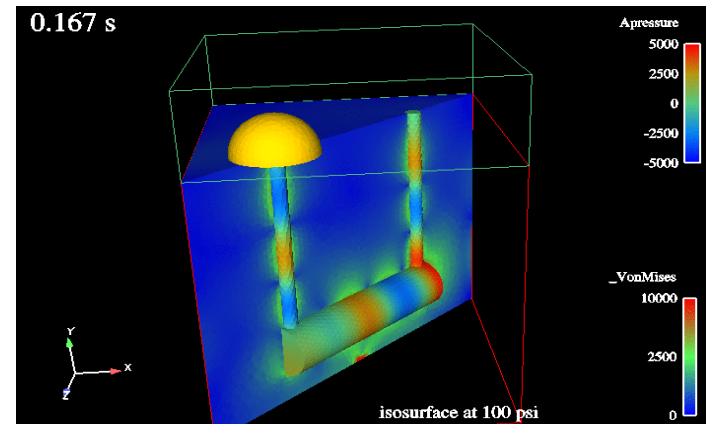
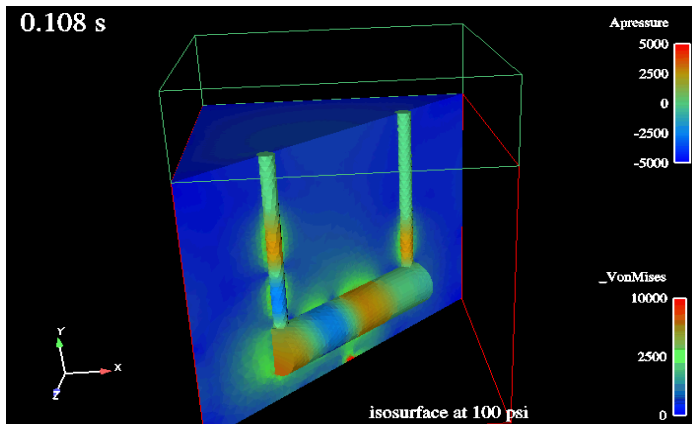
A Finite (Resonating) Tube



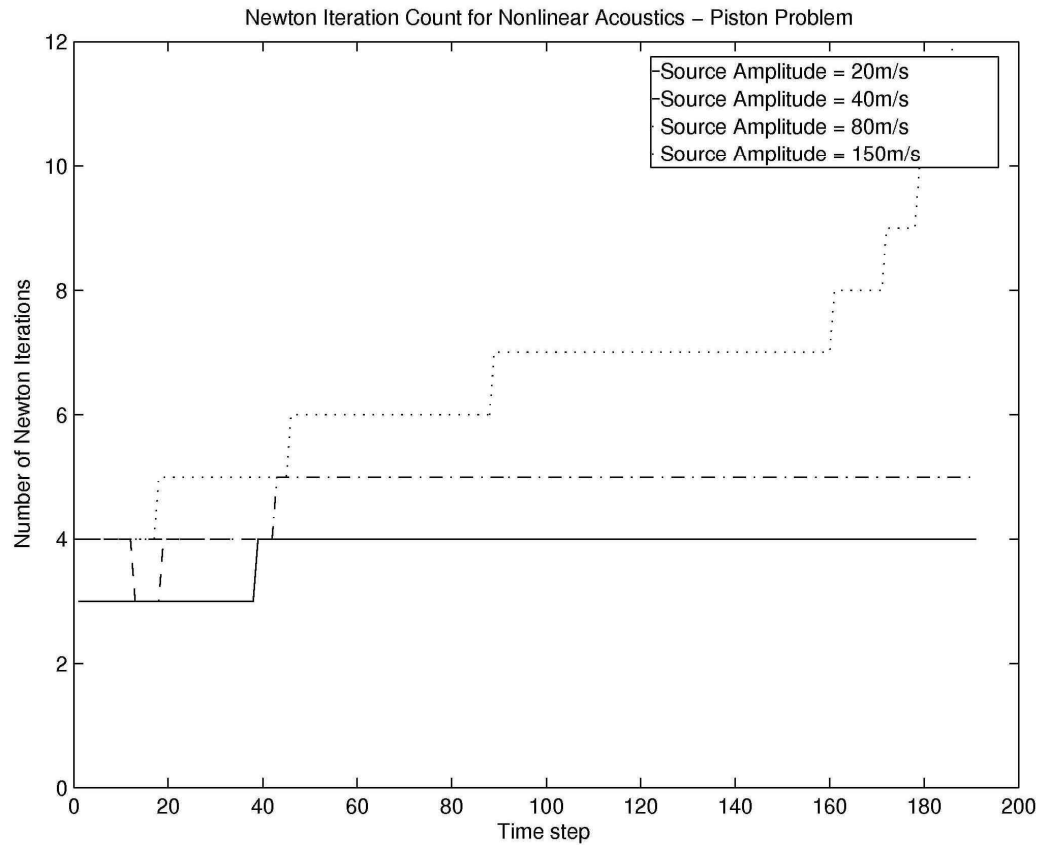
$$\omega_0 = 16.1 \text{ Hz}$$



3D Coupled Nonlinear Acoustic/Elastic Buried Tunnel Simulation – 1.5M unknowns



Newton Iteration Counts

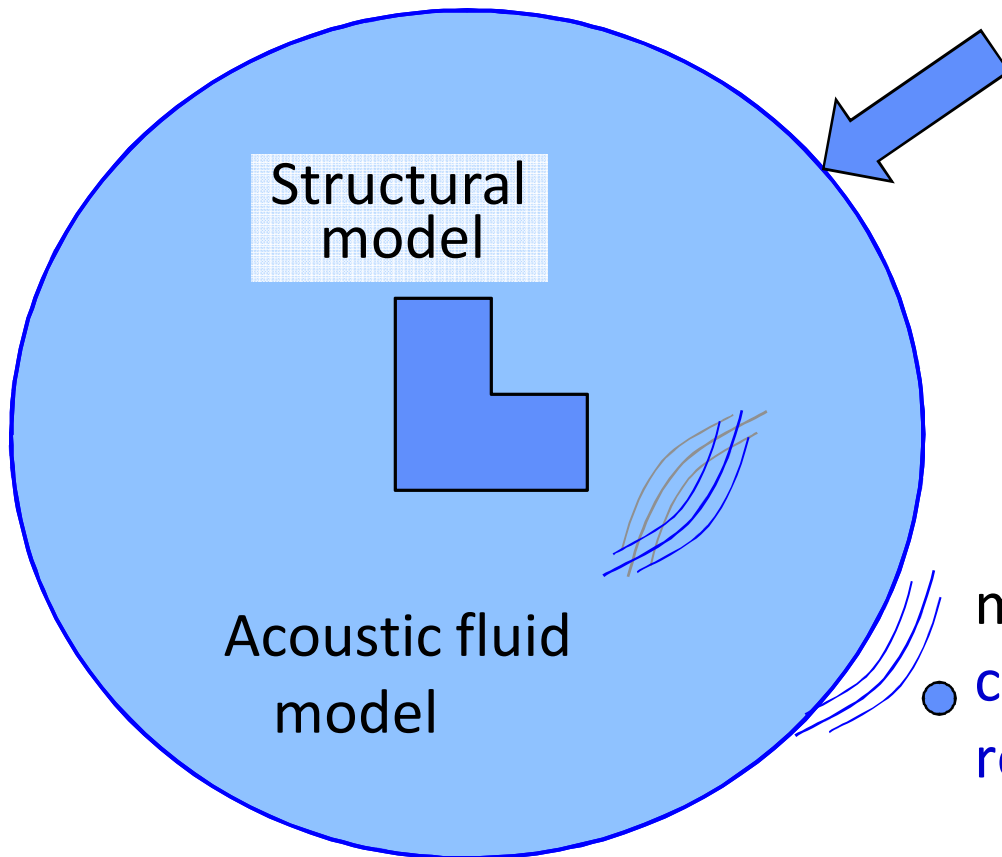




Research Areas in Sierra-SD

- **Some research areas in Sierra-SD**
 - Nonlinear acoustics
 - **Infinite elements and Perfectly Matched Layers (PML)**
 - Mismatched structural/acoustic meshes
 - Inverse problems

Far-Field Acoustics





Comparison of Infinite Elements and PML

Infinite Elements

- Uses analytical solution of wave equation as basis functions
- Time and frequency domain formulations are identical (same matrices)
- Built-in capability for computing far-field pressures (outside of acoustic mesh)
- Restricted to homogeneous media on ellipsoidal domains

PML

- Originally restricted to frequency domain solutions
- Works on arbitrarily shaped convex domains (with corners)
- Can also absorb evanescent waves, and in some cases works on heterogeneous domains
- No capability for computing far-field pressure



Brief History of Infinite Elements

Originally developed for frequency domain calculations

- Bettess, Burnett, Astley, Demkowicz, etc

Time-domain versions originated with “mapped wave envelope” elements by Astley et al. using a **conjugated formulation**

- **Complex conjugation applied to test functions, trial functions remain in unconjugated form**
- **Petrov-Galerkin method (non-symmetric linear systems)**
- **Later extended to time-domain infinite elements**



Time-Domain Far-Field Acoustics

Two separate requirements:

1. Absorbing boundary condition on exterior acoustic surface
2. Far-field post-processor to compute response outside of acoustic mesh

Two different approaches:

- **Absorbing boundary condition (PML, high-order absorbing boundary, etc) followed by Kirchoff integral postprocessor**
- **Time-domain infinite elements**



Comparison of Kirchoff Integral and Infinite Elements in the Time Domain

Kirchoff integral

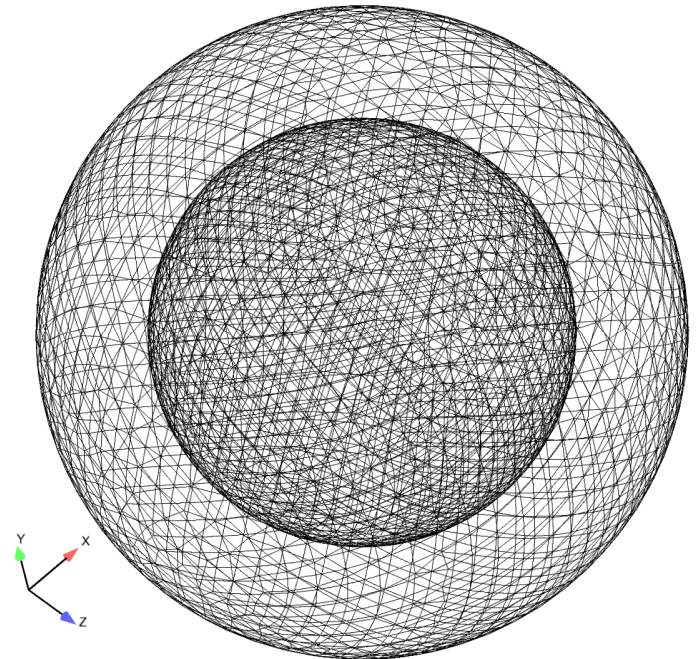
- Large data storage requirement (entire exterior boundary for all times)
- Potential numerical instabilities (similar to time-domain boundary elements)
- Requirement for spatial and temporal derivatives of finite element solution
 - Loss of accuracy

Infinite Element

- Low (or no) data storage required
- Numerically stable provided zero-mass condition is satisfied (Astley, 2006)
- Need to identify host infinite element of far-field point of interest, and master element coordinates
 - Nonlinear problem

A Comparison of Infinite Elements and PML

- Acoustic velocity condition applied to internal spherical surface
- Absorbing conditions applied to outer spherical surface
- Solver iterations compared for
 - Perfectly matched layers (PML),
 - infinite elements (IE),
 - absorbing boundary conditions (ABC)
- Frequency sweep from 10Hz to 100Hz in 10Hz increments
- Solved with 6 processor decomposition using GDSW Helmholtz solver





GDSW Convergence for Infinite Elements and PML

Infinite Elements

Solve	Iter	Total	Avg	Residual
1	20	20	20	7.61228e-10
2	19	39	19	3.66972e-10
3	20	59	19	7.19439e-10
4	22	81	20	4.12606e-10
5	23	104	20	9.94421e-10
6	25	129	21	5.12322e-10
7	27	156	22	4.11824e-10
8	29	185	23	3.81936e-10
9	30	215	23	9.1351e-10
10	32	247	24	7.04619e-10

PML

Solve	Iter	Total	Avg	Residual
1	26	26	26	6.71575e-10
2	19	45	22	5.42187e-10
3	18	63	21	6.71494e-10
4	18	81	20	5.35751e-10
5	18	99	19	4.72051e-10
6	18	117	19	5.02066e-10
7	18	135	19	5.87214e-10
8	18	153	19	9.06385e-10
9	19	172	19	6.71325e-10
10	28	200	20	3.39376e-10

On this example, PML required less iterations than infinite elements

Infinite Element Formulation

Acoustic wave equation for fluid

$$\frac{1}{c^2} p_{tt} - \Delta p = 0 \quad \Omega_x [0, T]$$

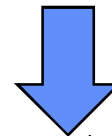
$$\frac{\partial p}{\partial n} = g(x, t) \quad \Gamma_x [0, T]$$

Weak formulation on exterior domain

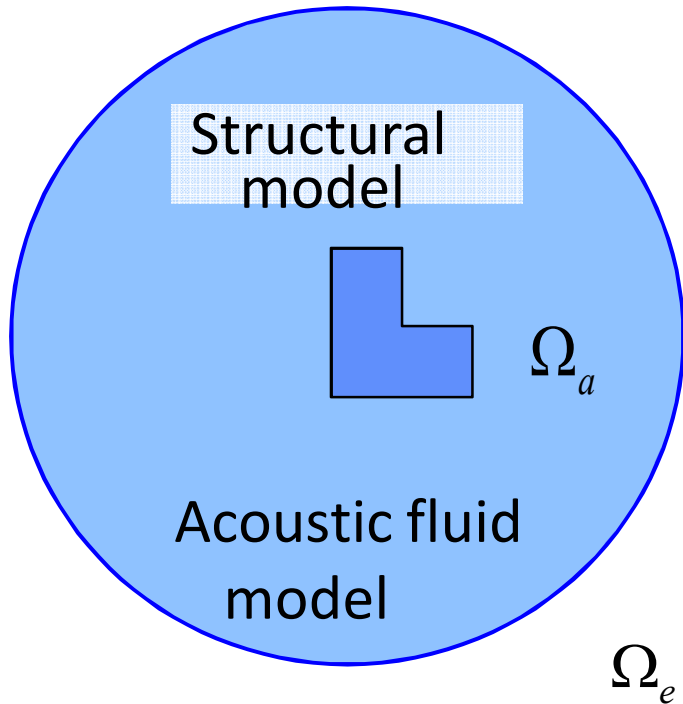
$$\int_{\Omega} \frac{1}{c^2} \ddot{p} q dV + \int_{\Omega} \nabla p \bullet \nabla q dV = \int_{\Gamma} g q dS$$

Trial and weight functions

$$\phi(x, \omega) = P(x) e^{-ik\mu(x)} \quad q = D(x) P(x) e^{ik\mu(x)}$$



$$(-\omega^2 M + i\omega C + K)p = f$$



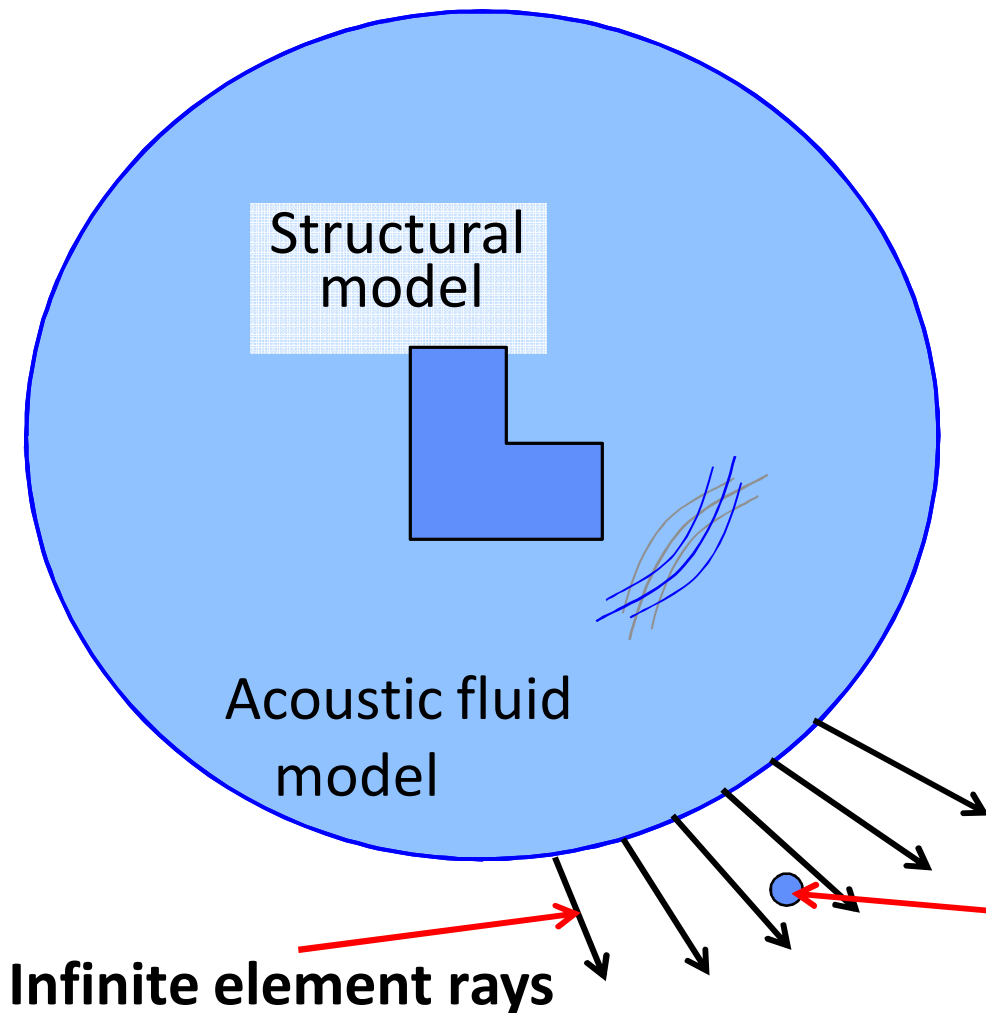
$$\Omega = \Omega_a + \Omega_e$$

Time-Domain Infinite Elements

$$P \approx \left(\frac{\alpha_1}{r} + \frac{\alpha_2}{r^2} + \dots + \frac{\alpha_n}{r^n} \right) e^{-ikr}$$

- Trial functions derived from expansion of exact far-field solution
- Singular Jacobian maps infinitely-long elements to unit master elements
- Order defined as how many terms kept in expansion
 - Conjugated (Petrov-Galerkin) $q \approx D(x)P(x)e^{-ikr}$
 - Unconjugated (Galerkin) $q \approx D(x)P(x)e^{ikr}$
- Conjugated leads to frequency-independent K , M , C
 $(-\omega^2 M + i\omega C + K)p = f \iff M \ddot{p} + C \dot{p} + Kp = f$

Comparison of Kirchoff Integral and Infinite Elements



Kirchoff integral:

1. Store entire time history of pressure and velocity on entire exterior surface
2. Evaluate Kirchoff integral

Infinite Elements:

1. Determine which infinite element owns microphone location
2. Element-level summation



Infinite Element Interpolation

- **Infinite elements discretize the entire (infinite) exterior region.**
- **A singular mapping is necessary to transform finite-size element face into prism of infinite length.**

$$x = \sum_{j=1}^N M_j(s, t, \nu) x_j$$

$$r = \frac{2a}{1 - \nu}$$

Given coordinates of far-field point, need to determine master element coordinates (s,t,v)

- Nonlinear problem



Infinite Element Interpolation

- **Given an arbitrary point outside of the acoustic mesh, we need to determine which infinite element owns the point.**
- **Current approach is to loop through all infinite elements (embarrassingly parallel operation) and do Newton iterations to see which one converges.**
- **Then, once (u,v,w) are known, acoustic pressure at far-field point computed with standard element-level interpolation**



Kirchoff Integral Formulation

$$\begin{aligned} p(x, t) = & \frac{\rho}{4\pi} \int_S \frac{a_n(x, t - R/c)}{R} H(t - R/c) dS \\ & + \frac{1}{4\pi c} \int_S e_R \cdot n_S \frac{\partial}{\partial t} \frac{p(x_S, t - R/c)}{R} H(t - R/c) dS \\ & + \frac{1}{4\pi c} \int_S \frac{c}{R^2} p(x_S, t - R/c) H(t - R/c) dS \end{aligned}$$

Required quantities:

$p(x, t)$ Acoustic pressure from finite element solution

$a_n(x, t)$ Spatial gradient of acoustic pressure

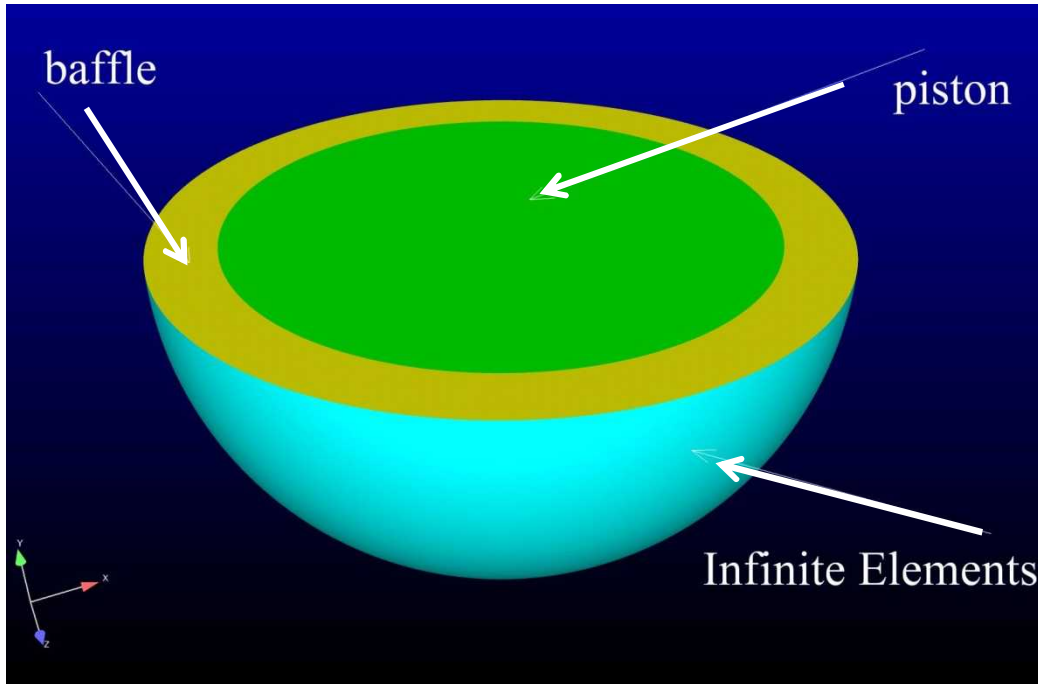
$\frac{\partial p(x, t)}{\partial t}$ Temporal gradient of acoustic pressure



Kirchoff Integral Formulation

- **Requirement for spatial and temporal gradients of acoustic pressure (from finite element calculation)**
 - **These do not come directly from the finite element calculation**
 - **Must be computed numerically after-the-fact**
 - **Loss of order of accuracy**

Piston on Infinite Baffle



Rigid piston, with plane harmonic wave applied

Baffle (symmetry BC)

Model synopsis:

Acoustic mesh : tets

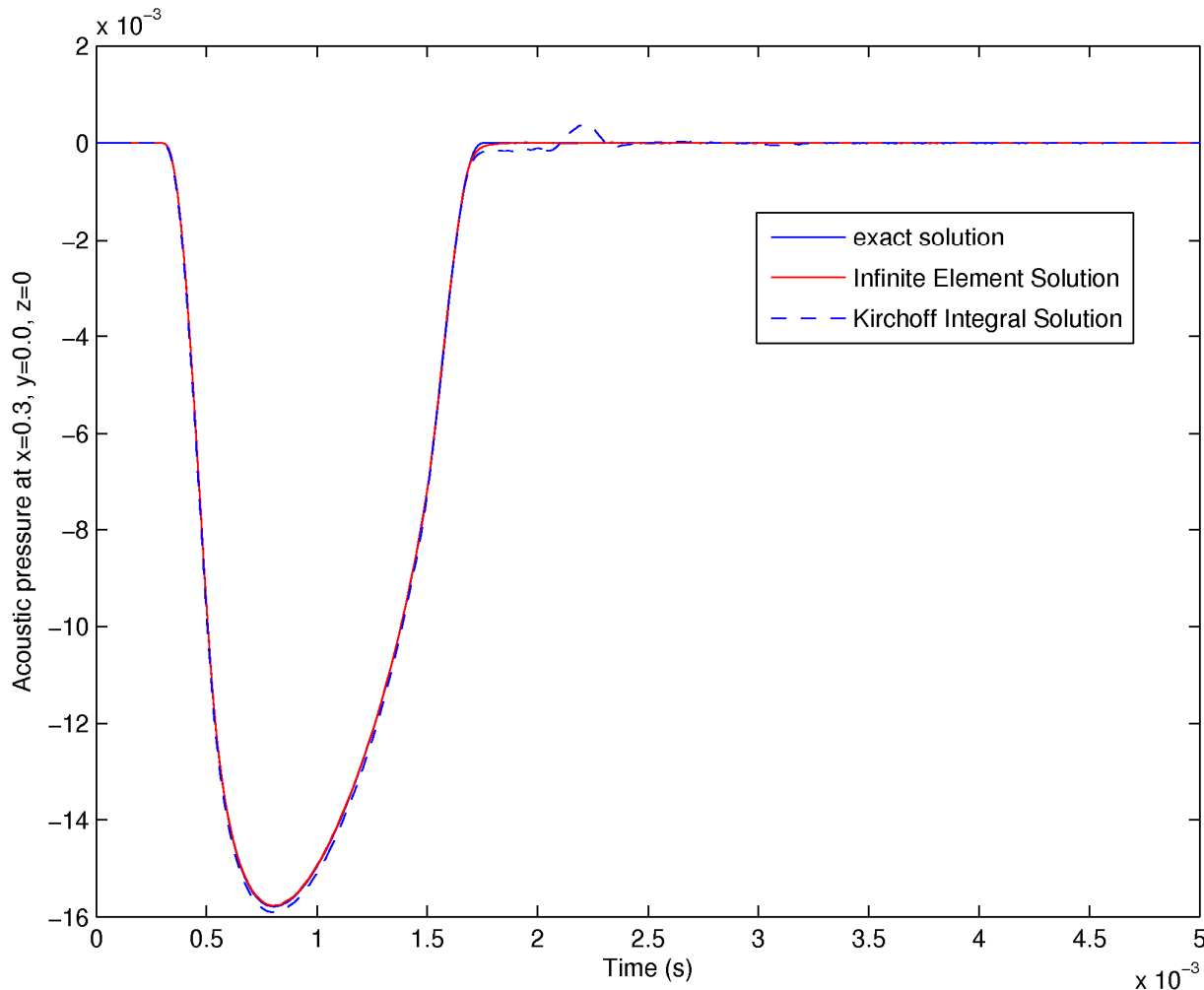
Fluid boundary verification

Analytic solution for acoustic pressure

[Ref]: Pierce, *Acoustics*,
McGraw-Hill, 1981.

$$p(x, t) = \frac{\rho}{4\pi} \int_S \frac{a_n(x, t - R/c)}{R} H(t - R/c) dS$$

Piston on Infinite Baffle

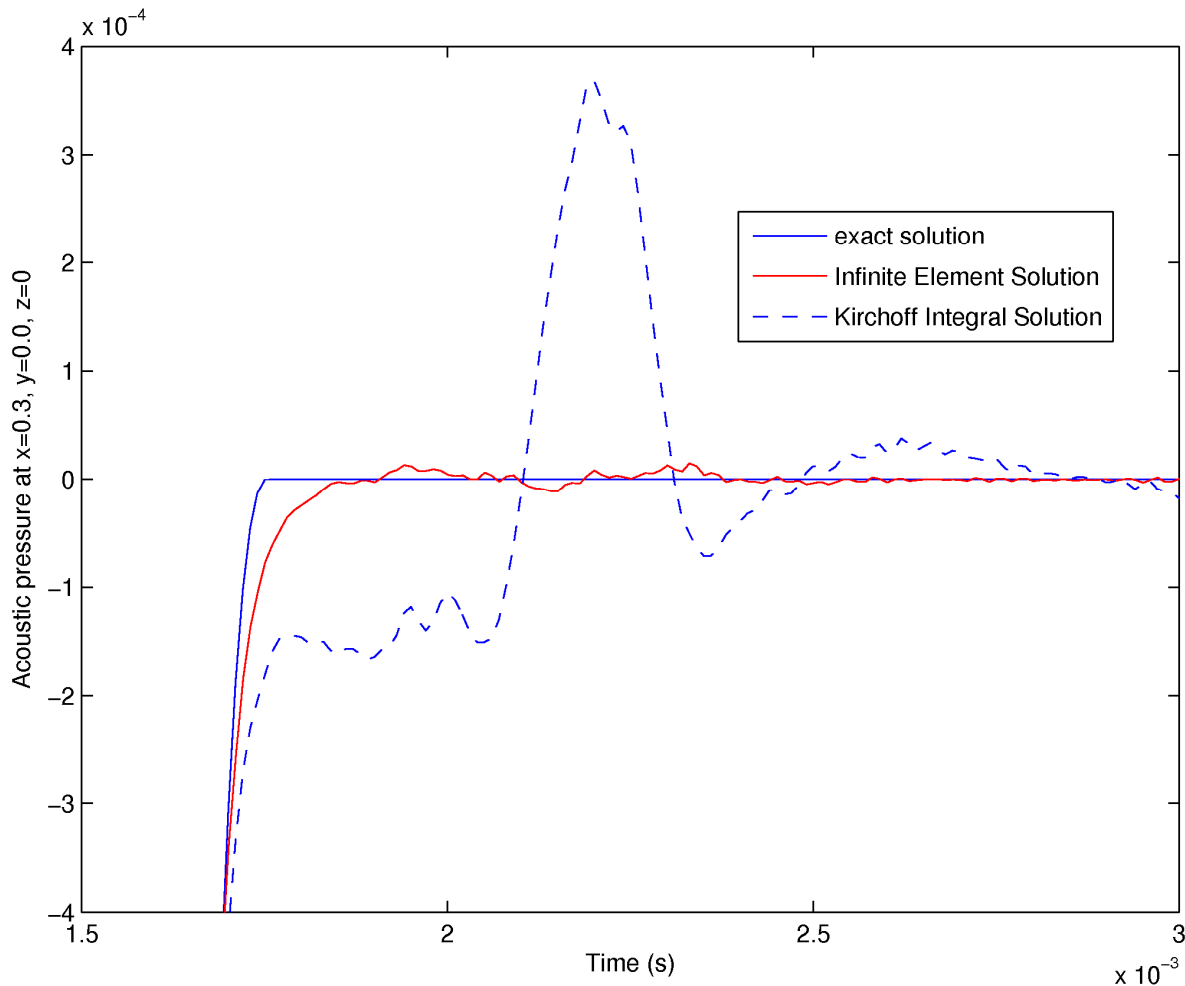


$$r/R = 1.1$$

r: radius to far-field point

R: radius of acoustic mesh

Piston on Infinite Baffle



$$r/R = 1.1$$

r : radius to far-field point

R : radius of acoustic mesh



Research Areas in Sierra-SD

- **Some research areas in Sierra-SD**
 - Nonlinear acoustics
 - Infinite elements and Perfectly Matched Layers (PML)
 - **Mismatched structural/acoustic meshes**
 - Inverse problems



Motivation

- **Acoustic and structural meshes typically generated independently**
 - e.g. ship in water
- **Acoustic and structural meshes almost always have different mesh density requirements**
- **Mesh tying methods have been researched extensively in solid mechanics – but not in acoustics or structural acoustics**
- **Fully coupled simulations are needed**
 - **Coupled modes, coupled frequency response**



Mesh tying methods are needed for nonconforming wet interface



Mesh Tying Methods for Structural Acoustics

1. **Conforming finite element approach**
 - **Requires matching meshes**
2. **Classical multipoint constraint equations with ghost nodes**
3. **Mortar method with ghost nodes**

In all cases we need to evaluate integrals of the $\int_{\Gamma} N_M N_S d\Gamma$ type:

N_M Surface shape function on master side

N_S Surface shape function on slave side

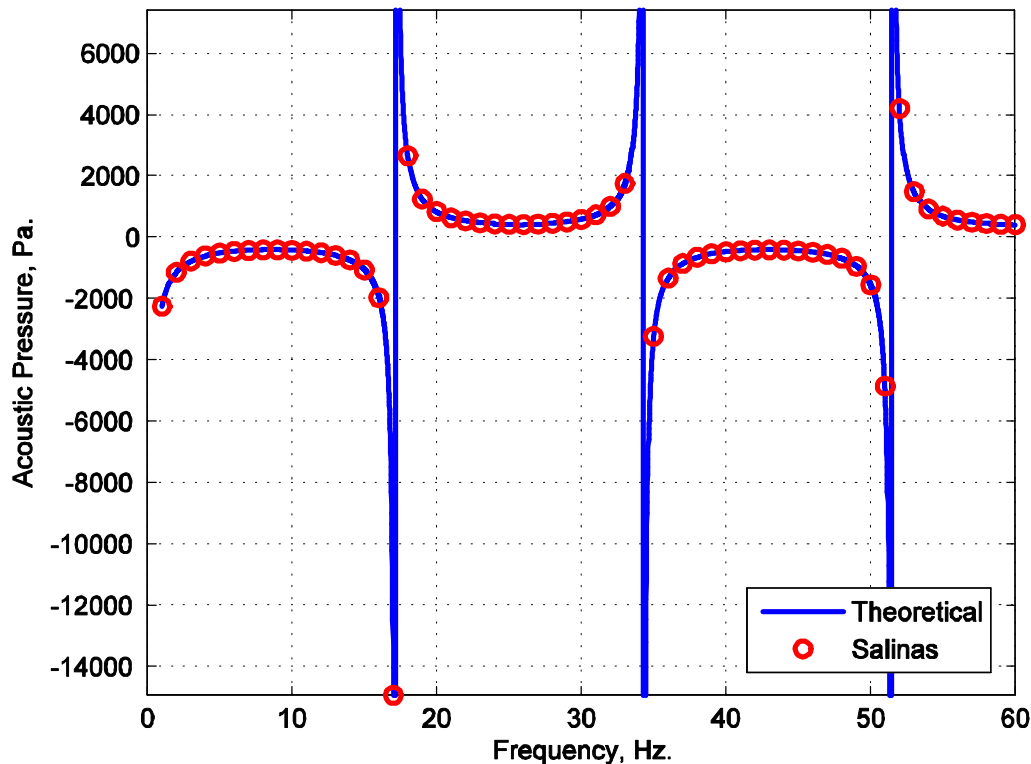


Mismatched Acoustic/Solid Meshes

(solid dof + ghost acoustic
dof)

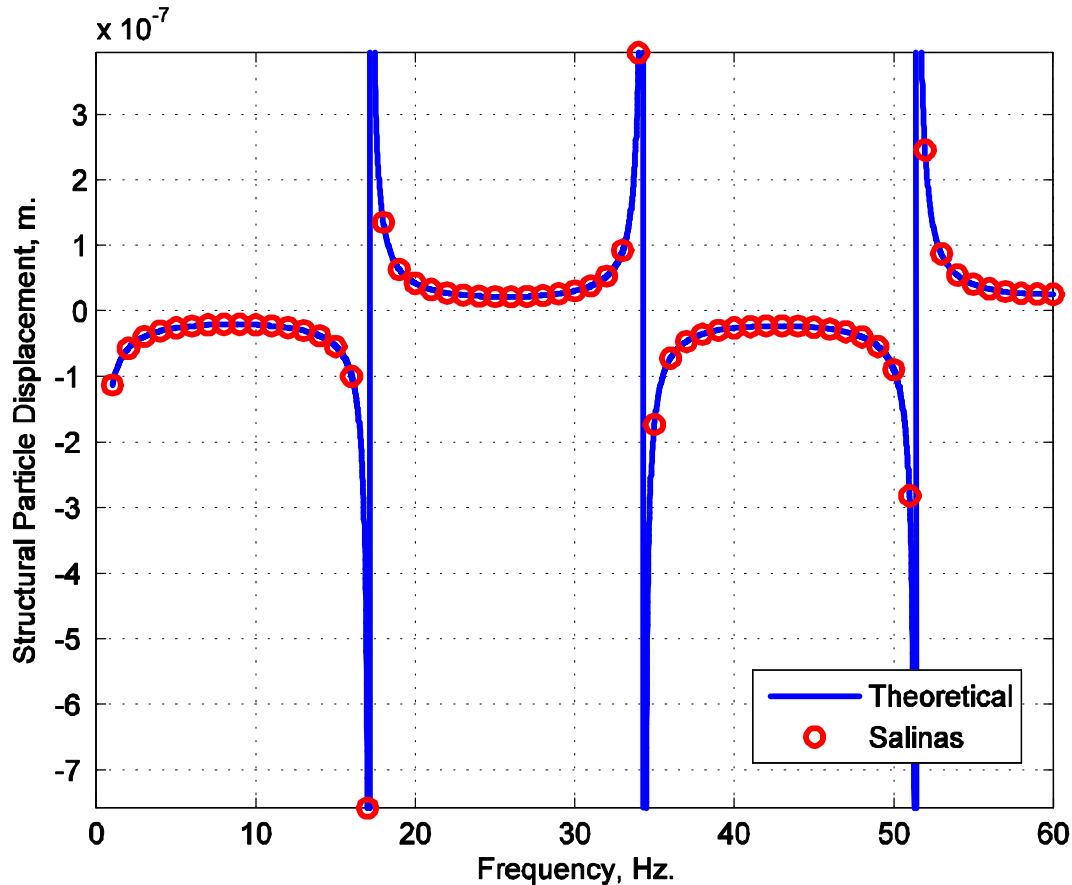
Mismatched Acoustic/Solid Meshes – Verification Example

Acoustic Pressure at Wet interface of Waveguide with Embedded Solid Structure



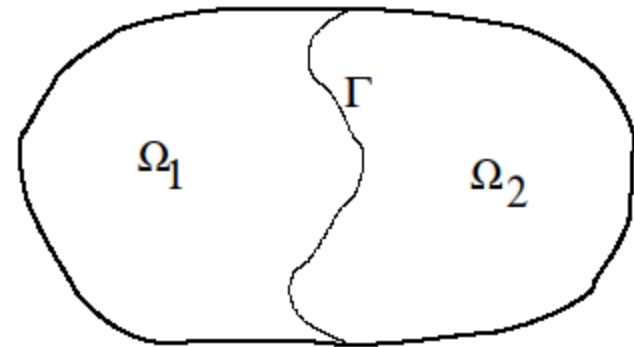
Mismatched Acoustic/Solid Meshes – Verification Example

Acoustic Pressure at Wet interface of Waveguide with
Embedded Solid Structure



Mesh Tying Methods for Acoustics

Weak formulations



$$\int_{\Omega_1} \left[\frac{1}{c^2} \ddot{\psi} \phi + \nabla \psi \cdot \nabla \phi \right] d\Omega_1 = 0 \quad \int_{\Omega_2} \left[\frac{1}{c^2} \ddot{\psi} \phi + \nabla \psi \cdot \nabla \phi \right] d\Omega_2 = 0$$

Constraint equations on interface

- Classical MPC equations
- Mortar method

$$\psi_S = \sum c_i \psi_M$$
$$\int_{\Gamma} (\psi_1 - \psi_2) n d\Gamma = 0$$



Discretization of Boundary Constraint

Boundary Constraint Equation:


$$\int_{\Gamma} (\psi_1 - \psi_2) \eta d\Gamma = 0$$

Discretization:

$$\psi_M = \sum N_{M_i}(x) \psi_{M_i} \quad \psi_S = \sum N_{S_i}(x) \psi_{S_i}$$

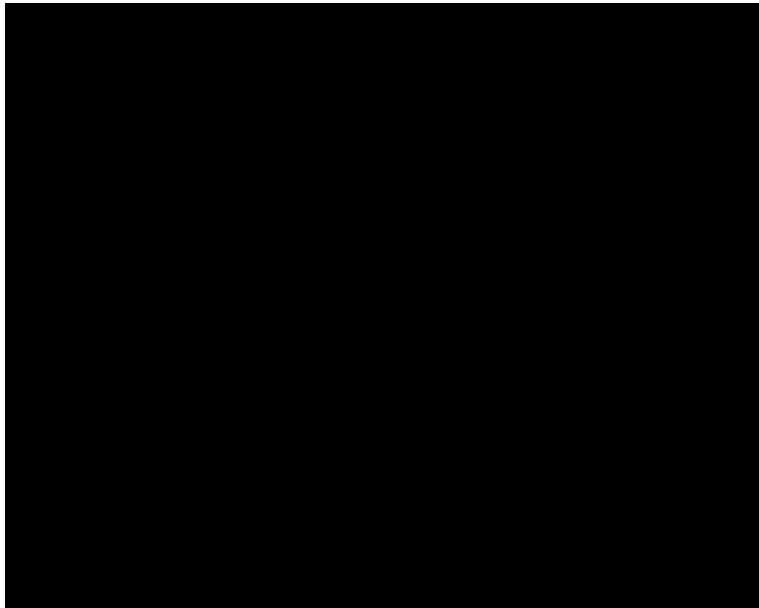
$$\eta = N_{S_j}$$

$$\int_{\Gamma} (\psi_1 - \psi_2) \eta d\Gamma = \sum_i \int_{\Gamma} N_{M_i} N_{S_j} d\Gamma - \sum_i \int_{\Gamma} N_{S_i} N_{S_j} d\Gamma$$

 Mortar method for acoustics involves same surface integrals as for conforming structural acoustics



Convergence Results for Water-Castor Oil System



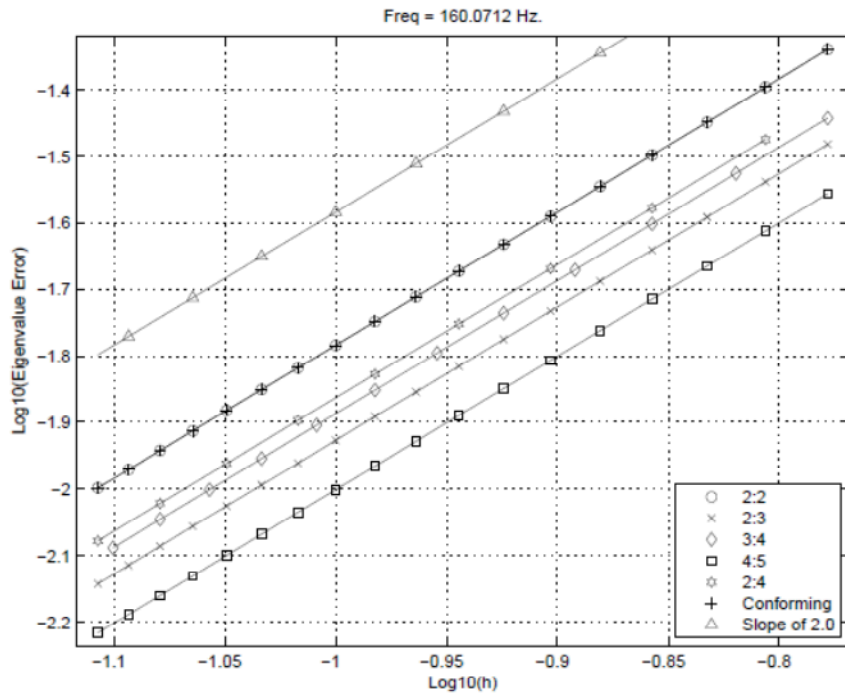
- **Two-fluid tank filled with water and castor oil**
- **Assumed that no mixing occurs**

We compare the results using 3 methods:

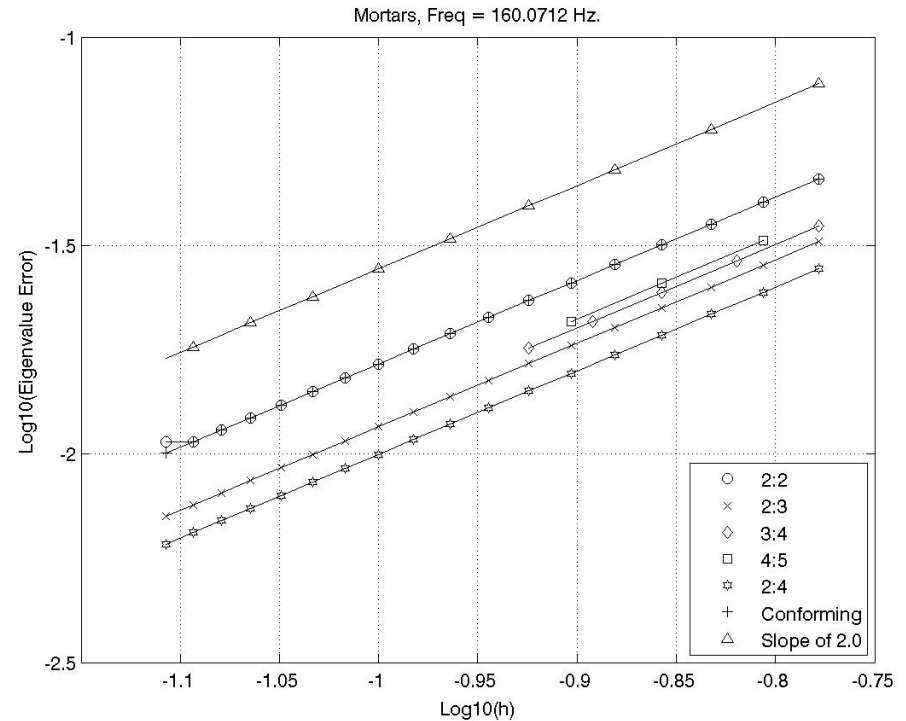
- **Conforming meshes**
- **Nonconforming meshes with ghost nodes and classical MPCs**
- **Nonconforming meshes with ghost nodes and mortar constraints**

Convergence Results Two-Fluid System

160 Hz



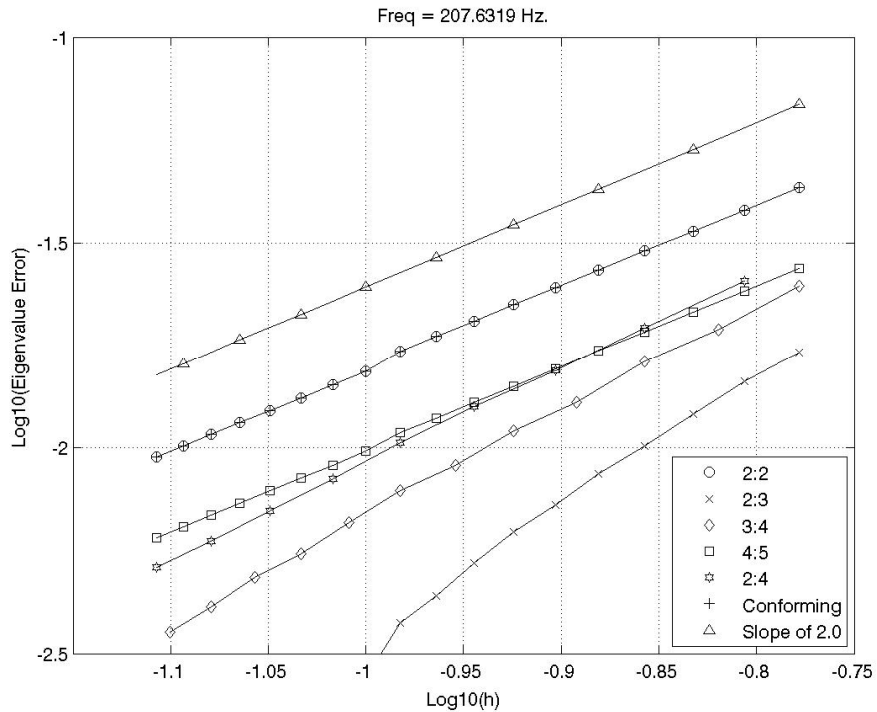
Classical MPCs



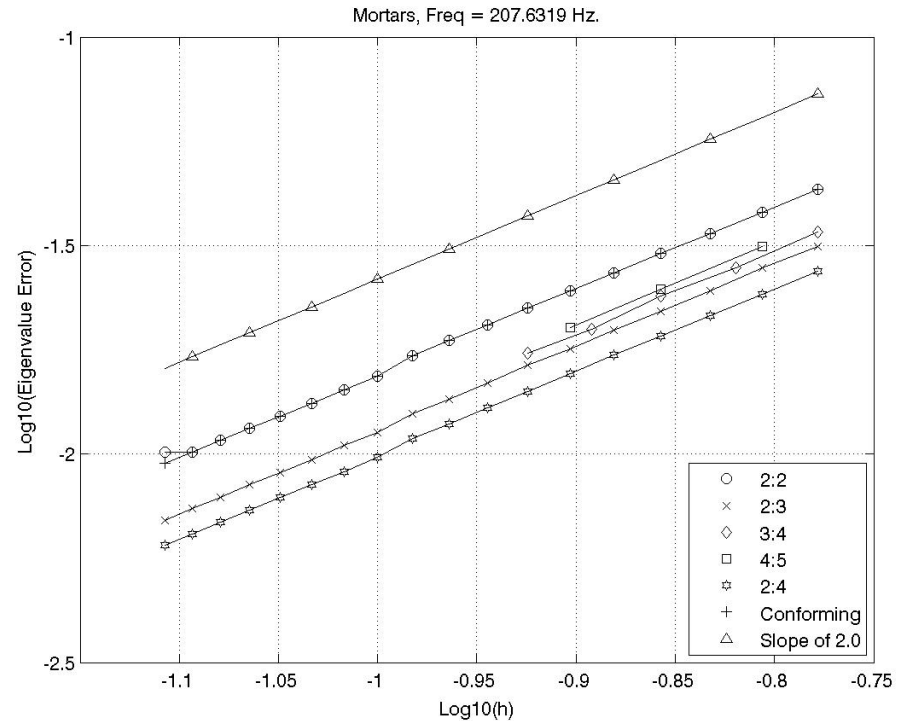
Mortar method

Convergence Results Two-Fluid System

207 Hz



Classical MPCs



Mortar method



Research Areas in Sierra-SD

- **Some research areas in Sierra-SD**
 - Nonlinear acoustics
 - Infinite elements and Perfectly Matched Layers (PML)
 - Mismatched structural/acoustic meshes
 - **Inverse problems**



Inverse Problems- Motivation

- **Characterizing energy sources from experimental measurements is a common need in structural acoustics**
 - Earthquake modeling, nonproliferation, acoustic testing, damage or defect identification from acoustic emission
- **Determining unknown material properties from measurements is a common need in model calibration**
 - Subsurface modeling, medical ultrasonics
- **For applications that involve complex geometries and/or sources, finite element modeling is needed for an accurate solution of the forward problem.**
- **Goal: leverage existing massively parallel finite element technology developed for forward problems to solve the inverse problem.**

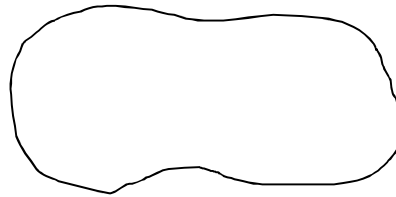


Inverse Problems: The physical View

The direct or forward problem

External inputs
(known)

e.g. forces,
fluxes, etc.



System response
(unknown)

e.g. displacements,
temperature,
concentrations, etc.

The System (known)

e.g. geometry, material
properties, etc.

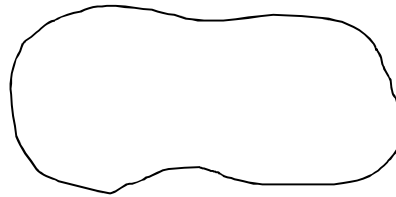


Inverse Problems: The physical View (2)

One type of inverse problem

External inputs
(unknown)

e.g. forces,
fluxes, etc.



System response
(partially known)

e.g. displacements,
temperature,
concentrations, etc.

The System (unknown)

e.g. geometry, material
properties, etc.



Inverse Problems (3) Challenges

- **Usually, inverse problems are ill-posed.**
 - **Solution may not exist.**
 - **Solution may not be unique.**
 - **Solution may be unstable. That is, it may be sensitive to small changes in the input data.**
- **Can be very computationally demanding.**



Inverse Problems: General Scenario

Experimental
measurements

+

Known system
or external inputs

Unknown system or
external inputs

Requires strong collaboration between experimental, analysis, and
code development groups



Source Inversion Methodology

- **PDE-constrained optimization approach**
 - Offers flexibility and extensibility
 - Applicable to time-domain, frequency-domain, and nonlinear problems. Can be tailored to each application.
 - Applicable to large numbers of design variables.
 - Allows significant code sharing with material inversion capability (backward time integrators for adjoint problems, experimental data manager, objective function, etc)
- **Massively parallel finite element code Sierra-SD is used for solving the forward and adjoint problems.**
- **Optimization code ROL/PEOpt is used for solving the optimization problem.**



Structural Acoustic Equations of Motion

acoustics

$$\nabla^2 \phi = \frac{1}{c^2} \ddot{\phi}, \quad \text{in } \Omega_f \times (0, T)$$

$$\nabla \phi \cdot \mathbf{n}_f = -\rho_f \ddot{u}_n, \quad \text{on } \partial\Omega_f^N \times [0, T]$$

$$\phi = 0, \quad \text{on } \partial\Omega_f^D \times [0, T]$$

$$\phi(0, T) = 0, \quad \text{in } \Omega_f$$

$$\dot{\phi}(0, T) = 0, \quad \text{in } \Omega_f$$

Time domain

$$[M]\mathbf{a}(t) + [C]\mathbf{v}(t) + [K]\mathbf{u}(t) = \mathbf{f}(t)$$

solid mechanics

$$\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\mathbf{u}}, \quad \text{in } \Omega \times (0, T)$$

$$\boldsymbol{\sigma} \mathbf{n} = \mathbf{h}, \quad \text{on } \partial\Omega^N \times [0, T]$$

$$\boldsymbol{\sigma} = \mathbf{D} : \nabla \mathbf{u}, \quad \text{in } \Omega \times [0, T]$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega^D \times [0, T]$$

$$\mathbf{u}(0, T) = \mathbf{0}, \quad \text{in } \Omega$$

$$\dot{\mathbf{u}}(0, T) = \mathbf{0}, \quad \text{in } \Omega$$

Frequency domain (Helmholtz)

$$[H(\omega)]\mathbf{z}(\omega) = \mathbf{F}(\omega)$$

$$[H(\omega)] = -\omega^2[M] + i\omega[C] + [K]$$



Structural Acoustic Equations of Motion

Fully coupled formulation

$$\begin{bmatrix} M_s & 0 \\ 0 & M_a \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} C_s & L^T \\ -L & C_a \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & K_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} = \begin{bmatrix} f_s \\ f_a \end{bmatrix}$$

Condensed notation

$$[M]\mathbf{a}(t) + [C]\mathbf{v}(t) + [K]\mathbf{u}(t) = \mathbf{f}(t)$$

We will use the condensed notation in following slides



Statement of Inverse Problem

Minimize objective function

$$J(\{\mathbf{u}\}, \{\mathbf{p}\}) = \frac{\kappa}{2} (\{\mathbf{u}\} - \{\mathbf{u}_m\})^T [\mathbf{Q}] (\{\mathbf{u}\} - \{\mathbf{u}_m\}) + \mathcal{R}(\{\mathbf{p}\}),$$

$\{\mathbf{u}\}$ State variables (displacement, pressure)

$\{\mathbf{u}_m\}$ Measured data (displacement, pressure)

$\{\mathbf{p}\}$ Unknown parameters (loads, material parameters)

$[\mathbf{Q}]$ Weight matrix

Subject to equations of motion

$$[\mathbf{M}]\mathbf{a}(t) + [\mathbf{C}]\mathbf{v}(t) + [\mathbf{K}]\mathbf{u}(t) = \mathbf{f}(t)$$

Statement of Inverse Problem (2)

The Lagrangian

$$\begin{aligned} \mathcal{L}(\{d\}, \{\hat{d}\}, \{p\}) = & \tilde{J}(\{p\}) + \hat{\mathbf{u}}_0^T \left([M] \mathbf{a}_0 + [C] \mathbf{v}_0 + [K] \mathbf{u}_0 - \mathbf{f}_0(\{p\}) \right) \\ & + \sum_{k=1}^N \left\{ \hat{\mathbf{u}}_k^T \left([M] \mathbf{a}_k + [C] \mathbf{v}_k + [K] \mathbf{u}_k - \mathbf{f}_k(\{p\}) \right) \right. \\ & + \hat{\mathbf{v}}_k^T [M] \left(\mathbf{v}_k - \mathbf{v}_{k-1} - \Delta t [(1 - \gamma) \mathbf{a}_{k-1} + \gamma \mathbf{a}_k] \right) \\ & \left. + \hat{\mathbf{a}}_k^T [M] \left(\mathbf{u}_k - \mathbf{u}_{k-1} - \Delta t \mathbf{v}_{k-1} - \frac{\Delta t^2}{2} [(1 - 2\beta) \mathbf{a}_{k-1} + 2\beta \mathbf{a}_k] \right) \right\} \end{aligned}$$

where

$$\{d(\{p\})\} = \{ \{u\}, \{v\}, \{a\} \}$$



Optimality conditions

- **Optimality is obtained by setting derivatives of Lagrangian to zero**
- **We will adopt a reduced space approach where we derive reduced gradients from full space approach**
- **Reduced space approach can be derived from full space**

Statement of Inverse Problem (3)

Gateaux derivatives of the Lagrangian with respect to adjoint variables

$$\nabla_{\mathbf{a}_0} \mathcal{L} \cdot \delta \mathbf{a}_0 = \delta \mathbf{a}_0^T \left([M] \hat{\mathbf{u}}_0 - \frac{\Delta t^2}{2} (1 - 2\beta) [M] \hat{\mathbf{a}}_1 - \Delta t (1 - \gamma) [M] \hat{\mathbf{v}}_1 \right),$$

$$\nabla_{\mathbf{u}_k} \mathcal{L} \cdot \delta \mathbf{u}_k = \delta \mathbf{u}_k^T \left([M] (\hat{\mathbf{a}}_k - \hat{\mathbf{a}}_{k+1}) + [K] \hat{\mathbf{u}}_k + \kappa [Q] (\mathbf{u}_k - \mathbf{u}_{m_k}) \right),$$

$$\nabla_{\mathbf{v}_k} \mathcal{L} \cdot \delta \mathbf{v}_k = \delta \mathbf{v}_k^T \left([C] \hat{\mathbf{u}}_k - \Delta t [M] \hat{\mathbf{a}}_{k+1} + [M] \hat{\mathbf{v}}_k - [M] \hat{\mathbf{v}}_{k+1} \right),$$

$$\begin{aligned} \nabla_{\mathbf{a}_k} \mathcal{L} \cdot \delta \mathbf{a}_k &= \delta \mathbf{a}_k^T \left([M] \hat{\mathbf{u}}_k - \beta \Delta t^2 [M] \hat{\mathbf{a}}_k - \frac{\Delta t^2}{2} [M] (1 - 2\beta) \hat{\mathbf{a}}_{k+1}, \right. \\ &\quad \left. - \Delta t [M] (\gamma \hat{\mathbf{v}}_k + (1 - \gamma) \hat{\mathbf{v}}_{k+1}) \right), \end{aligned}$$

$$\nabla_{\mathbf{u}_N} \mathcal{L} \cdot \delta \mathbf{u}_N = \delta \mathbf{u}_N^T \left([M] \hat{\mathbf{a}}_N + [K] \hat{\mathbf{u}}_N + \kappa [Q] (\mathbf{u}_N - \mathbf{u}_{m_N}) \right),$$

$$\nabla_{\mathbf{v}_N} \mathcal{L} \cdot \delta \mathbf{v}_N = \delta \mathbf{v}_N^T \left([C] \hat{\mathbf{u}}_N + [M] \hat{\mathbf{v}}_N \right),$$

$$\nabla_{\mathbf{a}_N} \mathcal{L} \cdot \delta \mathbf{a}_N = \delta \mathbf{a}_N^T \left([M] \hat{\mathbf{u}}_N - \Delta t^2 \beta [M] \hat{\mathbf{a}}_N - \Delta t \gamma [M] \hat{\mathbf{v}}_N \right).$$

Statement of Inverse Problem (4)

(i) Final conditions

$$\begin{aligned}[C] \hat{\mathbf{u}}_N + [M] \hat{\mathbf{v}}_N &= \mathbf{0} \\ \hat{\mathbf{u}}_N &= \Delta t^2 \beta \hat{\mathbf{a}}_N + \Delta t \gamma \hat{\mathbf{v}}_N \\ [M] \hat{\mathbf{a}}_N + [K] \hat{\mathbf{u}}_N &= \kappa [Q] (\mathbf{u}_{m_N} - \mathbf{u}_N)\end{aligned}$$

(ii) Backward transition equations

$$\begin{aligned}\hat{\mathbf{u}}_k - \beta \Delta t^2 \hat{\mathbf{a}}_k - \Delta t \gamma \hat{\mathbf{v}}_k &= \frac{\Delta t^2}{2} (1 - 2\beta) \hat{\mathbf{a}}_{k+1} + \Delta t (1 - \gamma) \hat{\mathbf{v}}_{k+1} \\ [C] \hat{\mathbf{u}}_k + [M] (\hat{\mathbf{v}}_k - \Delta t \hat{\mathbf{a}}_{k+1} - \hat{\mathbf{v}}_{k+1}) &= \mathbf{0} \\ [M] \hat{\mathbf{a}}_k + [K] \hat{\mathbf{u}}_k &= [M] \hat{\mathbf{a}}_{k+1} + \kappa [Q] (\mathbf{u}_{m_k} - \mathbf{u}_k)\end{aligned}$$

(iii) Last transition equation

$$\hat{\mathbf{u}}_0 = \frac{\Delta t^2}{2} (1 - 2\beta) \hat{\mathbf{a}}_1 + \Delta t (1 - \gamma) \hat{\mathbf{v}}_1$$



Statement of Inverse Problem (5)

Gateaux derivatives of the Lagrangian with respect to design variables

$$\nabla_{\{\mathbf{p}\}} \mathcal{L}(\{\mathbf{d}\}, \{\hat{\mathbf{d}}\}, \{\mathbf{p}\}) \cdot \{\delta \mathbf{p}\} = \nabla_{\{\mathbf{d}\}} \mathcal{L} \cdot \{\delta \mathbf{d}\} + \nabla_{\{\mathbf{p}\}} \mathcal{L} \cdot \{\delta \mathbf{p}\}$$

$$\nabla_{\{\mathbf{d}\}} \mathcal{L} = \mathbf{0} \quad (\text{from adjoint solution})$$

Gradient Equation

$$\nabla_{\{\mathbf{p}\}} \tilde{\mathcal{J}} = \nabla_{\{\mathbf{p}\}} \mathcal{L} = - \sum_{k=1}^N \hat{\mathbf{u}}_k^T \left(\nabla_{\{\mathbf{p}\}} \mathbf{f}_k(\{\mathbf{p}\}) \right) + \nabla_{\{\mathbf{p}\}} \mathcal{R}.$$



Solution of Inverse Problem

Do until tolerance $<$ eps

Solve forward problem

Solve adjoint problem

Compute gradients, Hessians

Optimization step

Receive design variable updates from optimization solver

end



Source Inversion Methodology - Summary

- **PDE-constrained optimization approach**
 - **Offers flexibility and extensibility**
 - **Applicable to time-domain, frequency-domain, and nonlinear problems. Can be tailored to each application.**
 - **Applicable to large numbers of design variables.**
 - **Allows significant code sharing with material inversion capability**
 - **Sierra-SD is used for solving the forward and adjoint problems.**
 - **Optimization code ROL is used for solving the optimization problem.**



Material Inversion Methodology

PDE-constrained optimization approach

- Offers flexibility and extensibility
- Applicable to time-domain, frequency-domain, and nonlinear problems.
- Parallelized to handle large number of design variables.

Objective function for MECE minimization:

$$U(\mathbf{u}, \boldsymbol{\sigma}; \mathbb{C}) := \frac{1}{2} \int_{\Omega} (\boldsymbol{\sigma} - \mathbb{C} : \boldsymbol{\epsilon}[\mathbf{u}]) : \mathbb{C}^{-1} : \overline{(\boldsymbol{\sigma} - \mathbb{C} : \boldsymbol{\epsilon}[\mathbf{u}])} d\Omega$$

$$\Lambda(\mathbf{u}, \boldsymbol{\sigma}; \mathbb{C}) = U(\mathbf{u}, \boldsymbol{\sigma}; \mathbb{C}) + \frac{\kappa}{2} \|\mathbf{u} - \mathbf{u}^m\|_{L^2(\Omega_m)}^2$$

Objective function for L2 minimization:

$$\mathcal{J}(\mathbf{u}, \mathbb{C}) = \frac{1}{2} \|\mathbf{u} - \mathbf{u}^m\|_{L^2(\Omega_m)}^2 + \mathcal{R}(\mathbb{C})$$



Source Inversion – Research Directions

- Full space methods (presented methods were reduced space)
- UQ and stochastic inversion
- Continue to optimize linear solvers
- Continue to develop better optimization algorithms

Total solution time \approx Number of function evaluations \times linear solve time



Other Potential Applications for Source Inversion

- Aircraft cabin noise
 - How much energy coming from mechanical vs air?
- Speaker design
 - Need to use microphones to infer a very detailed description of sources
- Magnetic Resonance Imaging machines
 - Potential cause hearing damage if sources not understood
- Nuclear reactors
 - Measure noise instead of temperatures (thermoacoustics)
 - Need to understand the acoustic sources in order to control them

In all cases, can we use mics and accelerometers to infer information about the sources?

Structural Acoustics in Sierra-SD

Use Case: Acoustic source inversion

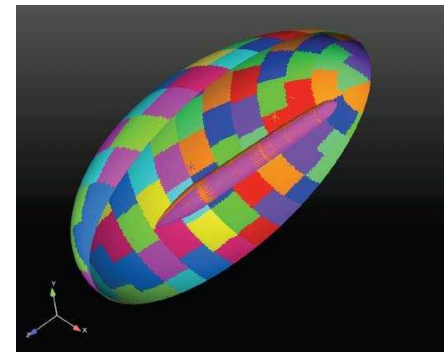
Goal:

Solve inverse problem to obtain acoustic patch inputs that produce the given microphone measurements.

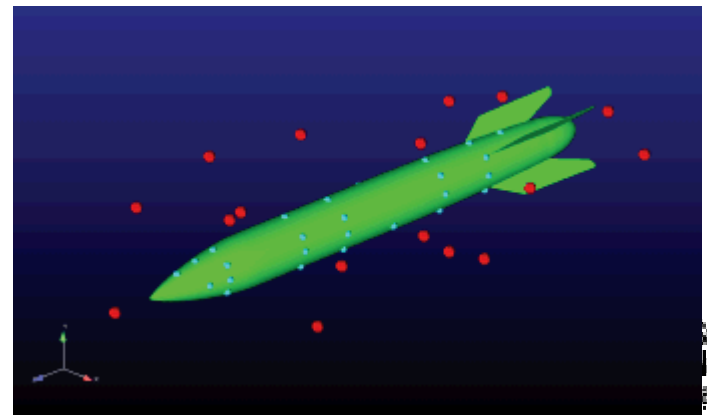
2 approaches:

1. Frequency domain
 - broadband frequency sweep
2. Time domain
 - implicit time integration that covers frequency range of interest

Surface with 172 acoustic patches

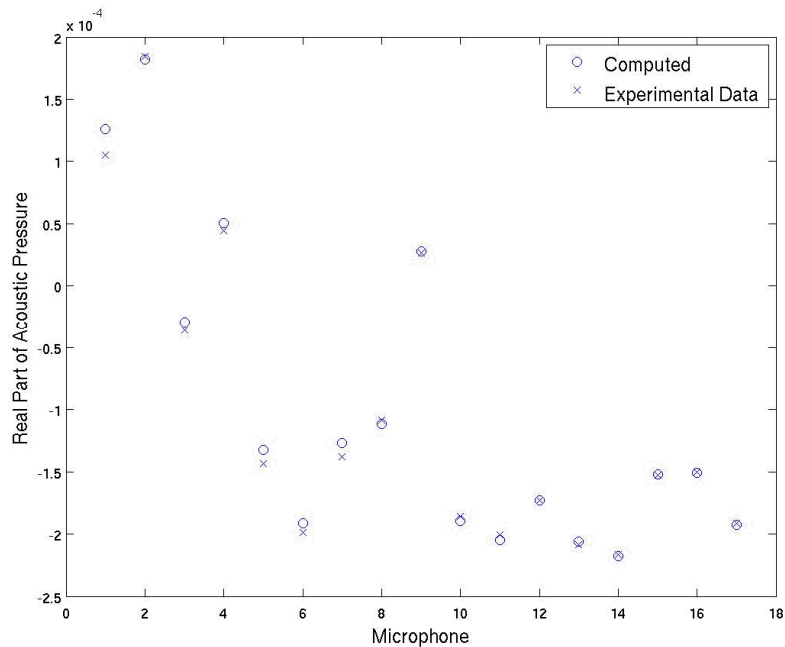


Microphone locations

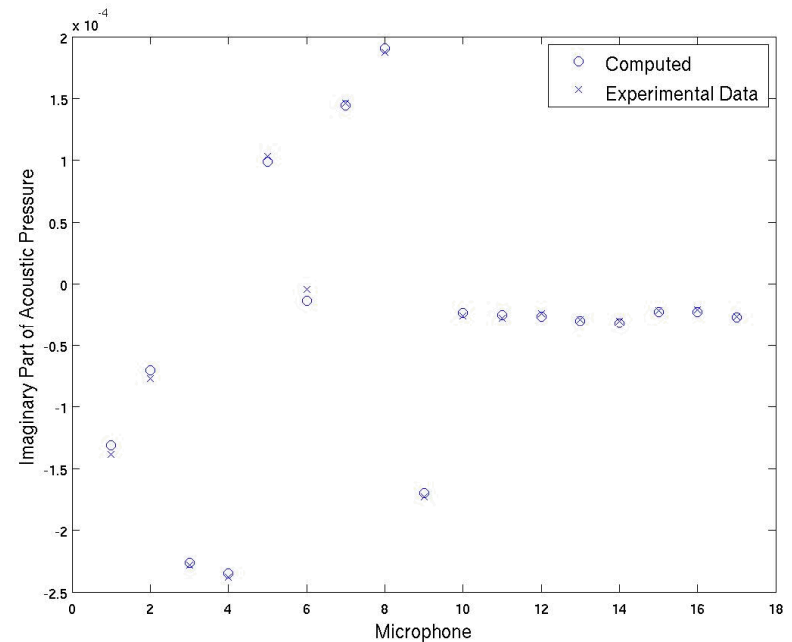


Frequency Domain Source Inversion

Single Frequency Results



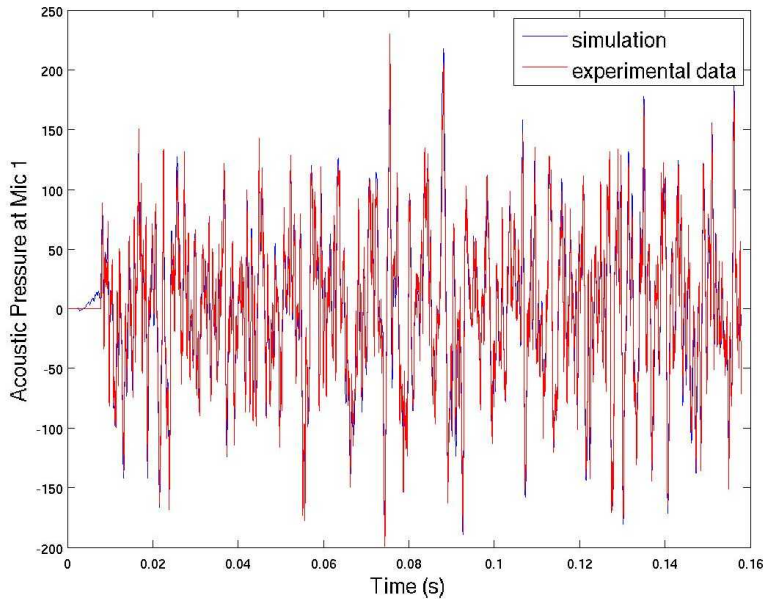
Real part of acoustic pressure



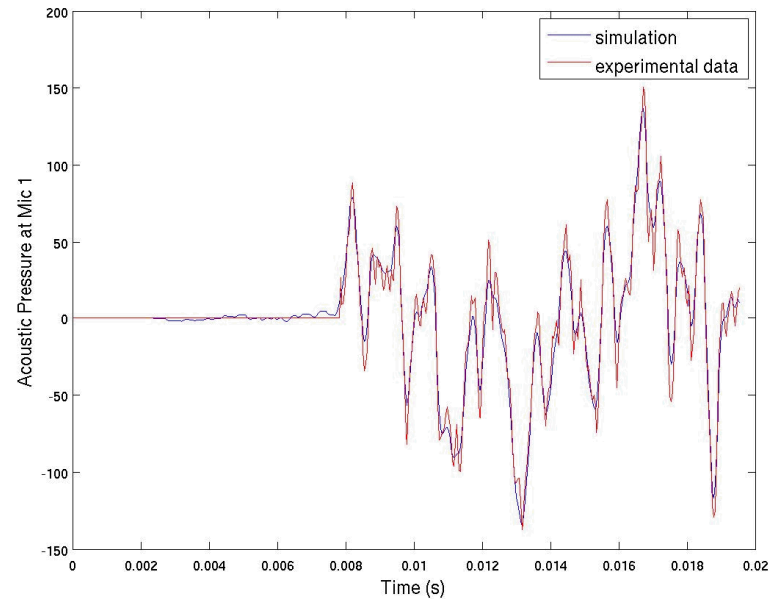
Imaginary part of acoustic pressure

Time Domain Source Inversion

Results for Microphone 1 (other mics were similar)



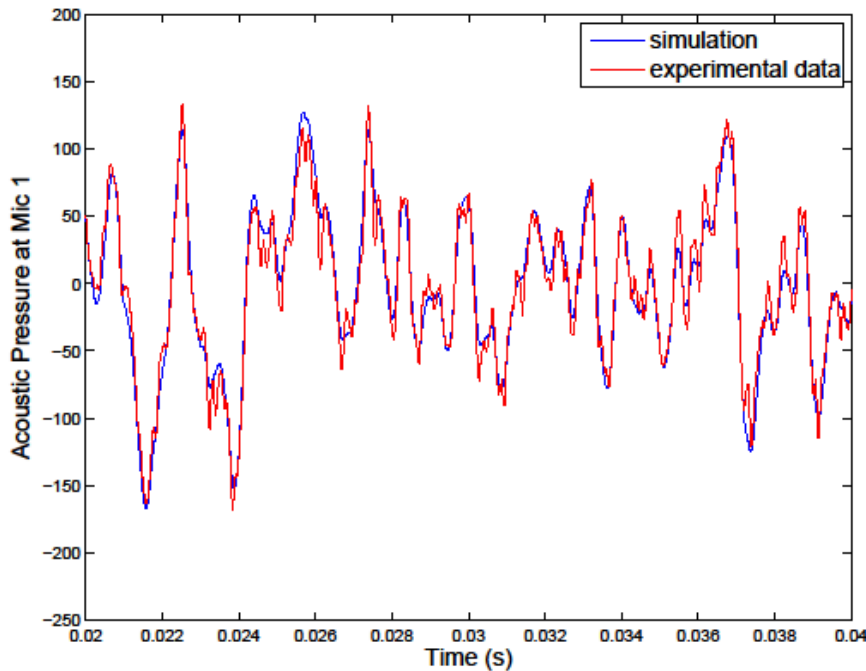
Full time history



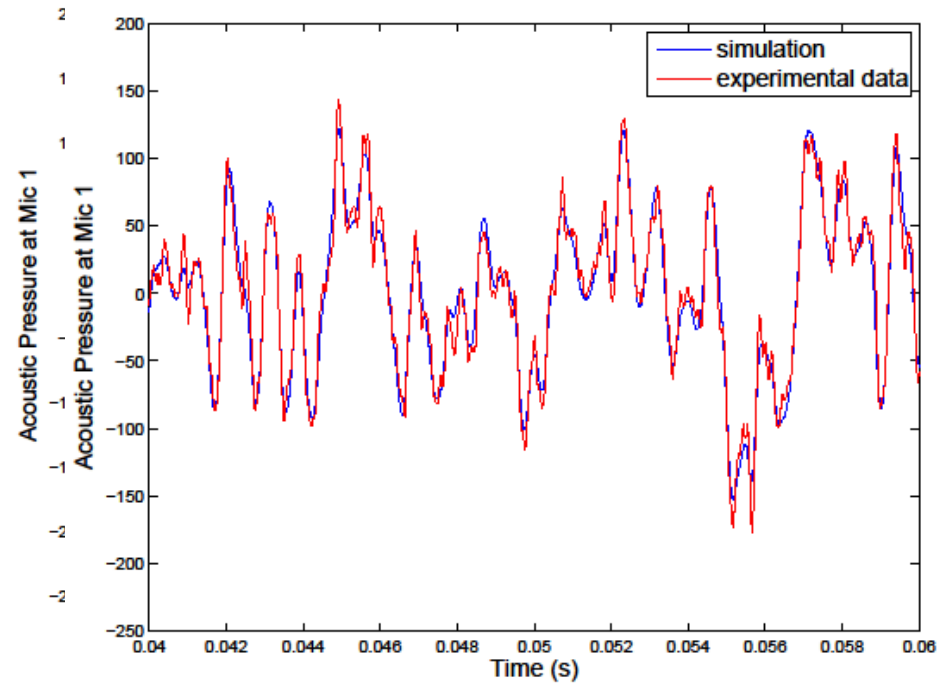
Blow-up near origin of wave

Time Domain Source Inversion

Results for Microphone 1



$0.02(s) < t < 0.04(s)$



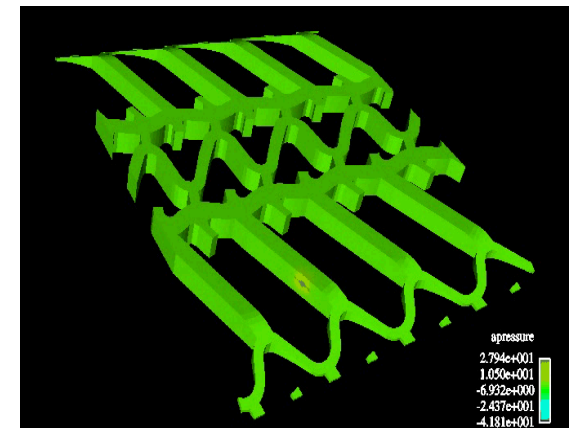
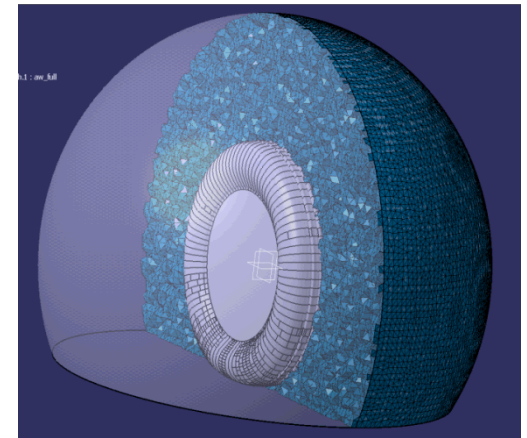
$0.04(s) < t < 0.06(s)$

Structural Acoustics in Sierra-SD

Use Case: Tire Noise Modeling with Sierra Mechanics

Large-scale computational approach for tire noise modeling

- Infinite elements
- Exterior meshing around tire surface
- Far-field acoustic calculations





Conclusions

- **Massively parallel finite element structural acoustics and optimization codes have been loosely coupled for the solution of source and material inversion problems.**
- **Adjoint methods have been implemented in Sierra-SD in both time and frequency domains.**
- **Applicable to large-scale models with many degrees of freedom.**
- **The method allows flexibility to work with both time and frequency domain, and nonlinear problems.**
- **Method has been applied to solve source and material inversion on problems of interest.**