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Symmetric Linear Programming Formulations for Minimum Cut with Applications to TSP

Robert D. Carr* Benjamin Moseley†

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Abstract

In this paper we consider new linear programming relaxations for the minimum cut problem. We introduce a new LP formulation, which is the smallest known formulation for minimum cut which has symmetricity. We show that this formulation for minimum cut can either be used to give a stronger LP relaxation for the traveling salesman problem than the subtour relaxation or gives the optimal value for the minimum cut problem.

1 Introduction

The minimum cut problem is one of the most fundamental problems in computer science and has numerous applications in other fields. Consequently, there has been a vast amount of research on the topic. See [6] for an overview. In this problem you are given a graph G on n nodes and a cost function $c : V \times V$ denoting the capacity of the edges in the graph. In the global cut problem the goal is to find a non-empty set $S \subset V$ such that the sum of the edge weights crossing the cut $(S, V \setminus S)$ is minimized. In the s,t cut problem it must be the case that $s \in S$ and $t \in V \setminus S$ where $s,t \in V$.

It is well known that the minimum cut problem yields polynomial time algorithms and a variety of efficient algorithms are known [6, 7] as well as efficient parallel algorithms [8, 9]. One widely used technique is mathematical programming. A variety of linear programs are known for the minimum cut problem [6, 4] and several are known to give exact solutions to the problem. The smallest known linear program relaxation, which we call the w-LP, was given in [4]. This formulation has $O(n^2)$ variables and $O(n^3)$ constraints. From a theoretical viewpoint, it is interesting to determine the bounds of the smallest linear program for a given problem. Practically, small linear programs can lead to more efficient algorithms in practice. For the minimum cut problem, finding a smaller relaxation not only can improve the performance for finding the minimum cut of a graph, but also minimum cut is a subroutine for other fundamental problems such as the Traveling Salesman problem (TSP) [4]. In the TSP problem, the input is the same as in minimum cut, but the objective is to find a tour of the graph of minimum cost.

Currently, it is unknown whether or not all problems in P admit a polynomial sized linear program. However, there is evidence that this may not be the case. Currently, all known linear programming formulations for weighted perfect matching have exponential size, it was shown by

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†Toyota Technological Institute at Chicago

Yannakakis in [12] that no symmetric LP can have polynomial size for this problem. An LP is said to be symmetric if the variables of the LP can be permuted and not change the feasible region. Further, it is known that certain classes of LP's can be lifted to give symmetry [12]. This gives evidence that the perfect matching problem may not have a polynomial size LP, but it is known to be in P.

Results: In this work, we consider linear program relaxations for the global minimum cut problem. We begin by giving a symmetric relaxation assuming one knows $\alpha := |S|$, the size of one side of the cut. This relaxation is then shown to give the optimal solution on graphs which consist of a Hamiltonian cycle for any fixed $\alpha > 0$. Using this property, we can show that either this linear program is exact or using a lift and project method we can show that this LP can be used as a subroutine for the TSP problem to give the tightest known relaxation to TSP which does not use cutting planes. In the case where the LP is exact, this would give the smallest known LP for minimum cut which has symmetricity. The LP has $O(n^3)$ variables and $O(n^4)$ constraints. This compares to the previous best known relaxation with $O(n^4)$ variables and $O(n^4)$ constraints which can be obtains by extending the standard formulation of s, t minimum cut. Beyond being interesting from a theoretical viewpoint, symmetric LPs are also typically easier to analyze than other LPs.

Next we extend this relaxation to give another relaxation which we can show is $(1-\epsilon)$ approximate for the minimum cut problem while having $O(n^2 \log_\epsilon n)$ variables and $O(n^3 \log_\epsilon n)$ constraints for any $\epsilon > 0$. Although we can show that this LP is not exact for the minimum cut problem one can use this relaxation to improve the solution for the TSP problem, while yielding a significantly smaller LP.

2 Preliminaries

We are going to construct symmetricity LPs that solve or approximate minimum (global) cut. A cut in an undirected graph $G = (V, E(V))$ partitions the vertex set V of n vertices into a set of top nodes and a set of bottom nodes, with the edges in the cut going between these two sets. Given a vector $c_{ij} \geq 0$ of capacities and edge variables x_{ij} indicating whether edge $\{i, j\}$ is in a cut for $\{i, j\} \in E(V)$ we want to minimize $c \cdot x$. Minimum s-t cut where s is specified to be on the bottom and t is specified to be on the top is the most usual type of minimum cut problem (without such a specified pair of nodes it is the minimum global cut problem). Let node variables h_i indicate whether i is on top or on bottom in a cut. Minimum s-t cut has an easy (maturally integer) LP formulation:

$$\begin{array}{lll} \text{minimize} & c \cdot x & \\ \text{subject to} & & \\ & x_{ij} \geq h_j - h_i & \forall ij \in E(V) \\ & x_{ij} \geq h_i - h_j & \forall ij \in E(V) \\ & 0 = h_s \leq h_i \leq h_t = 1 & \forall i \in V. \end{array}$$

One can also formulate the NP-hard maximum cut problem. A naive IP formulation and LP relaxation is

$$\begin{array}{lll} \text{maximize} & c \cdot x & \\ \text{subject to} & & \\ & x_{ij} \leq h_j + h_i & \forall ij \in E(V) \\ & x_{ij} \leq 2 - h_j - h_i & \forall ij \in E(V) \\ & 0 \leq h_i \leq 1,\ h \text{ integer} & \forall i \in V. \end{array}$$

However, setting $h = 1/2$ shows that this is a weak LP relaxation.

Supposing one constrained minimum cut so that there were exactly $\alpha \leq n/2$ nodes on top. Call this the minimum α-cut problem, which is also an NP-hard problem. We will now construct an LP relaxation for this problem. For $i \in V$ consider $x(\delta(i))$. Suppose $h_i = 0$. Then there are α nodes opposite i, so $x(\delta(i)) = \alpha$. Now, suppose $h_i = 1$. Then there are $n - \alpha$ nodes opposite i, so $x(\delta(i)) = n - \alpha$. Hence, it is always the case that

$$x(\delta(i)) = (n - 2\alpha)h_i + \alpha.$$

Also, for distinct $i, j, k \in V$ we have the (metric) triangle inequality property that

$$x_{ij} \leq x_{ik} + x_{kj}.$$

Here is our first LP relaxation:

$$\begin{aligned}
&\text{minimize} c \cdot x \\
&\text{subject to} \\
&\sum_{i=1}^{n} h_i &&= \alpha && && (1)\\
&x(\delta(i)) &&= (n-2\alpha)h_i + \alpha && \forall i \in V && (2)\\
&x_{ij} &&\leq h_i + h_j && \forall \{i,j\} \in E && (3)\\
&x_{ij} &&\leq x_{k,i} + x_{k,j} && \text{distinct } i,j,k \in V && (4)\\
&1 \geq \quad x, h &&\geq 0. && && (5)
\end{aligned}$$

Denote this α-LP relaxation's feasible region by

$$A^\alpha z^\alpha \geq b^\alpha, z^\alpha \geq 0,$$

where $z^\alpha = [x, h]$. Let c^H be the incidence vector of an arbitrary Hamilton cycle H. In the next section we will bound the objective value of this LP. Before we do that, we introduce the w-LP of [4] which our LP can be compared to. In this linear program $w_k = 1$ when k is the last node (numerically) on top in the cut.

$$\begin{array}{lrcll}
\text{minimize} & c \cdot x & & & \\
\text{subject to} & & & & \\
 & x_{ij} + 2w_k & \leq & x_{ki} + x_{kj} & \forall k < i < j \\
 & x_{ki} & \geq & w_k & \forall k < i \\
 & \sum_{k=1}^{n-1} w_k & = & 1, & \\
 & 1 \geq x, w & \geq & 0. &
\end{array}$$

3 Bounding the objective of α-LP on Hamiltonian Cycles

In this section we show when the input graph is a single cycle on all of the nodes of the graph and all edges of the graph have unit capacity that the α-LP gives an exact solution. Further we give bounds on the solution to the α-LP when the graph consists of a Hamiltonian cycle and a set of β singleton nodes which are adjacent to all 0 capacity edges. For the remainder of this section, we assume that we are given a graph G on n nodes indexed $v_1, v_2, \ldots, v_n$. Further this graph consists of β which have all 0 capacity edges adjacent to them indexed $v_{n-\beta+1}, \ldots v_n$ and the remaining $n - \beta$ nodes forming a Hamiltonian cycle. For the nodes on the Hamiltonian cycle, $v_1, \ldots, v_{n-\beta}$, there is an edge of capacity 1 between nodes v_i and v_{i+1} as week as $v_{n-\beta}$ and v_1. All other edges adjacent to these nodes have capacity 0.

Consider any optimal solution to the α-LP x^*, h^*. We are now going to transform this solution into another feasible α-LP solution. Let $h'_i = \frac{1}{n-\beta}\sum_{j\leq n-\beta} h_j$ for all $i \leq n-\beta$. We will define α' to be $\sum_{j\leq n-\beta} h_j$. Let C_k denote the total of the x variables between nodes distance k apart on the Hamiltonian cycle. We set x' for two nodes distance k apart on the Hamiltonian cycle to be $\frac{C_k}{n-\beta}$. Let $x_{i,k} = \frac{1}{(n-\beta)}\sum_{j=1}^{n-\beta} x_{j,k}$ for $i \leq n-\beta$ and $k \geq n-\beta+1$. The remaining variables stay the same.

Lemma 3.1. *The solution x', h' is a valid solution to the LP.*

Proof. Consider the vertices on the Hamiltonian Cycle. Let solution x', h' distributes the values of the variables on these nodes evenly. This can be done because of the symmetry in the LP. Indeed, if one were to reindex the nodes on the cycles such that i is $i+1$ for all $i \leq n-\beta-1$ and n with 1 then this would be a valid solution to the LP. The solution x' is the average over all $n-\beta$ reindexing of the nodes on the Hamiltonian cycle. □

Now our goal is give a bound on $x'_{i,i+1}$ for all $i \leq n-\beta-1$. This will give us a bound on the objective of the LP. Notice that by the definition of x' it must be the case that $x_{i,i+1} = x_{i+1,i+2}$ for all $i \leq n-\beta-1$ and therefore it suffices to bound $x_{1,2}$. Let $\epsilon = x_{1,2}$. We begin by showing an upper bound on the amount the edges adjacent to v_1 can be cut.

Lemma 3.2. $\sum_{j=2}^{n} x_{1,j} \leq \alpha' + \alpha - \frac{4\alpha'^2}{(n-\beta)^2\epsilon} + \frac{\beta\alpha'}{n-\beta}$

Proof. To show the lemma, we will first show that $x_{1,j} \leq (j-1)\epsilon$ for all $j \leq n-\beta$. We will prove this by induction. For the base case consider when $j = 2$. We have that $x_{1,2} = \epsilon$ by definition. Now consider an arbitrary $2 < j \leq n-\beta$. By the inductive hypothesis we have that $x_{1,j-1} = (j-2)\epsilon$. Further $x_{j-1,j} = \epsilon$ by definition. Thus, by constraint 2 in the LP we have that $x_{1,j} \leq x_{1,j-1} + x_{j-1,j} = j\epsilon + \epsilon = (j-1)\epsilon$. Thus, $x_{1,j} \leq (j-1)\epsilon$. Also notice that since the Hamiltonian cycle and the solution x', h' are symmetric it is the case that $x_{1,n-j+1} \leq (j-1)\epsilon$. Using this we can show the following:

$$\begin{aligned}
\sum_{j=2}^{\frac{2\alpha'}{(n-\beta)\epsilon}+1} x_{i,j} + \sum_{j=n-\beta-\frac{2\alpha'}{(n-\beta)\epsilon}+1}^{n-\beta} x_{i,j} &\leq 2\epsilon \sum_{j=1}^{\frac{2\alpha'}{(n-\beta)\epsilon}} j \quad [x_{1,j} \leq (j-1)\epsilon \text{ and } x_{1,n-j+1} \leq (j-1)\epsilon] \\
&= 2\epsilon\left(\frac{\frac{2\alpha'}{(n-\beta)\epsilon}\left(\frac{2\alpha'}{(n-\beta)\epsilon}+1\right)}{2}\right) \\
&= \frac{2\alpha'}{(n-\beta)}\left(\frac{2\alpha'}{(n-\beta)\epsilon}+1\right) \\
&= \frac{4\alpha'^2}{(n-\beta)^2\epsilon} + \frac{2\alpha'}{(n-\beta)}
\end{aligned}$$

Now notice that $x_{1,j} \leq \frac{2\alpha'}{n-\beta}$ for all $j \leq n-\beta$. This is because h_j and h_i are equal to $\alpha'/(n-\beta)$ and we can apply constraint 3. Using this we can show the following:

$$\begin{aligned}
\sum_{j=\frac{2\alpha'}{(n-\beta)\epsilon}+1}^{n-\beta-\frac{2\alpha'}{(n-\beta)\epsilon}} x_{i,j} &\le \sum_{j=\frac{2\alpha'}{(n-\beta)\epsilon}+1}^{n-\beta-\frac{2\alpha'}{(n-\beta)\epsilon}} \frac{2\alpha'}{n-\beta} \\
&= (n-\beta-\frac{4\alpha'}{(n-\beta)\epsilon}-1)\frac{2\alpha'}{n-\beta} \\
&= 2\alpha' - \frac{8\alpha'^2}{(n-\beta)^2\epsilon} - \frac{2\alpha'}{n-\beta}
\end{aligned}$$

Now we can bound the total amount the edges between v_1 and the β singleton nodes are cut.

$$\begin{aligned}
\sum_{j=n-\beta+1}^{n} x_{1,j} &\le \sum_{j=n-\beta+1}^{n} h_1 + h_j \quad \text{[Constraint 3]} \\
&= \frac{\beta\alpha'}{n-\beta} + \sum_{j=n-\beta+1}^{n} h_j \\
&= \frac{\beta\alpha'}{n-\beta} + (\alpha - \alpha') \quad \text{[Definition of } \alpha'\text{]}
\end{aligned}$$

Combining these three terms we have the lemma.

□

Now we are ready to show a lower bound the on how much the edges adjacent to node v_1 are cut.

Lemma 3.3. $\sum_{j=2}^{n} x_{1,j} \ge \alpha + \frac{n\alpha'}{n-\beta} - \frac{2\alpha\alpha'}{n-\beta}$

Proof. By considering constraint 2 in the LP we see that

$$\begin{aligned}
\sum_{v_j \in V, j\neq i} x_{1,j} &= (n-2\alpha)h'_1 + \alpha = \sum_{v_j \in V, j\neq i} x_{i,j} = (n-2\alpha)\frac{\alpha'}{n-\beta} + \alpha \quad [h'_1 = \tfrac{\alpha'}{n-\beta}] \\
&= \frac{n\alpha'}{n-\beta} + \alpha - \frac{2\alpha\alpha'}{n-\beta}
\end{aligned}$$

□

By combining the previous lemmas, we have the following.

Lemma 3.4. $\frac{2\alpha'}{(n-\beta)\alpha} \le \epsilon$

Proof. From Lemmas 3.3 and 3.2 we have that,

$$\alpha + \frac{n\alpha'}{n-\beta} - \frac{2\alpha\alpha'}{n-\beta} \le \alpha' + \alpha - \frac{4\alpha'^2}{(n-\beta)^2\epsilon} + \frac{\beta\alpha'}{n-\beta}$$

This implies that,

$$\begin{aligned}\frac{4\alpha'^2}{(n-\beta)\epsilon} &\leq -n\alpha' + 2\alpha\alpha' + (n-\beta)\alpha' + \beta\alpha' \\ &= 2\alpha\alpha'\end{aligned}$$

The lemma follows from the above and some algebra.

□

This gives rise to the following.

Theorem 3.5. *The objective value of the LP is at least* $\frac{2\alpha'}{\alpha}$.

4 Applications to TSP

A consequence of Theorem 3.5 is that for any α the α-LP achieves the optimal solution on the Hamiltonian cycle when $\beta = 0$.

Corollary 4.1. *The minimum of* $c^H \cdot x^\alpha$ *over the* α*-LP relaxation is* 2 *when* $\beta = 0$.

This leads to an interesting relaxation of the TSP using the trick relating compact separation to compact optimization, [11, 5]. Let c^{obj} be the objective function and t be the TSP edge variables. Recall the α-LP's objective $c \cdot x$. Consider the dual of the α-LP with $c := t$ for the columns in A corresponding to the x variables and $t := 0$ otherwise, which is

$$\begin{array}{ll}\text{maximize} & y^\alpha \cdot b^\alpha \\ \text{subject to} & \\ & y^\alpha \cdot A^\alpha \leq t \\ & y^\alpha \geq 0.\end{array}$$

Note that c^{obj} and t are set to 0 on the columns corresponding to the h variables. Then the α-TSP relaxation is

$$\begin{array}{ll}\text{minimize} & c^{obj} \cdot t \\ \text{subject to} & \\ & y^\alpha \cdot A^\alpha \leq t \\ & y^\alpha \cdot b^\alpha \geq 2 \\ & y^\alpha \geq 0.\end{array}$$

How this compares to the subtour relaxation of the TSP is of interest. Compact subtour relaxations can be found in [1, 3]. We have the following surprise:

Theorem 4.2. *If the* α*-LP gives an answer of strictly less than* 2 *for a subtour extreme point* c^**, then* α*-TSP can be made stronger than the subtour relaxation! (after adding a compact subtour relaxation to it).*

Proof: One can see that $t := c^*$ and $t \cdot x^* < 2$ for a feasible α-LP solution x^* is contradicted by weak duality and $y^\alpha \cdot b^\alpha \geq 2$. Therefore, c^* is an infeasible value for t. □

The companion theorem that uses Lavasz splitting [10] is:

Theorem 4.3. *If the* α*-LP gives an answer of strictly less than the minimum cut for a cost function* c'*, then* α*-LP gives an answer of strictly less than* 2 *for a subtour extreme point* c^*.

Proof: Normalize c' so that the minimum cut is 2. Perform Lovasz splitting off getting intermediate vectors c'' and ending with c^* that has fractional degree 2 at every node. Recall that Lovasz splitting off keeps the minimum global cut the same, so c^* is a feasible subtour point. Due to the triangle inequalities, at each step $c'' \cdot x^* < 2$ because $c' \cdot x^* < 2$ and the objective decreases at each step. In the end $c^* \cdot x^* < 2$, and without loss of generality one can choose c^* to be a subtour extreme point. □

Combining these two theorems gives us the following corollary:

Corollary 4.4. *Either α-LP always gives an answer of at least the minimum global cut or we have produced, using α-TSP, a compact relaxation stronger than the subtour relaxation.*

5 Removing Knowledge of α

Recall that $z^\alpha = [x^\alpha, h^\alpha]$ in α-LP. We can now use the ideas of lift-and-project [2] to obtain our first minimum global cut LP formulation that has symmetricity:

$$\begin{array}{ll} \text{minimize} & c \cdot x \\ \text{subject to} & \\ & A^\alpha z^\alpha \geq b^\alpha \lambda_\alpha \\ & z^\alpha \geq 0 \\ & z = \sum_\alpha z^\alpha \\ & \sum_\alpha \lambda_\alpha = 1, \ \lambda \geq 0. \end{array}$$

This $n^4 \times n^3$ LP is 2 orders of magnitude bigger than our SODA minimum cut LP ($n^3 \times n^2$). We can however shed almost all of this extra size if we sacrifice exactness by a small amount. To make things easy, assume the number of vertices in the mincut instance is m that is a power of 2. Let β be a power of 2. In a single LP we will account for all possible α' nodes on top between β and 2β. In order to do this, we create β dummy nodes, so $n := m + \beta$. We then define $\alpha := 2\beta$. Then solve α-LP with these values. If all of the dummy nodes end up on top, then $\alpha' = \beta$. If none of the dummy nodes end up on top, then $\alpha' = 2\beta$. If only some of the dummy nodes end up on top, then α' is between β and 2β. Call this new LP the β-LP. The feasible region of the β-LP is denoted by

$$A^\beta z^\beta \geq b^\beta, \ z^\beta \geq 0,$$

where $z^\beta = [x^\beta, h^\beta]$. Consider a Hamiltonian objective function c^H on nodes $1, \ldots, m$. We have this interesting result as a consequence of Theorem 3.5.

Corollary 5.1. *The Hamiltonian objective function is at least 1 for β-LP when $\alpha = 2\beta$*

Define β-TSP analogously to how α-TSP was defined before. Just like before we have the following results.

Theorem 5.2. *If the β-LP with $\alpha = 2\beta$ gives an answer of strictly less than 1 for a subtour extreme point c^*, then β-TSP can be made stronger than the subtour relaxation! (after adding a compact subtour relaxation to it).*

Theorem 5.3. *If the β-LP gives an answer of strictly less than $1/2$ times the minimum cut for a cost function c', then β-LP gives an answer of strictly less than 1 for a subtour extreme point c^*.*

Corollary 5.4. *Either β-LP always gives an answer of at least $1/2$ times the minimum global cut or we have produced, using β-TSP, a compact relaxation stronger than the subtour relaxation.*

Using lift-and-project ideas for the $\log_\epsilon m$ choices of β, $(1+\epsilon)^i$ for i from 1 to $\log_\epsilon m$, we obtain an approximate mincut LP with symmetricity:

$$
\begin{array}{lll}
\text{minimize} & c \cdot x & \\
\text{subject to} & & \\
& A^\beta z^\beta \geq b^\beta \lambda_\beta & \\
& z^\beta \geq 0 & \\
& z = \sum_\beta z^\beta & \text{only when indices are in } \{1, \ldots, m\} \\
& \sum_\beta \lambda_\beta = 1,\ \lambda \geq 0. &
\end{array}
$$

We note that Theorem 3.5 gives the following.

Corollary 5.5. *The β-LP with $\log_\epsilon m$ choices for β has an objective of at least $(1-\epsilon)$ times the minimum cut.*

6 Doing away with Symmetricity

Supposing one does not care about the symmetricity of our main mincut LP of this paper that yields a minimum global cut relaxation. Then our smallest LP formulation of minimum global cut from 2007 is still the champion in many respects. But, consider the main mincut LPs of this paper for the range of α' (the number of vertices in the smaller of the two partitions of the cut) going say from β to 2β for a fixed choice of β. Now, our smallest LP formulation of minimum global cut and these symmetricity LPs can be combined into single LPs with virtually the same number of constraints as these symmetricity LPs have. Our capacitated graph that we are taking the minimum global cut of is on a vertex set of nodes 1 to m. We tack on $\beta \leq m/4$ dummy nodes to this graph, so the number of nodes in the resulting graph is $n := m + \beta$. Then the number of nodes on top is $\alpha := 2\beta \leq n/2$. The resulting combined LPs for fixed βs are

$$
\begin{array}{lrcll}
\text{minimize} & & c \cdot x & & \\
\text{subject to} & & & & \\
& \sum_{i=1}^n h_i & = & \alpha & \\
& x(\delta(i)) & = & (n-2\alpha)h_i + \alpha & i = 1, \ldots, n \\
& x_{ij} & \leq & h_i + h_j & \{i,j\} \in E \\
& x_{ij} & \leq & x_{k,i} + x_{k,j} & \text{distinct } i,j,k \in V \\
& & & & k > \min(i,j) \text{ or } \max(i,j) > m \\
& 2w_k + x_{ij} & \leq & x_{k,i} + x_{k,j} & \text{distinct } i,j,k \in V \\
& & & & k < \min(i,j) \text{ and } \max(i,j) \leq m \\
& x_{km} & \geq & w_k & k = 1, \ldots, m-1 \\
& \sum_{k=1}^{m-1} w_k & = & 1 & \\
& 1 \geq x, h, w & \geq & 0. &
\end{array}
$$

Note that this truly is a combination of our two LPs (β fixed) since the constraints $x_{ki} \geq w_k$ for all $k < i$ can be derived by adding $2w_k + x_{in} \leq x_{\{k,i\}} + x_{\{k,n\}}$ and $x_{k,n} \leq x_{\{in\}} + x_{\{k,i\}}$.

One may want all of the triangle inequalities in the above formulations anyways to reach a more natural metric constraint on the x variables. We put the above LPs for fixed choices of β (say the powers of 2 that are at most $m/2-1$) together. The formulations above are of the form $A^\beta z^\beta \geq b^\beta$

where $z = [x, w, h]$. The final combined LP is then

$$\begin{array}{lll} \text{minimize} & c \cdot x & \\ \text{subject to} & & \\ & A^\beta z^\beta \geq b^\beta \lambda_\beta & \\ & z = \sum_\beta z^\beta & \text{only when indices are in } \{1, \ldots, m\} \\ & \sum_\beta \lambda_\beta = 1,\ \lambda \geq 0. & \end{array}$$

As before, lift-and-project ideas are used here.

Consider a Hamilton cycle H and its incidence vector c^H. From the w-LP, we can derive $c^H x \geq 2$ by dual arguments. But we can also derive almost that $c^H x^\beta \geq 2$ (at least 1 but LP runs suggest 1.777) by completely different dual arguments that involve inequalties from the maxcut side of the fixed β polytopes! Having both of these dual arguments for this class $c^H x \geq 2$ (or 1 or more) of inequalties on the mincut side of the combined polytope is a difficult to quantify but interesting form of tightness of the formulation.

Take the easily understood objective function $x(E(\{1, \ldots, m\}))$ from the maxcut side. The combined LP gives the correct answer $m^2/4$ for this LP, whereas the w-LP gives the obvious $m^2/2$ ($x = 1$ on all edges). Clearly the combined LP projected into the x variable space is a strictly tighter formulation than the LPs that it came from.

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