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Two Domain Decomposition Algorithms for Problems in H(curl)

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Joint work with
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Thanks

- Scientific Committee
- Conference Organizers

OUTLINE

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  - Coarse Space
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  - Examples

OUTLINE

- **3D Algorithm**
  - Face Decomposition Lemma
  - Coarse Spaces
  - Local Spaces & Hybrid Method
  - Theory Overview
  - Examples

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Introduction

Problem Statement:

Find $\boldsymbol{u} \in H_0(\text{curl};\Omega)$ such that

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_\Omega \quad \forall \boldsymbol{v} \in H_0(\text{curl},\Omega),$$

where

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega [(A\nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}) + (B\boldsymbol{u} \cdot \boldsymbol{v})]\, dx,$$

$$(\boldsymbol{f},\boldsymbol{v})_\Omega := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

Goal: DD algorithms for edge element approximations

One Challenge: large near null space as B → 0

Introduction

Some Applications:

- **Implicit time integration of eddy current model of Maxwell's equations**
- **Solution of linear, magnetostatics problems ($B \to 0$)**
- **Solution of electromagnetic eigenvalue problems for cavities and waveguides (related saddle-point systems)**

Introduction

Brief Review:

- **Key Points of Following Slides:**
  - No comprehensive theory with favorable bounds is available in either 2D or 3D
  - Overlapping Schwarz methods presuppose a coarse mesh, while Neumann-Neumann or FETI methods require subdomain matrices
  - Choices very limited in 3D if no coarse mesh available

Introduction

Brief Review:

- **Overlapping Schwarz (3D):**
  - **Toselli, *Numer. Math*. (2000) 86:733-752**
    - o **Coarse space from coarse finite elements**
    - o **Quasi-uniform coarse triangulation**
    - o **Convex domains**
    - o **Constant material properties**
  - **Pasciak & Zhao, *J. Numer. Math*. (2002) 10:221-234**
    - o **Coarse space from coarse finite elements**
    - o **Non-convex domains**
    - o **Constant material properties**

Introduction

Brief Review:

- **Iterative Substructuring (2D):**
  - **Toselli, Widlund, Wohlmuth, *Math. Comp.* (2000) 70:935-949**
    - **Coarse space from coarse finite elements**
    - **Material property jumps allowed between subdomains**

$$\kappa(M^{-1}A) \leq C\eta(1 + \log(H/h))^2$$

$$\eta = \max_i(1 + H_i^2 \beta_i/\alpha_i)$$

$$a_{\Omega_i}(\boldsymbol{u}, \boldsymbol{u}) = \int_{\Omega_i} [\alpha_i(\nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}) + \beta_i \boldsymbol{u} \cdot \boldsymbol{v}]\, dx$$

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Introduction

Brief Review:

- **Neumann-Neumann (2D):**
  - **Toselli, *ETNA* (2000) 11:1-24**
- **FETI (2D):**
  - **Toselli and Klawonn, *SINUM* (2001) 39:932-956**
- **FETI-DP (2D):**
  - **Toselli and Vasseur, SINUM (2005) 42:2590-2611**

– **Theory for all three allows material property jumps between subdomains. Need subdomain matrices.**

$$\kappa(M^{-1}A) \leq C\eta(1 + \log(H/h))^2$$

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Introduction

Brief Review:

- **Iterative Substructuring (3D):**
  - **Hu and Zou, *SINUM* (2003) 41:1682-1708**
    - **Tetrahedral subdomains**
    - **Very generous coarse space**
      - **All edges of Γ incident to a wire basket node**
    - **Could not conclude if condition number estimate is independent of property jumps between subdomains**

$$\kappa(M^{-1}A) \leq C(1 + \log(H/h))^3$$

Introduction

Brief Review:

- **FETI-DP (3D):**
  - **Toselli, *IMA J. Numer. Anal.* (2003) 41:1682-1708**
    - **Requires change of basis along subdomain edges**
    - **Theory assumes either all α_i or all β_i the same for each subdomain**
    - **2 coarse dofs per subdomain needed to obtain good performance in theory and practice**

$$\kappa(M^{-1}A) \leq C\eta(1 + \log(H/h))^4$$

Introduction

Brief Review:

- **Geometric Multigrid (3D):**
  - **Hiptmair, *SINUM* (1998) 36:204-225**
  - **Arnold, Falk, Winther, *Numer. Math*. (2000) 85:197-217**
    - **Convex polyhedron**
    - **Quasi-uniform meshes**
    - **Constant material properties**
- **Algebraic Multigrid**
  - **Auxiliary Space Preconditioners (Hiptmair, Xu, Beck, Widmer, Zou, LLNL group, …)**
  - **Smoothed Aggregation (Sandia group, …)**

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Introduction

Some Observations:

- **Overlapping Schwarz algorithms presuppose a coarse mesh and theory is restricted to constant material properties**
- **FETI-like methods require subdomain matrices**
- **Choices are very limited in 3D if coarse mesh not available**
- **No comprehensive theory with favorable condition number bounds appears available in 2D or 3D**
- **Current theory restricted to regular-shaped subdomains**

Introduction

Some Goals:

- Develop new coarse spaces for H(curl) problems
  - Algebraic approach that does not require geometric information or subdomain matrices
  - Based on energy minimization
  - Automatic generation of coarse spaces for either iterative substructuring, overlapping Schwarz, or hybrid combination
- Extend theory
  - More favorable bounds (address the η issue)
  - Accommodate irregular-shaped subdomains

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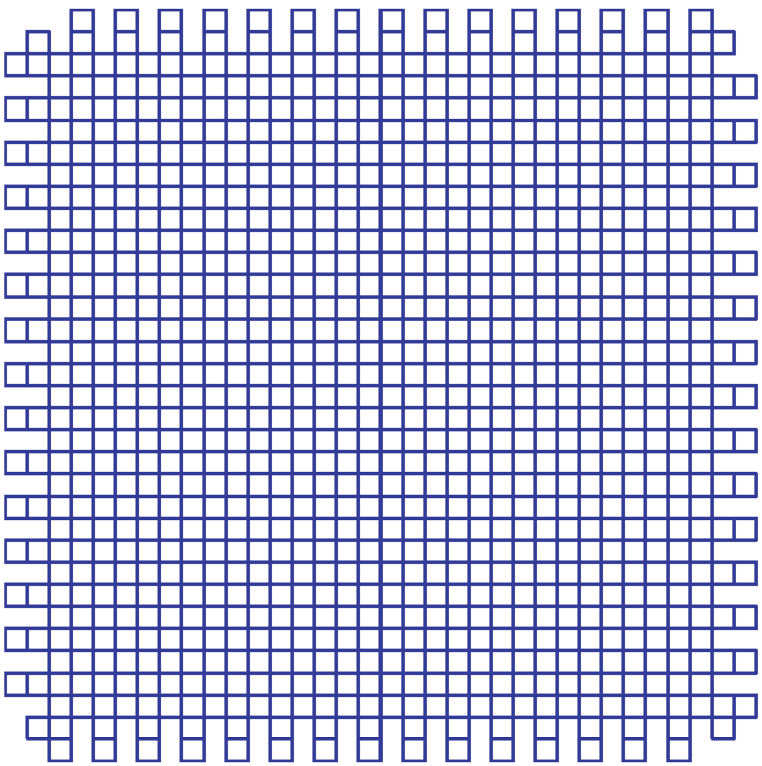
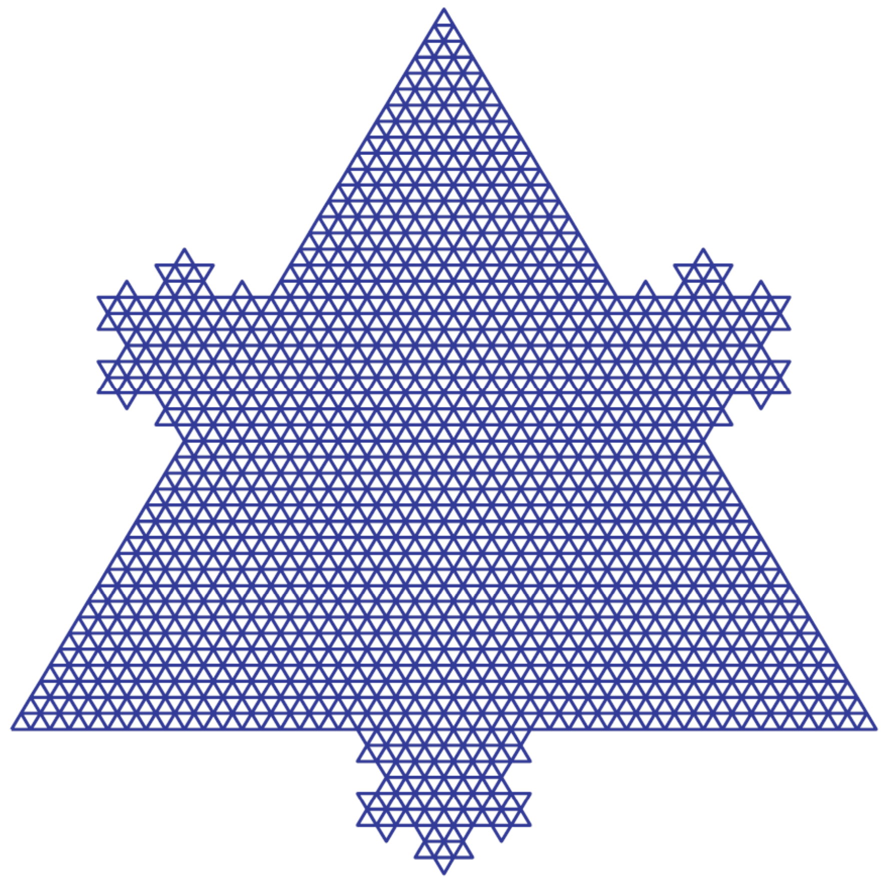
2D Algorithm

Uniform Domains: some examples w/ nice C_U



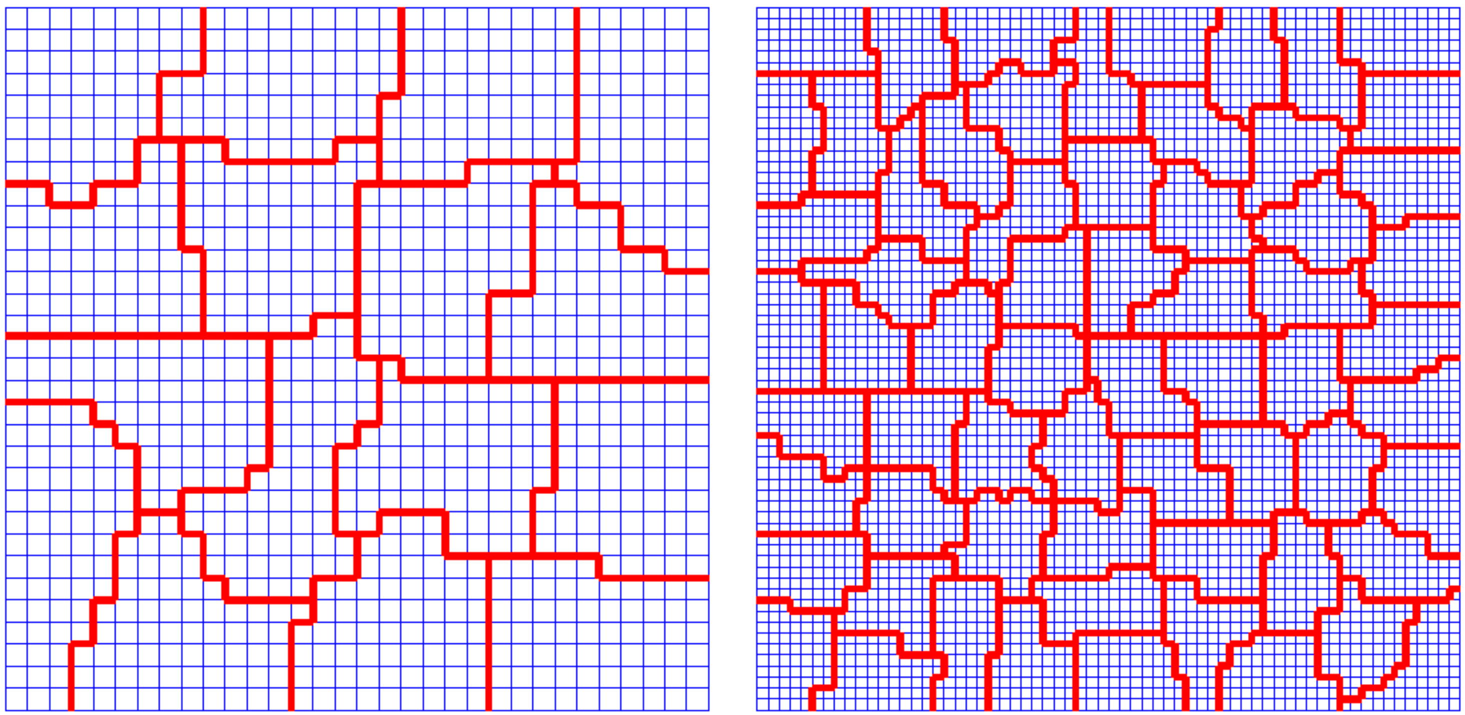
Type 2 subdomain



Type 3 subdomain

2D Algorithm

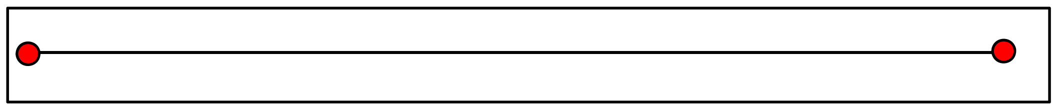
Uniform Domains: some examples w/ nice C_U

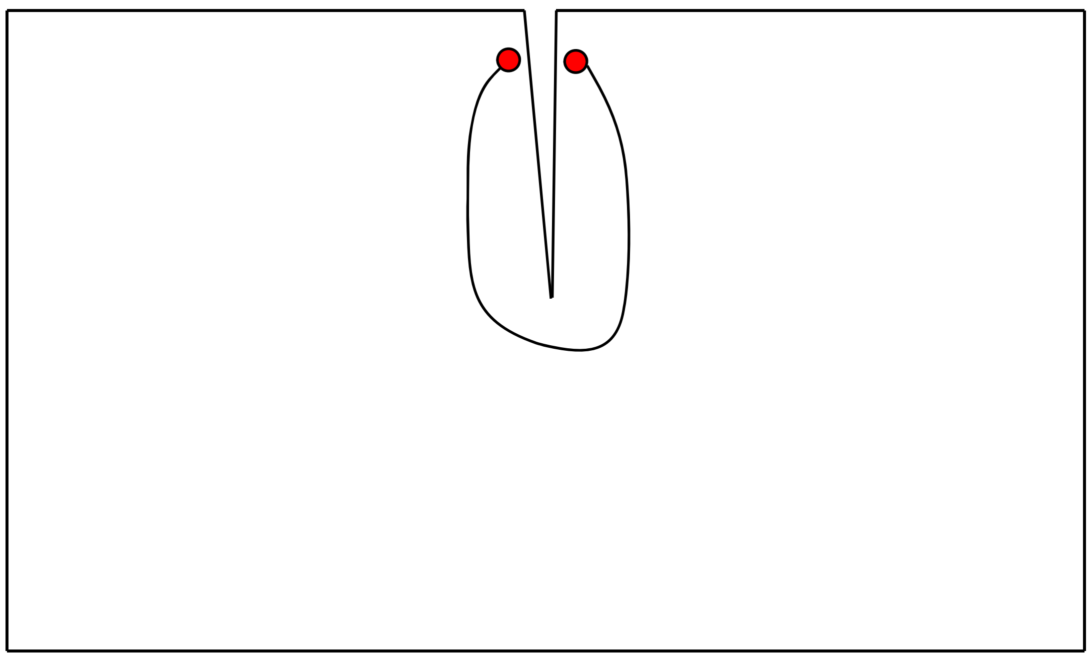


From graph-based mesh partitioner (Metis)

2D Algorithm

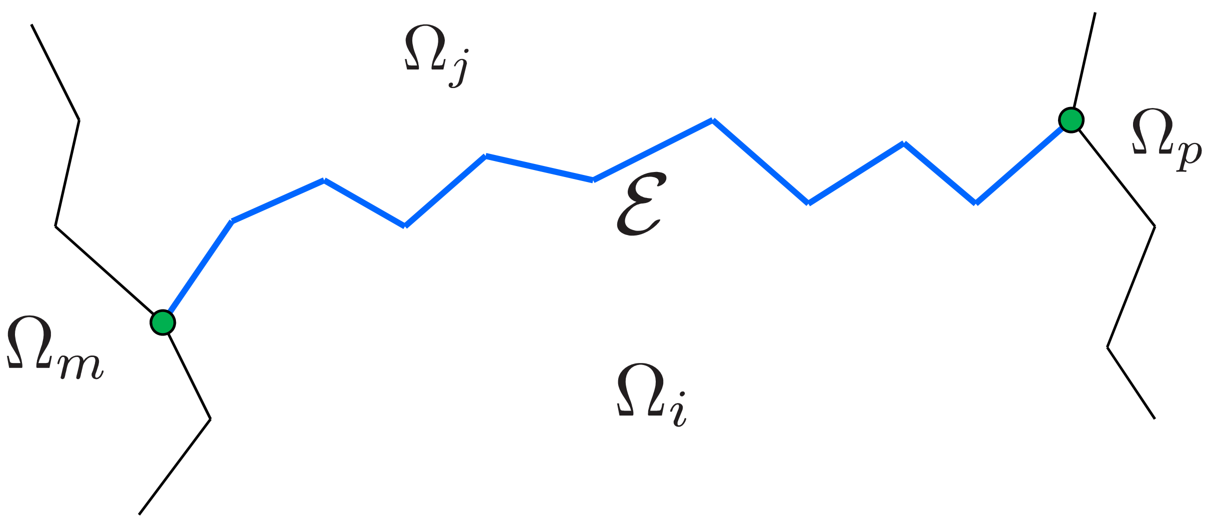
Uniform Domains: some examples with large C_U





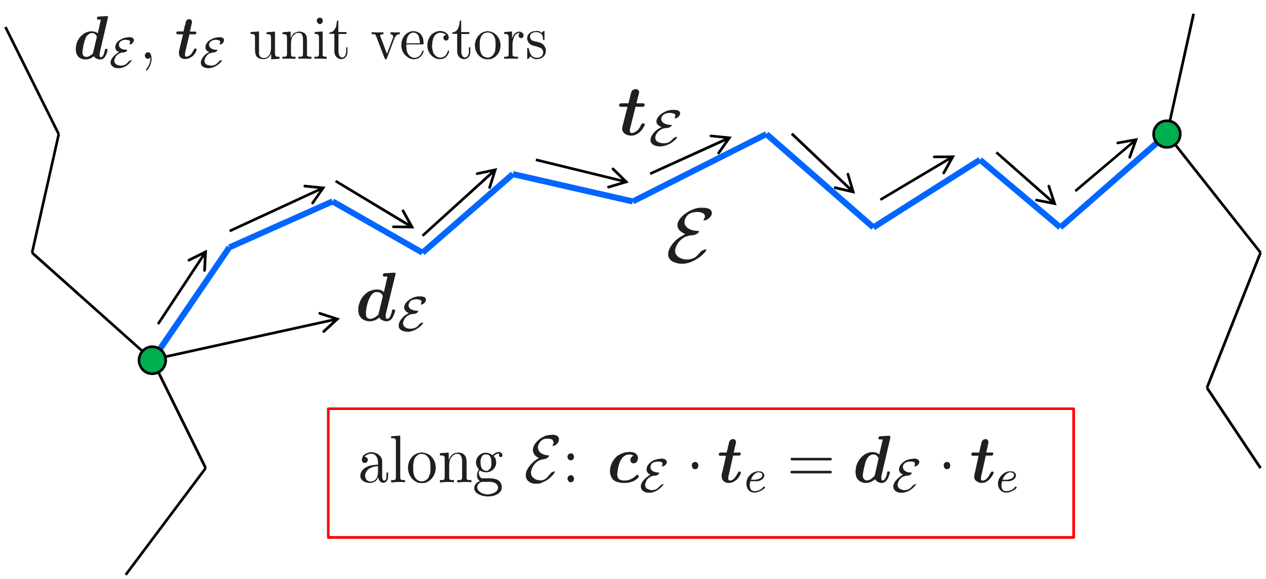
2D Algorithm

- **Coarse Space**
  - **One coarse basis function $c_{\mathcal{E}}$ for each $\mathcal{E}$**



2D Algorithm

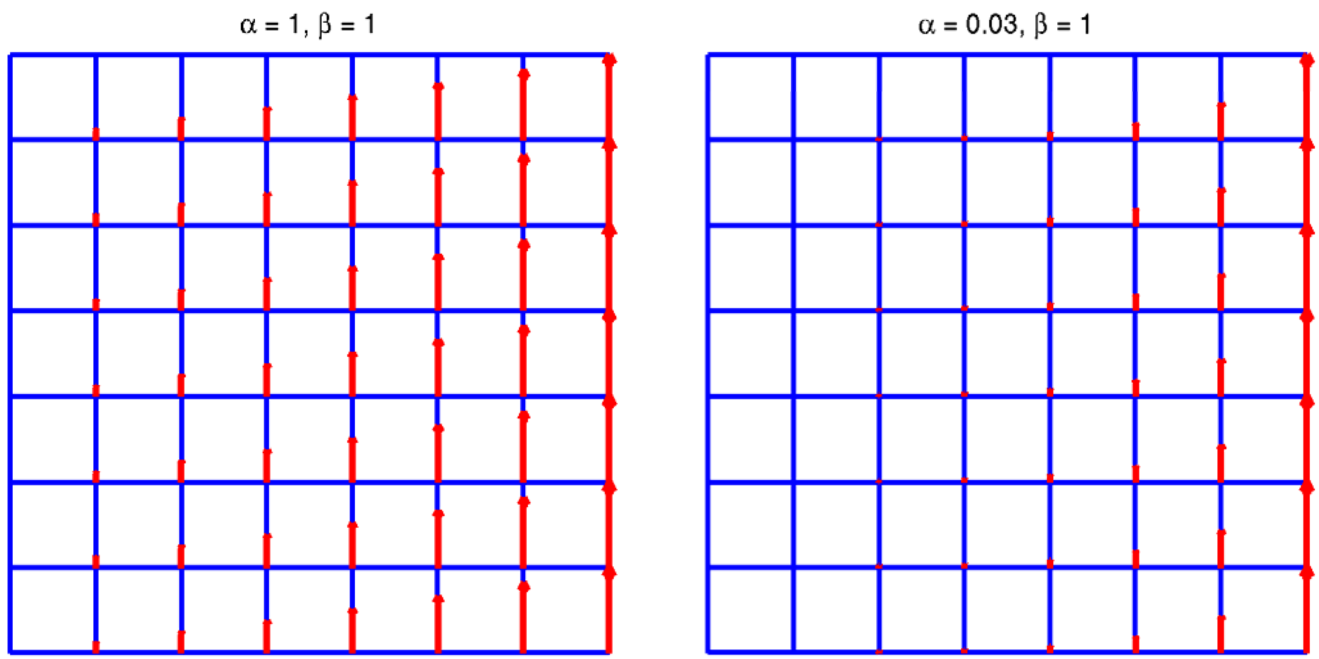
- **Coarse Space**
  - **Coefficients of $c_{\mathcal{E}}$ nonzero only along $\mathcal{E}$ and interiors of two neighboring subdomains**



2D Algorithm

- **Coarse Space**
  - **Coefficients of $c_{\mathcal{E}}$ in subdomain interiors chosen to minimize energy**

$$E_i(\boldsymbol{c}_{\mathcal{E}}) = \int_{\Omega_i} (\alpha_i \nabla \times \boldsymbol{c}_{\mathcal{E}} \cdot \nabla \times \boldsymbol{c}_{\mathcal{E}} + \beta_i \boldsymbol{c}_{\mathcal{E}} \cdot \boldsymbol{c}_{\mathcal{E}})\, dx$$



2D Algorithm

Recent Tools:

- **HX Decomposition**

For any $\boldsymbol{u}_h \in W^{h_i}_{curl}$, there exist $\boldsymbol{q}_h \in W^{h_i}_{curl}$, $\boldsymbol{\Psi}_h \in (W^{h_i}_{grad})^2$, and $p_h \in W^{h_i}_{grad}$ such that

$$\boldsymbol{u}_j = \boldsymbol{q}_h + \Pi^{h_i}(\boldsymbol{\Psi}_h) + \nabla p_h,$$

$$\|\nabla p_h\|^2_{L^2(\Omega_i)} \leq C(\|\boldsymbol{u}_h\|_{L^2(\Omega_i)} + H_i^2\|\nabla \times \boldsymbol{u}_h\|^2_{L^2(\Omega_i)}),$$

$$\|h_i^{-1}\boldsymbol{q}_h\|^2_{L^2(\Omega_i)} + \|\boldsymbol{\Psi}_h\|^2_{H^1(\Omega_i)} \leq C\|\nabla \times \boldsymbol{u}_h\|^2_{L^2(\Omega_i)}$$

- **Useful to DD and MG**

*** See Hiptmair & Xu, SINUM (2007) 45:2483-2509, Lemmas 5.1, 5.2, see also Hiptmair, Widmer, Zou, Numer. Math. (2006) 103:435-459.**

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2D Algorithm

Local Spaces/Solves:

- **Iterative Substructuring:**
  - One solve for each subdomain edge $\mathcal{E}$
  - Unknowns are edge coefficients for $\mathcal{E}$ and interiors of two subdomains sharing $\mathcal{E}$

- **Overlapping Schwarz**
  - One solve for each overlapping subdomain
  - Overlapping subdomains obtained by extending original ones an integer layer of elements

2D Algorithm

Theory Overview*:

- **Iterative Substructuring:**

$$\kappa(M^{-1}A) \leq C\chi(1 + \log(H/h))^2$$

$$\chi = 1 \text{ for straight edges}$$

$$\chi \leq (4/3)^{\log(H/h)} \text{ for snowflake curve edges}$$

- **See talk Thursday morning in M14 by Olof Widlund for more details.**

*** An iterative substructuring algorithm for two-dimensional problems in H(curl), TR-936, 2010, Department of Computer Science, Courant Institute, NYU.**

2D Algorithm

Examples: Scalability

No big surprises, but OS performs better

Results for unit square domain decomposed into N subdomains, each with $H/h = 4$. Numbers of iterations and condition number estimates (in parenthesis) are reported for a relative residual tolerance of 10^{-8}. Subdomain material properties given by $\alpha_i = 1$ and $\beta_i = \beta$.

		classical iterative substructuring			overlapping Schwarz ($H/\delta = 4$)		
Type	N	$\beta = 10^{-3}$	$\beta = 1$	$\beta = 10^{3}$	$\beta = 10^{-3}$	$\beta = 1$	$\beta = 10^{3}$
1	16	18(16.7)	15(16.3)	8(3.8)	14(5.1)	12(5.0)	8(4.6)
	64	25(18.6)	21(18.3)	10(6.1)	13(5.2)	12(5.2)	9(4.5)
	144	28(19.1)	22(18.9)	12(8.1)	13(5.1)	12(5.1)	10(4.6)
square	256	30(19.4)	23(19.0)	14(9.9)	12(5.1)	12(5.1)	10(4.7)
	400	30(19.5)	25(19.3)	15(11.6)	12(5.0)	12(5.0)	10(4.7)
	576	30(19.5)	25(19.3)	16(12.7)	12(5.0)	12(5.0)	11(4.8)
	784	30(19.5)	25(19.3)	16(13.7)	12(5.0)	12(5.0)	11(4.8)
	1024	30(19.5)	25(19.3)	17(14.5)	12(5.0)	12(5.0)	11(4.8)
2	16	26(30.0)	20(28.5)	8(3.7)	14(5.0)	12(4.8)	8(4.6)
	64	36(33.6)	29(33.0)	11(6.8)	17(6.9)	14(7.0)	9(4.5)
	144	40(34.2)	31(33.8)	14(10.0)	19(7.6)	15(7.6)	10(4.5)
	256	42(34.5)	33(34.1)	17(13.1)	19(7.5)	15(7.5)	10(4.5)
ragged	400	43(34.6)	34(34.3)	18(15.8)	20(8.0)	16(7.9)	10(4.5)
	576	43(34.7)	34(34.3)	20(18.5)	20(8.0)	16(8.0)	11(4.6)
	784	44(34.8)	35(34.6)	21(20.8)	20(7.9)	16(7.9)	11(4.6)
	1024	44(34.8)	36(34.6)	22(22.6)	20(8.1)	17(8.2)	11(5.0)

2D Algorithm

Examples: Material Property Jumps

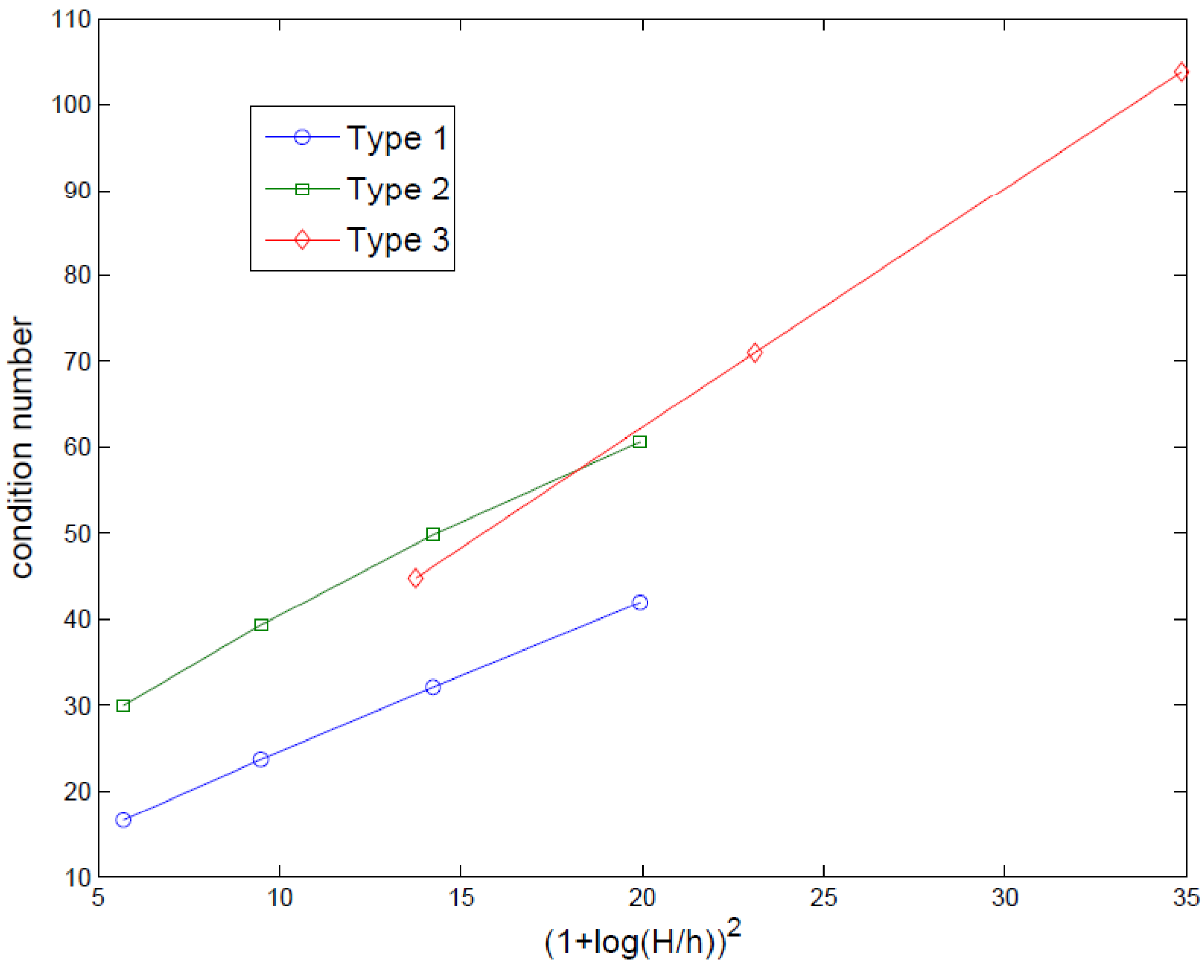
Classical iterative substructuring (CIS) and overlapping Schwarz (OS) results for unit square domain decomposed into 64 subdomains, each with $H/h = 8$ and $H/\delta = 4$ for OS. The eight subdomains along the diagonal from $(0,0)$ to $(1,1)$ have $\alpha_i = \alpha$ and $\beta_i = \beta$, while the remaining subdomains have $\alpha_i = 1$ and $\beta_i = 1$.

		Type 1		Type 2	
α	β	CIS	OS	CIS	OS
10^{-3}	10^{-3}	25(26.2)	14(5.7)	35(46.7)	15(6.6)
10^{-3}	1	24(26.0)	13(5.1)	34(43.8)	13(5.4)
10^{-3}	10^{3}	21(25.1)	13(5.5)	33(43.2)	15(9.8)
1	10^{-3}	26(26.3)	13(6.5)	36(46.7)	15(8.0)
1	1	24(26.3)	12(5.1)	32(44.5)	13(5.4)
1	10^{3}	24(24.7)	12(5.2)	32(41.9)	13(5.6)
10^{3}	10^{-3}	31(27.3)	13(6.4)	40(48.7)	15(7.9)
10^{3}	1	25(27.2)	12(5.1)	34(46.2)	13(5.4)
10^{3}	10^{3}	26(26.8)	14(6.4)	34(47.2)	15(6.8)

Covered by theory

2D Algorithm

Examples: Sharpness of Theory



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2D Algorithm

Examples: Mesh Partitioner

Results for unit square domain decomposed into N subdomains. There are 64 elements per subdomain for the Type 1 (square) subdomains and approximately 64 elements per subdomain for subdomains obtained from the mesh partitioner. Material properties are homogeneous with $\alpha_i = 1$ and $\beta_i = \beta$.

		classical iterative substructuring			overlapping Schwarz ($H/\delta = 4$)		
Type	N	$\beta = 10^{-3}$	$\beta = 1$	$\beta = 10^{3}$	$\beta = 10^{-3}$	$\beta = 1$	$\beta = 10^{3}$
1	16	20(23.7)	17(23.3)	9(5.8)	12(5.1)	12(5.1)	8(4.6)
	64	29(26.6)	24(26.3)	12(9.2)	12(5.1)	12(5.1)	10(4.5)
	144	31(27.1)	25(26.8)	14(12.2)	12(5.1)	12(5.1)	10(4.5)
	256	34(27.4)	26(27.1)	16(14.7)	12(5.1)	11(5.0)	11(4.7)
	400	35(27.6)	26(27.3)	17(16.6)	11(5.0)	11(5.0)	11(4.7)
Metis	16	30(27.8)	23(25.4)	10(5.2)	13(6.5)	13(6.5)	9(5.0)
	64	40(33.7)	30(32.2)	13(8.8)	13(5.5)	12(5.4)	11(4.8)
	144	42(37.4)	33(36.2)	15(11.9)	18(8.0)	16(7.9)	12(5.9)
	257	45(38.6)	35(36.8)	17(13.3)	16(7.2)	16(17.1)	13(6.3)
	400	46(41.9)	36(40.8)	17(13.7)	16(7.1)	15(6.9)	13(6.1)

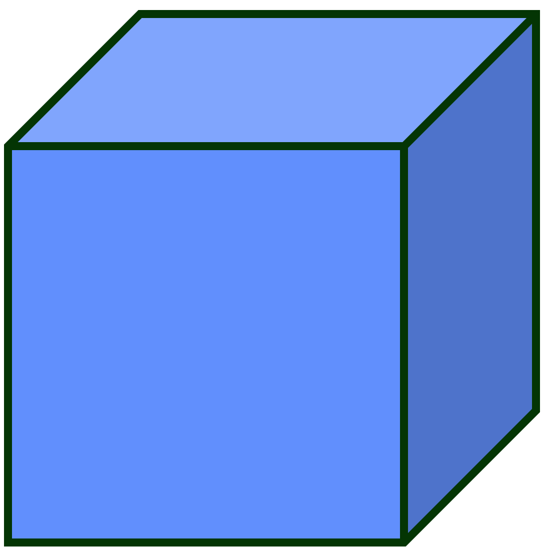
Mesh partitioner results not as good, but not bad

3D Algorithm

Face Decomposition Lemma:

For any $\boldsymbol{u}_i \in W^{h_i}_{curl}$ *with* $\boldsymbol{u}_i \cdot \boldsymbol{t}_e$ *vanishing along all subdomain edges of* Ω_i, *there exists a* $\boldsymbol{u}_{i\mathcal{F}} \in W^{h_i}_{curl}$ *such that* $\boldsymbol{u}_{i\mathcal{F}} \cdot \boldsymbol{t}_e = \boldsymbol{u}_i \cdot \boldsymbol{t}_e$ *for all* $e \in \mathcal{M}_{\mathcal{F}}$, $\boldsymbol{u}_{i\mathcal{F}} \cdot \boldsymbol{t}_e = 0$ *for all remaining edges of* $\mathcal{M}_{\partial\Omega_i}$, *and*

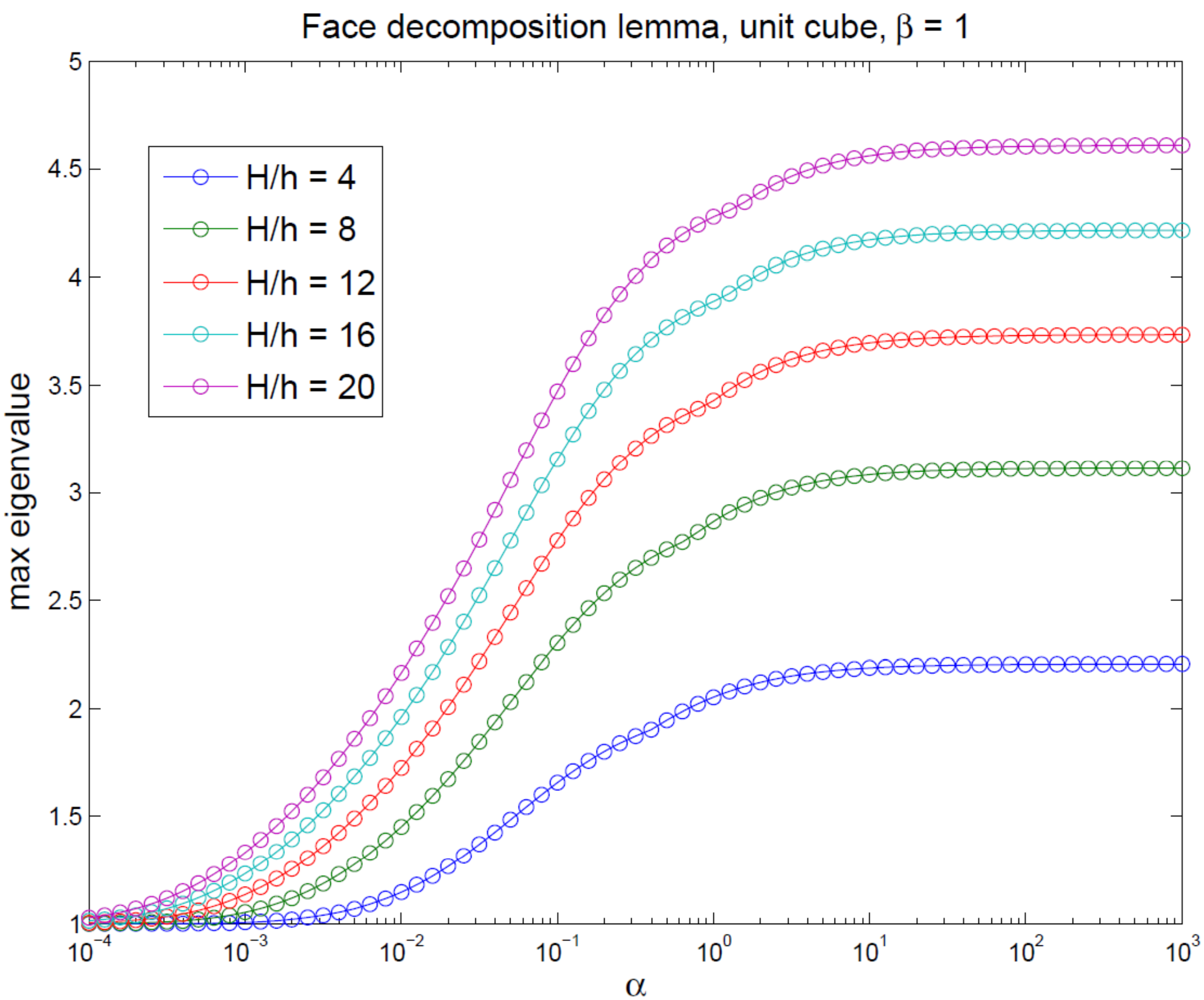
$$E_i(\boldsymbol{u}_{i\mathcal{F}}) \leqslant C(1+\log(H/h))^2 E_i(\boldsymbol{u}).$$



Example: tangential trace 0 along all 12 edges

$$E_i(\boldsymbol{u}) = \int_{\Omega_i} (\alpha_i \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} + \beta_i \boldsymbol{u} \cdot \boldsymbol{u})\, dx$$

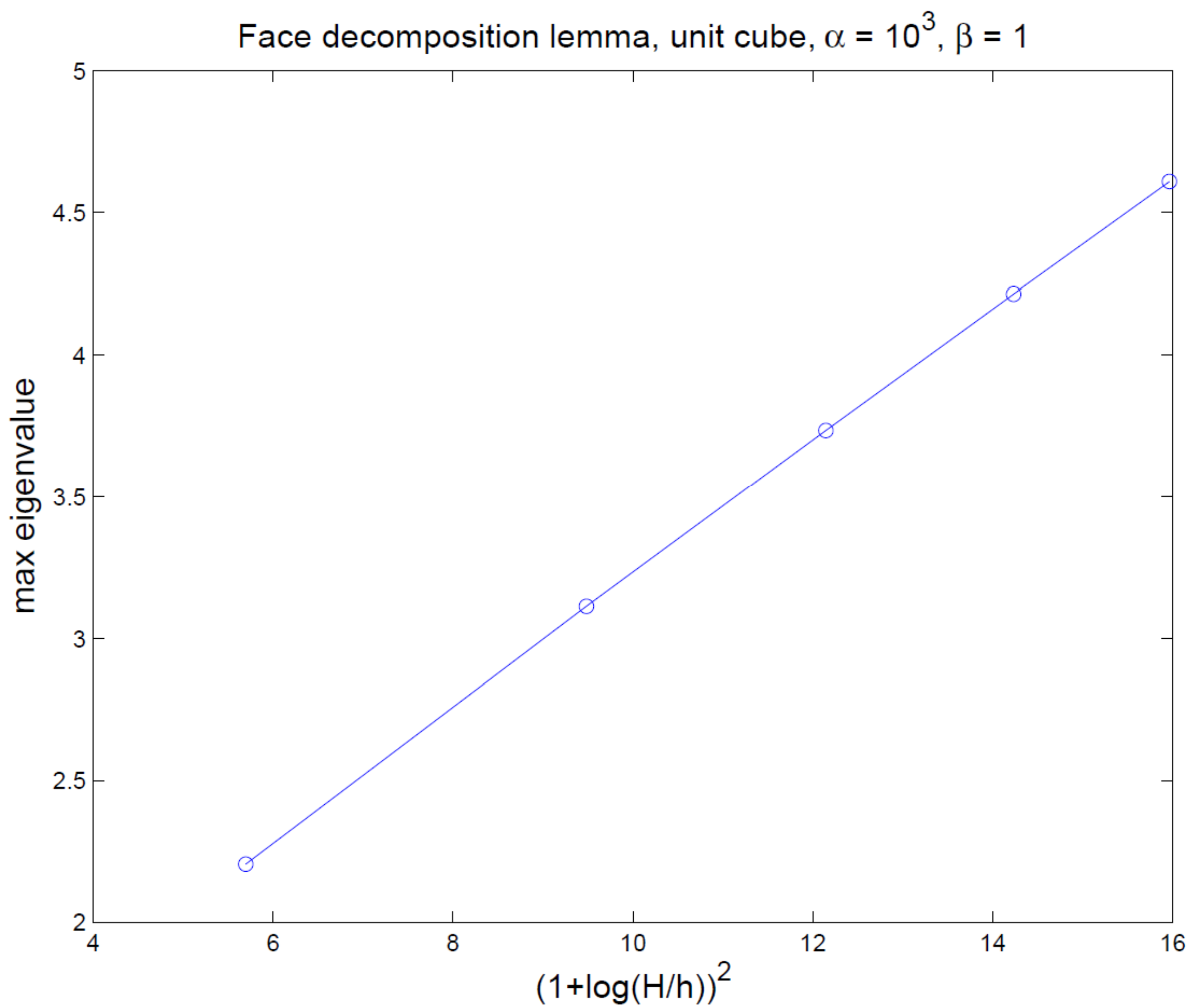
3D Algorithm



$$\frac{E_i(\boldsymbol{u}_{i\mathcal{F}})}{E_i(\boldsymbol{u})}$$

Computations can help early on

3D Algorithm



$$\frac{E_i(\boldsymbol{u}_{i\mathcal{F}})}{E_i(\boldsymbol{u})}$$

Computations can guide theory dev

3D Algorithm

- **Material Property Assumptions for Theory**

Assumption 1 *If Ω_i and Ω_j have a subdomain edge in common, then either*

$$\alpha_i \geqslant \alpha_j \quad \textit{and} \quad \beta_i \geqslant \beta_j$$

or

$$\alpha_i \leqslant \alpha_j \quad \textit{and} \quad \beta_i \leqslant \beta_j.$$

Assumption 2 *Let $\mathcal{I}_\mathcal{V}$ denote the set of all indices j such that Ω_j contains subdomain vertex $\mathcal{V}$. Further, let $j^\star \in \mathcal{I}_\mathcal{V}$ be chosen so that $\beta_{j^\star} \geqslant \beta_j$ for all $j \in \mathcal{I}_\mathcal{V}$. For each $j \neq j^\star$ there exists a sequence $S_{j\mathcal{V}} = \{j_0 = j, j_1, \ldots, j_{m-1}, j_m = j^\star\}$ such that Ω_{j_k} and $\Omega_{j_{k+1}}$ have a subdomain edge in common and $\beta_j \leqslant \beta_{j_k}$ for $k = 1, \ldots, m$.*

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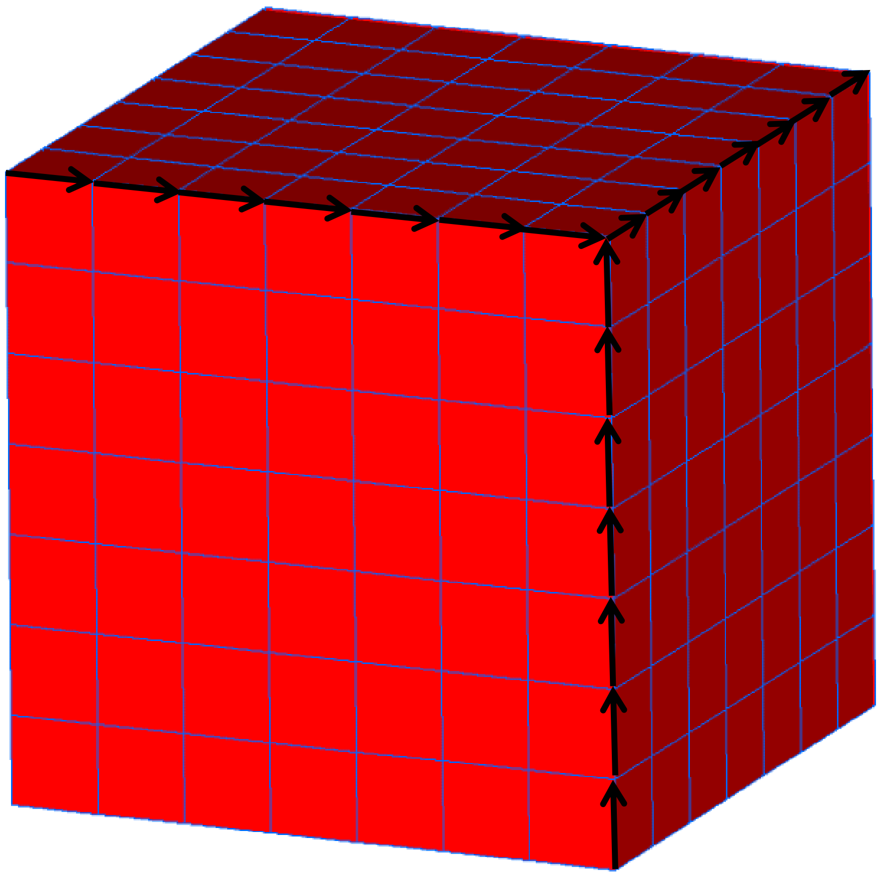
3D Algorithm

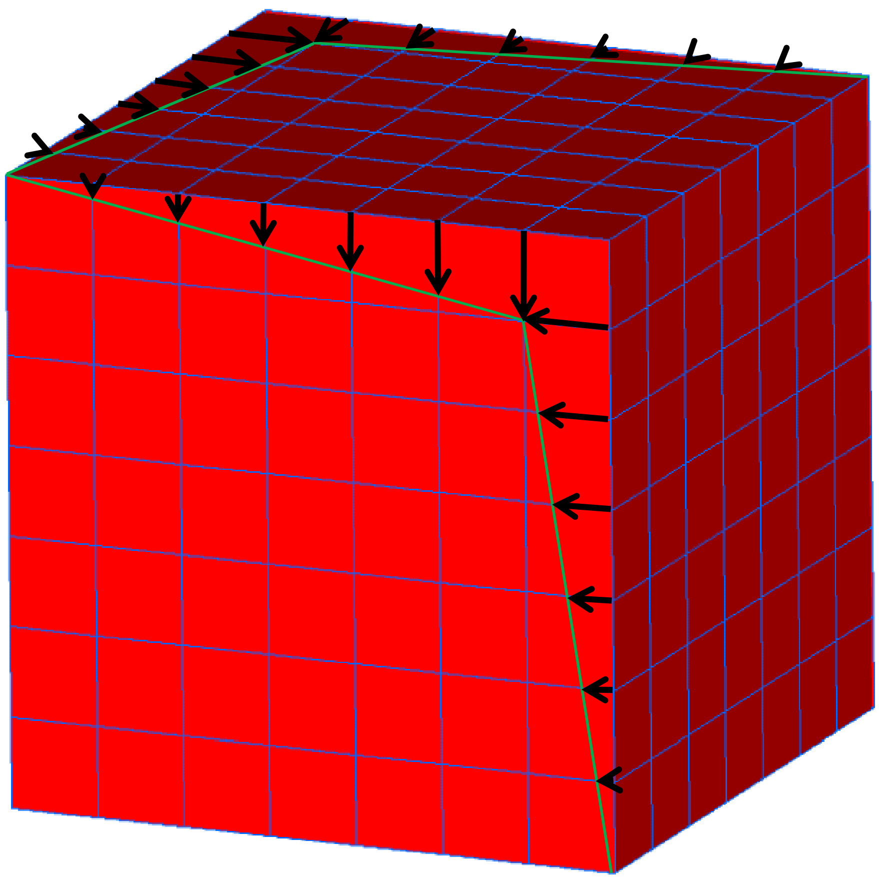
- **Coarse Space: Edge functions**



3 coarse basis functions shown

3D Algorithm

- **Coarse Space: Vertex/Face functions**



2 coarse basis functions shown

3D Algorithm

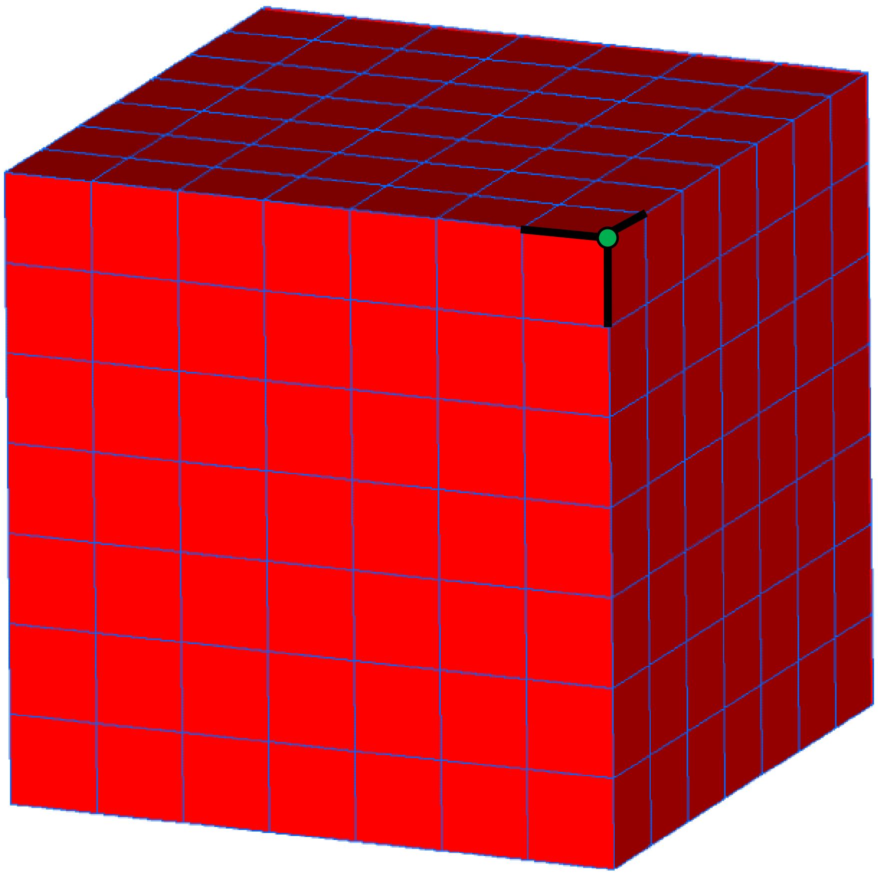
- **Full Coarse Space**
  - **1 coarse dof for each subdomain edge**
  - **1 coarse dof for each vertex of each face**

$$E(I^H(\boldsymbol{u})) \leq C(1 + \log(H/h))^3 E(\boldsymbol{u})$$

- **Reduced Coarse Space**
  - **1 coarse dof for each subdomain edge**

3D Algorithm

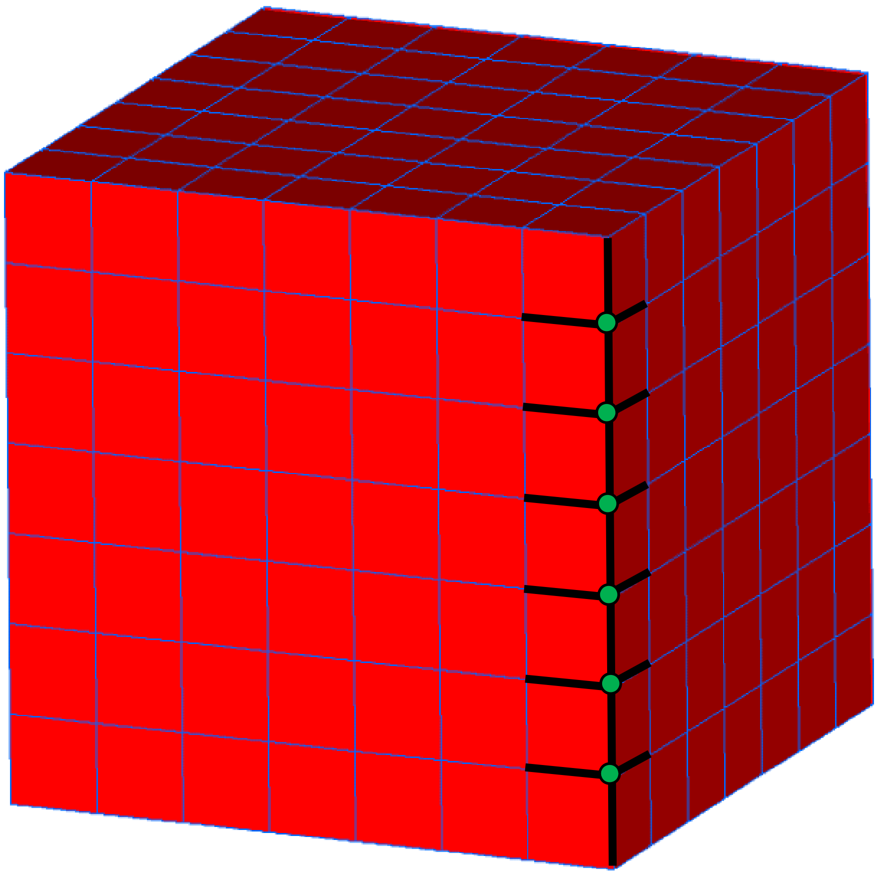
- **Iterative Substructuring Local Spaces: Vertex**



All edges incident to a subdomain vertex node

3D Algorithm

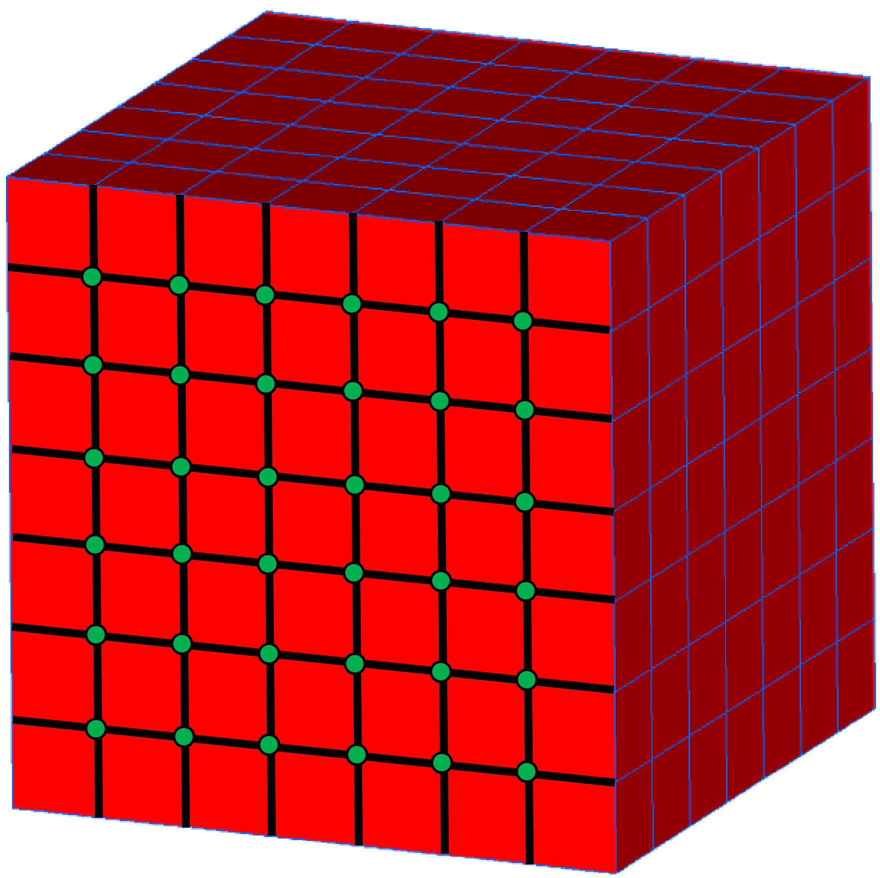
- **Iterative Substructuring Local Spaces: Edge**



All edges incident to one or more subdomain edge nodes

3D Algorithm

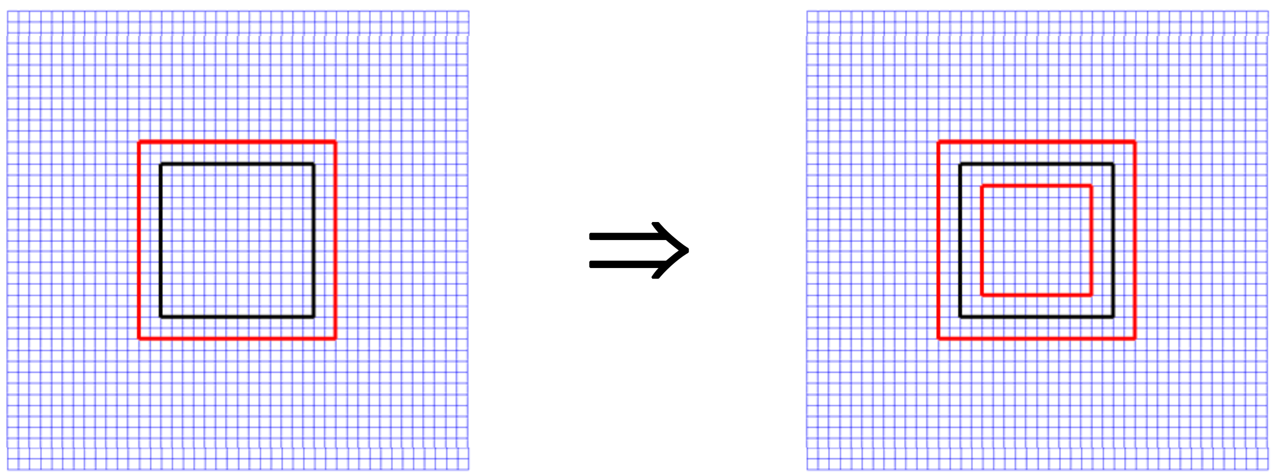
- **Iterative Substructuring Local Spaces: Face**



All edges on a face and in the interiors of two subs sharing the face

Hybrid Overlapping Schwarz/Iterative Substructuring

- **Just like overlapping Schwarz, but**
  - **add static condensation correction so residuals are always zero in subdomain interiors**
  - **use "boundary layer" subdomains to reduce cost of solves on overlapping subdomains**



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3D Algorithm

Theory Overview:

- **Iterative Substructuring:**

$$\kappa(M^{-1}A) \leq C(1 + \log(H/h))^3$$

- **Theory only accommodates convex polyhedral subdomains at this time**

- **Jumps in material properties between subdomains allowed per Assumptions 1 and 2**

3D Algorithm

Examples: Scalability

Results for unit cube domain decomposed into N cube subdomains, each with $H/h = 4$, for iterative substructuring and hybrid overlapping Schwarz (HOS) algorithms. Numbers of iterations and condition number estimates (in parenthesis) are reported for a relative residual tolerance of 10^{-8}. Subdomain material properties given by $\alpha_i = \alpha$ and $\beta_i = 1$. No Dirichlet BCs, full coarse space, additive corrections.

	Iterative Substructuring			HOS ($H/\delta = 4$)		
N	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$
4^3	31 (8.8)	28 (8.7)	21 (5.6)	32 (11.7)	29 (11.4)	20 (5.8)
6^3	32 (9.1)	30 (9.1)	23 (6.5)	35 (13.2)	32 (13.0)	23 (7.1)
8^3	33 (9.3)	33 (9.3)	25 (7.3)	38 (13.9)	34 (13.7)	26 (8.5)
10^3	34 (9.4)	31 (9.3)	26 (7.8)	38 (14.2)	35 (14.1)	28 (9.7)

HOS: not the likeable person on Bonanza

3D Algorithm

Examples: Additive vs. Multiplicative

Results for unit cube domain decomposed into N cube subdomains, each with $H/h = 4$, for iterative substructuring algorithm. Numbers of iterations and condition number estimates (in parenthesis) are reported for a relative residual tolerance of 10^{-8}. Subdomain material properties given by $\alpha_i = \alpha$ and $\beta_i = 1$. No Dirichlet BCs, full coarse space.

	additive			multiplicative		
N	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$
4^3	31 (8.8)	28 (8.7)	21 (5.6)	8 (1.38)	8 (1.37)	5 (1.08)
6^3	32 (9.1)	30 (9.1)	23 (6.5)	8 (1.42)	8 (1.41)	6 (1.15)
8^3	33 (9.3)	33 (9.3)	25 (7.3)	8 (1.43)	8 (1.43)	7 (1.23)
10^3	34 (9.4)	31 (9.3)	26 (7.8)	8 (1.44)	9 (1.44)	7 (1.29)

like Jacobi — like Gauss-Seidel

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3D Algorithm

Examples: Full vs. Reduced Coarse Spaces

Results for unit cube domain decomposed into N cube subdomains, each with $H/h = 4$, for iterative substructuring algorithm. Numbers of iterations and condition number estimates (in parenthesis) are reported for a relative residual tolerance of 10^{-8}. Subdomain material properties given by $\alpha_i = \alpha$ and $\beta_i = 1$. Additive corrections, no Dirichlet BCs.

	full coarse				reduced coarse			
N	$ncdof$	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$	$ncdof$	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$
4^3	648	31 (8.8)	28 (8.7)	21 (5.6)	108	29 (8.1)	27 (8.0)	20 (5.2)
6^3	2550	32 (9.1)	30 (9.1)	23 (6.5)	450	31 (8.5)	29 (8.5)	23 (6.3)
8^3	6468	33 (9.3)	33 (9.3)	25 (7.3)	1176	33 (8.7)	30 (8.7)	25 (7.1)
10^3	13122	34 (9.4)	31 (9.3)	26 (7.8)	2430	33 (8.8)	30 (8.8)	26 (7.6)

Reduced coarse space gets the job done

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Examples: Neumann vs. Dirichlet BCs

Results for unit cube domain decomposed into N cube subdomains, each with $H/h = 4$, for iterative substructuring algorithm. Numbers of iterations and condition number estimates (in parenthesis) are reported for a relative residual tolerance of 10^{-8}. Subdomain material properties given by $\alpha_i = \alpha$ and $\beta_i = 1$. Additive corrections, reduced coarse space.

	Neumann BCs			Dirichlet BCs		
N	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$	$\alpha = 10^2$	$\alpha = 1$	$\alpha = 10^{-2}$
4^3	29 (8.1)	27 (8.0)	20 (5.2)	45 (35.6)	41 (32.5)	23 (6.8)
6^3	31 (8.5)	29 (8.5)	23 (6.3)	58 (48.7)	54 (46.0)	29 (10.6)
8^3	33 (8.7)	30 (8.7)	25 (7.1)	67 (54.4)	62 (52.5)	34 (14.8)
10^3	33 (8.8)	30 (8.8)	26 (7.6)	72 (57.4)	66 (56.0)	39 (19.2)

Both scalable, but expected Dirichlet BCs to be easier.

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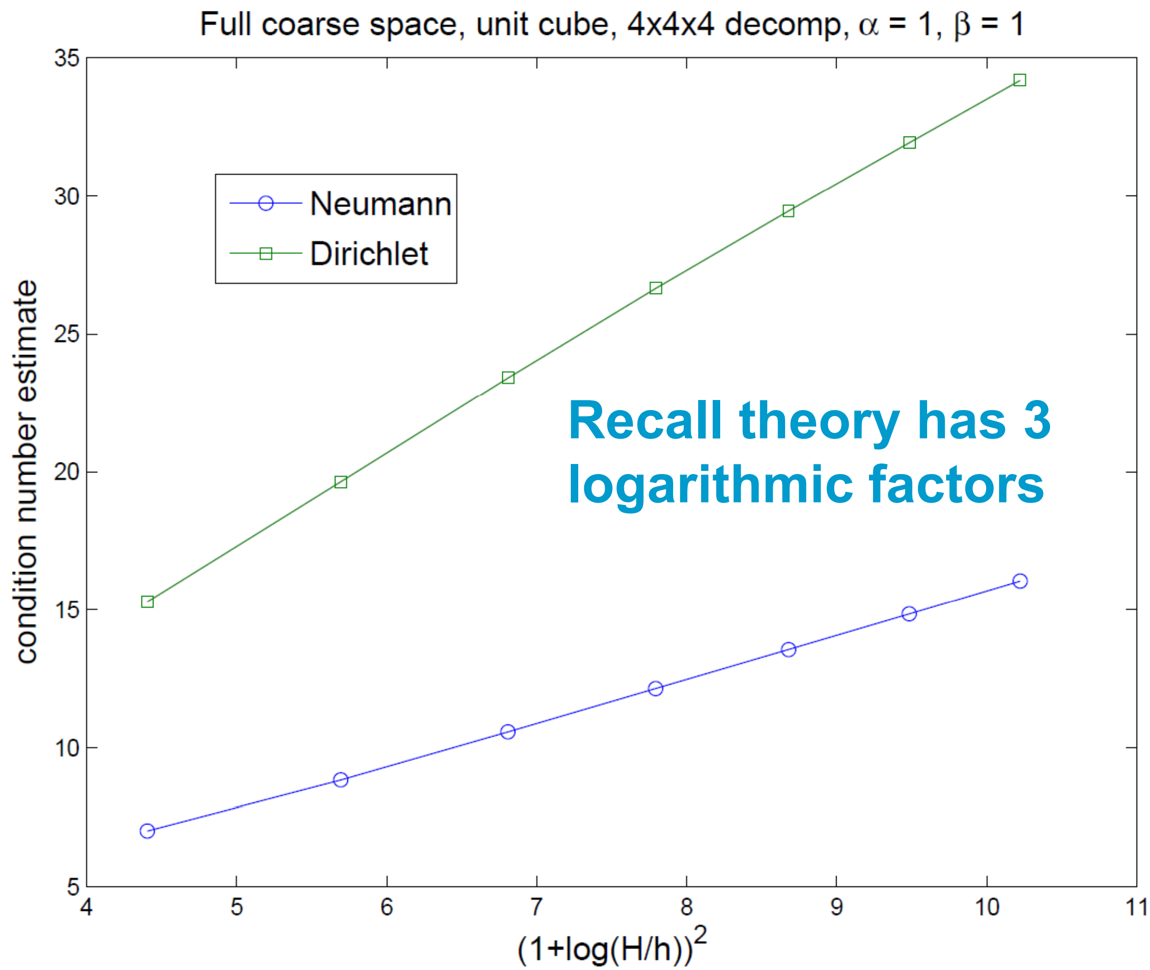
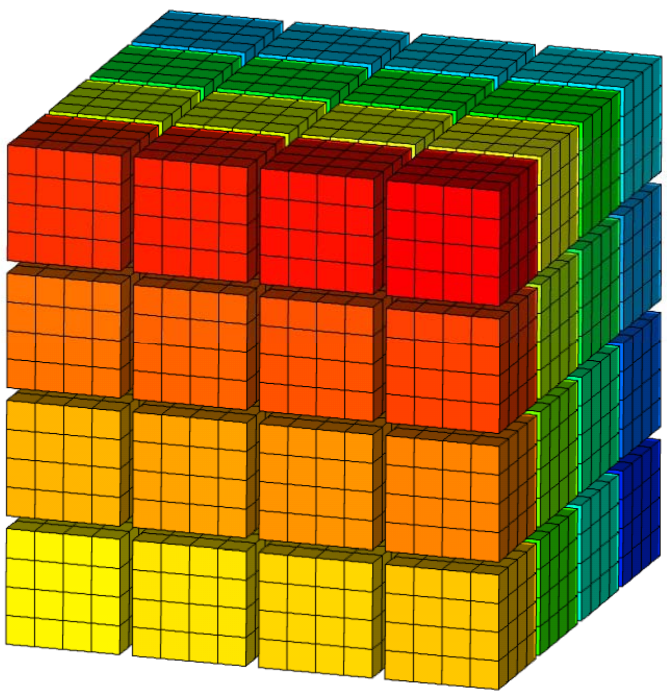
Examples: Scalability with respect to H/h



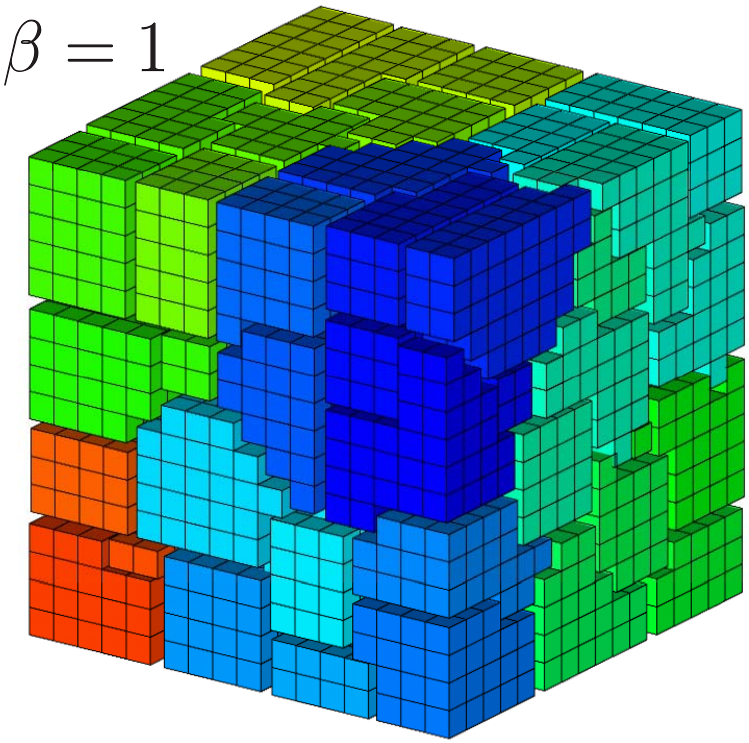
3D Algorithm

Examples: Regular vs. Irregular Decompositions

$$\alpha = \beta = 1$$



HOS, iter=29, $\kappa \approx 12$, $n_c = 108$, $H/\delta = 4$

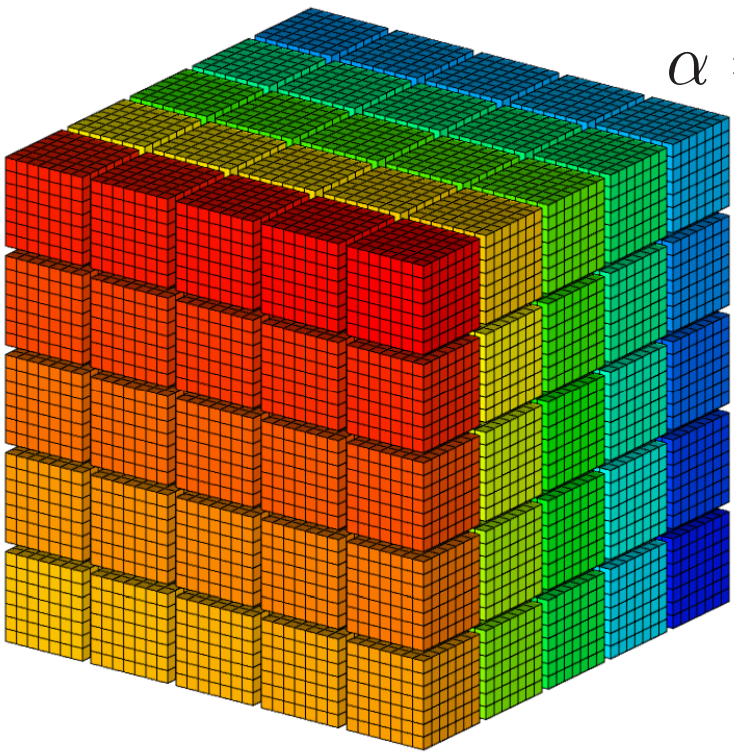


HOS, iter=34, $\kappa \approx 14$, $n_c = 422$, $H/\delta = 4$

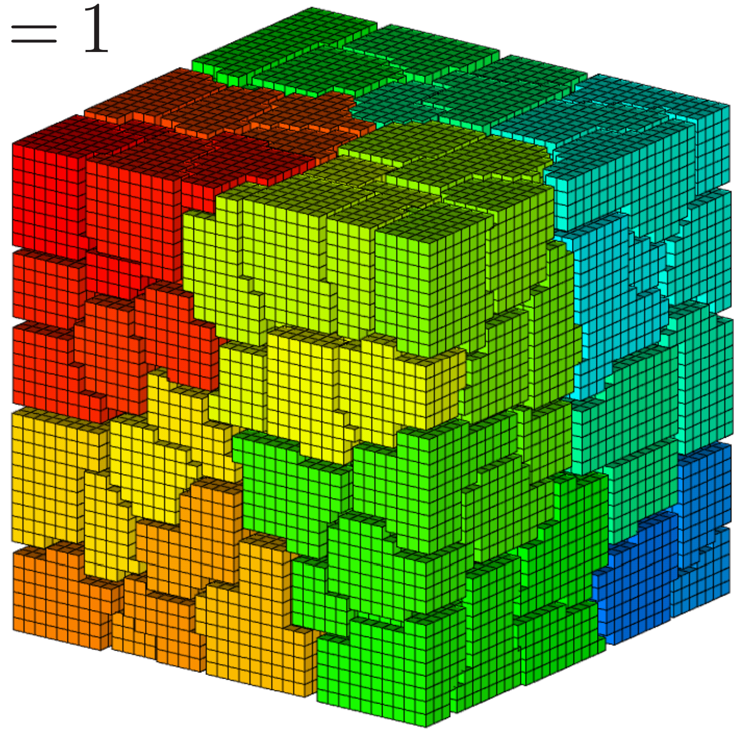
3D Algorithm

Examples: Regular vs. Irregular Decompositions

$$\alpha = \beta = 1$$



HOS, iter=40, $\kappa \approx 20$, $n_c = 240$, overlap = 1 layer



HOS, iter=46, $\kappa \approx 21$, $n_c = 1002$, overlap = 1 layer

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3D Algorithm

Examples: Material Property Jumps (Aligned)

Iterative substructuring results for unit cube with 4x4x4 subdomain decomposition. Numbers of iterations and condition number estimates (in parenthesis) are reported for a relative residual tolerance of 10^{-8}. Subdomain material properties are for a 4x4x4 checkerboard distribution with $\beta_i = 1$ and $\alpha_i = \alpha$ or $\alpha_i = \hat{\alpha}$. Additive corrections, full coarse space, no Dirichlet BCs.

$\hat{\alpha}$	$H/h = 4$	$H/h = 6$	$H/h = 8$
10^4	33 (9.1)	40 (12.5)	44 (15.3)
10^2	31 (9.1)	37 (12.5)	41 (15.2)
1	28 (8.7)	34 (12.0)	37 (14.6)
10^{-2}	29 (9.2)	34 (12.7)	38 (15.5)
10^{-4}	29 (9.9)	36 (14.3)	41 (18.1)

Covered by theory

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3D Algorithm

Examples: Material Property Jumps (Not Aligned)

Iterative substructuring results for unit cube domain with 4x4x4 subdomain decomposition. Numbers of iterations and condition number estimates (in parenthesis) are reported for a relative residual tolerance of 10^{-8}. Subdomain material properties are for an *approximate* 5x5x5 checkerboard distribution with $\beta_i = 1$ and $\alpha_i = \alpha$ or $\alpha_i = \hat{\alpha}$. Additive corrections, full coarse space, no Dirichlet BCs.

$\hat{\alpha}$	$H/h = 4$	$H/h = 6$	$H/h = 8$
10^4	36 (10.2)	43 (14.3)	46 (16.1)
10^2	33 (10.0)	39 (13.8)	42 (16.0)
1	28 (8.7)	34 (12.0)	37 (14.6)
10^{-2}	29 (9.4)	35 (12.9)	38 (14.6)
10^{-4}	35 (13.3)	40 (18.1)	44 (22.2)

Not covered by theory

Closing Remarks

- **2D Algorithm:**
  - Does not require subdomain matrices and accommodates irregular-shaped subdomains
  - More comprehensive theory than before
    - Favorable estimates over broader range of props
    - Allows broader class of subdomains (uniform)
- **3D Algorithm**
  - Same practical advantages as 2D algorithm
  - New theory and tools, but some work remains
    - Less restrictive assumptions on material properties
    - Less restrictive assumptions on subdomain shapes
    - Adaptive coarse spaces