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The Maximum Value Method:

A Node-movement Strategy for Simultaneous Mesh Untangling & Improvement

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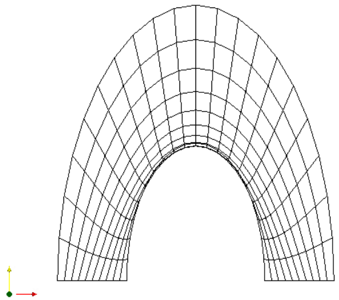
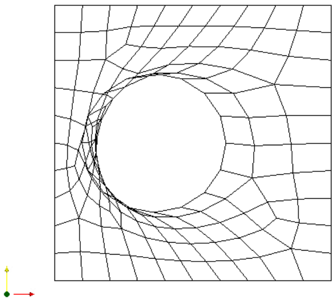
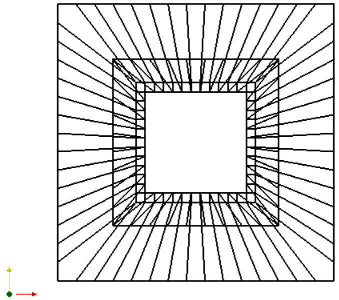
Examples of Tangled Meshes



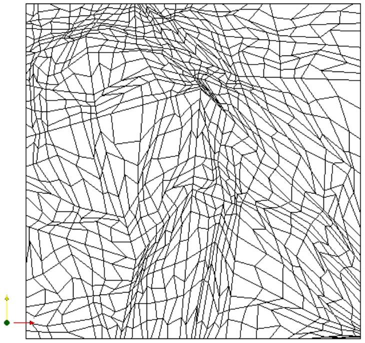
a) Horseshoe



b) Hole-in-Square

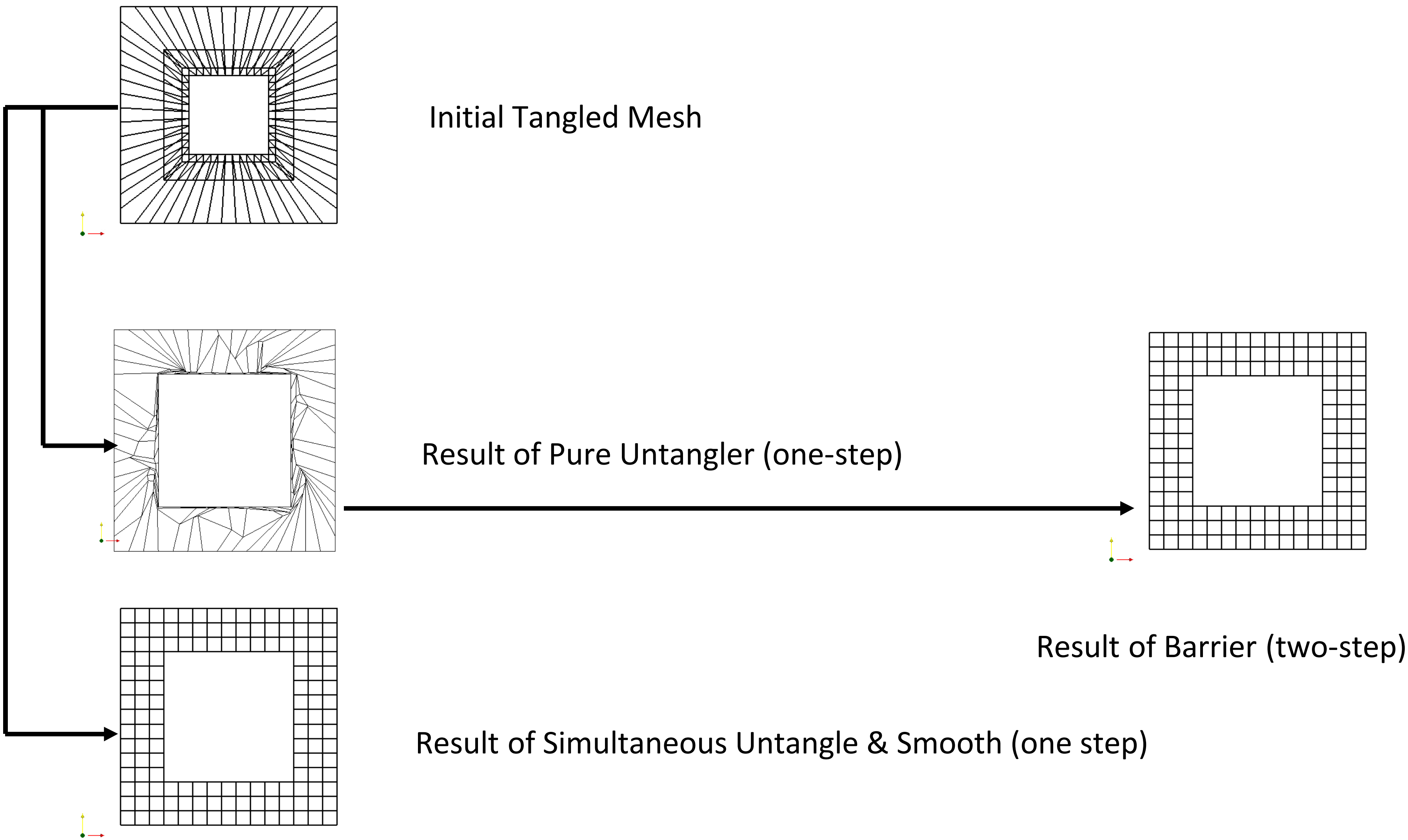


c) Inverted Hole



d) Shest Grid

Pure vs. Simultaneous Untanglers



Limitations

Pure Untanglers – None guarantee result is untangled
(sometimes they work & sometimes they don't)
- Even when result is untangled, the shape is usually poor.

Pure Untangle + Barrier - Second step improves shape, but requires the result of the first step to be untangled.

Simultaneous Untanglers – Although they improve shape, none guarantee result is untangled

Why do Untanglers fail?

-- i) the mesh cannot be untangled via node movement, or
-- ii) an untangled mesh exists but, although an untangled mesh is guaranteed if the *global* minimum of the objective function is attained, the optimization procedures often only find a local minimum. The guarantee does not hold for local minimums.

The "Pure" Untanglers

I. Freitag & Plassman (2001) – Maximize the minimum Jacobian determinant on a series of local mesh patches

II. Shashkov, Vachal (2001) – Place mesh vertices at the centroid of the feasible region of a series of local mesh patches.

III. Knupp (2001) - Minimize $\left\{ \left|\tau - \beta\right| - \left(\tau - \beta\right) \right\}$ on a local or global patch.

All of these were quite effective, some of the time.

None of them guaranteed that the result would be untangled.

Two Types of ‘Simultaneous’ Untanglers

I. ‘Incidental’ – those which improve shape or angles, without any explicit mechanism to encourage the mesh to be untangled.

Example: Laplace Smoothing

II. ‘Intentional’ – those which improve shape or angles, and contain an explicit mechanism to encourage the resulting mesh to be untangled.

Examples: (1) The Moving Barrier Method
(2) The Pseudo-Barrier Method
(3) The “Maximum Value” Method

1. The Moving Barrier Method

The Barrier Method

$$F = \sum_k \left(\frac{\ell_1^2 + \ell_2^2}{V} \right)_k$$

If all the initial V's are positive, then the resulting V's are guaranteed to be positive. If some of the V's are positive and some are negative, then the method cannot be used.

The Shifted-Barrier Method

$$F = \sum_k \left(\frac{\ell_1^2 + \ell_2^2}{V - V_{\min}} \right)_k \qquad V_{\min} = \min_k \{V_k\}$$

Since all of the initial V's are greater than V-min, the method can be used.

The resulting V's are guaranteed to be greater than V-min.

The Moving-Barrier Method.

Use the shifted-Barrier Method iteratively, updating V-min, until V-min becomes positive.

Barrera-Sanchez, Tinoco-Ruiz (1998). Often effective, but relatively expensive.

2. The Pseudo-Barrier Method

Objective Function:

$$F = \sum_k \left(\frac{\ell_1^2 + \ell_2^2}{\frac{1}{2}\left(V + \sqrt{V^2 + \left(V_{ref}\right)^2} \right)} \right)_k$$

V-ref << V.

There is no Barrier in this method, but the OF does tend to become large as V becomes negative. When V is positive, F is approx. the Barrier Method.

Escobar, et. al. (2003) No guarantee that result will be non-inverted but method can be used whether or not initial mesh is tangled.

3. The Maximum Value Method

Objective Function: $$F = \sum_k f\left(\mu\left(T_k\right)\right)$$

with f, a function of a local quality metric, given by

$$f(\mu) \equiv \left\{\left|[1-\varepsilon]-\mu\right| - \left([1-\varepsilon]-\mu\right)\right\}^2$$

(μ is a local quality metric satisfying a certain property.)

Knupp, Franks (2010) No guarantee result will be inverted.

The Maximum Value Method

I. The Target-matrix Paradigm
II. Two ‘Incidental’ Simultaneous Untangle Metrics from TMOP
III. Mathematical Propositions about the Metrics
IV. The function f(μ)
V. Results

I. The Target-Matrix Optimization Paradigm (TMOP)

I. For every mesh element, define a mapping from a master element to the physical element, in terms of the element vertices.

II. Certain points Ξ_k within the master element, called sample points, are selected.

III. Let A_k be the Jacobian matrix of the map at the k-th sample point. det(A_k) is a measure of the local volume at the sample point.

IV. Definition: in TMOP, a mesh is inverted if there exists any sample point in the mesh for which det(A_k) <=0. An inverted element is an element which contains a sample point at which the local volume is negative.

V. Let W_k be the reference or taget-Jacobian matrix at the sample point.

W_k represents the desired or ideal Jacobian matrix that we wish to create in the optimal mesh. Let $T_k = A_k (W_k)^{-1}$

VI. $\tau_k = \det(T_k) > 0$ iff $\det(A_k) > 0$

II. The TMOP "Relative Volume" Metric

The metric: $$\mu_{RV}(T) = (\tau - 1)^2$$

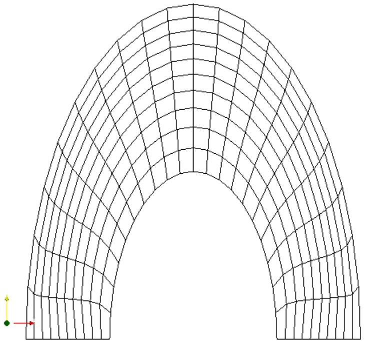
Metric is non-negative, and is zero if and only if τ=1. If τ=1,
Then det(A) = det(W) > 0 (by construction). The metric does not have a Barrier.

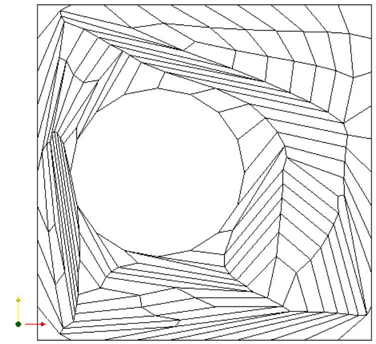
Choice of Target matrix: $W = \overline{\Lambda} S_{ideal}$ (and for all subsequent)

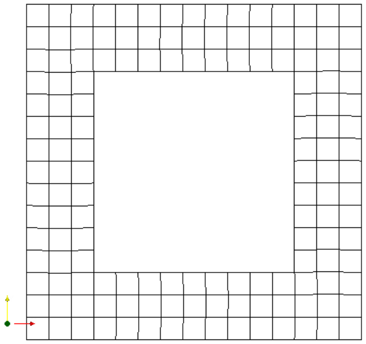
This metric is the basis for a mesh untangler. It is not a 'pure' untangler because it encourages more than positive volume; it encourages the volume to be close to det(W). Perhaps it is best classified as an 'intentional' simultaneous mesh untangler.

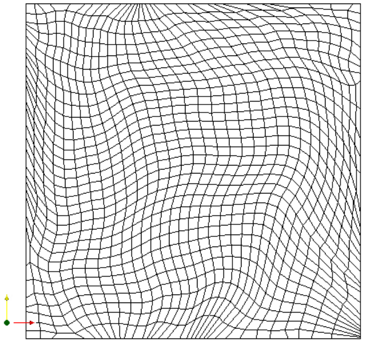
Results: The metric creates nearly equal-sized, non-smooth meshes.
It was able to untangle 3 of our 4 test meshes.

Optimal Meshes Created via the Relative Volume Metric









II. The TMOP "Size & Shape" Metric

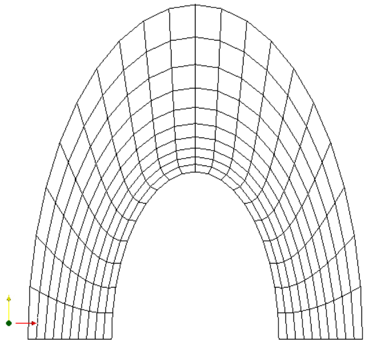
The metric: $$\mu_{SS}(T) = |T|^2 - \sqrt{|T|^2 + 2\tau} + 2$$

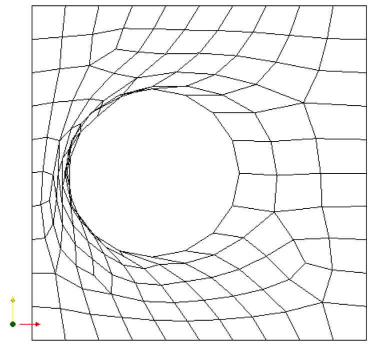
The metric is non-negative and is zero if and only if T=R, with R an undetermined rotation matrix. The metric thus controls local shape and size (relative volume). The metric has no barrier and thus can be used on initially tangled meshes.

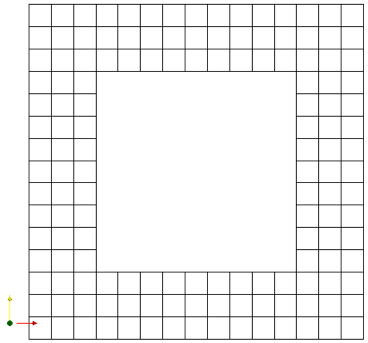
The metric is best classified as an 'incidental' simultaneous mesh untangler because there is no explicit mechanism to untangle and because it can do much more than untangle.

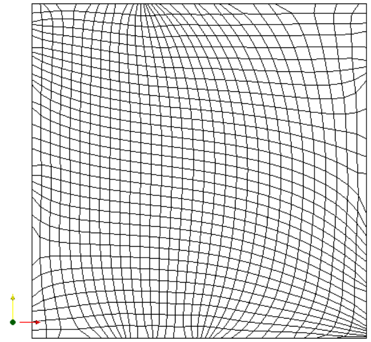
Results: The metric creates well-shaped elements with near-equal sizes.
It was able to untangle 3 of our 4 test meshes.

Optimal Meshes Created via the 'Shape & Size" Metric









Proposition I

Let T be a d x d real matrix. If $\mu_{RV} < 1$ then $0 < \tau < 2$

Proposition II

Let T be a d x d real matrix. If $\mu_{SS} < 1$ then $\tau > 0$

Corollary to Both

If $\max_k \{\mu(T_k)\} < 1$ then the mesh is non-inverted.

Not all TMOP metrics have propositions like this.

Van der Zee showed that if $|T - I|^2 < 1$, then $\tau > 0$.

(Note that $\mu > 1$ does not necessarily mean that $\tau < 0$)

III. Direct Application of the Propositions

Minimize over the global mesh:

$$\max_k \{\mu(T_k)\}$$

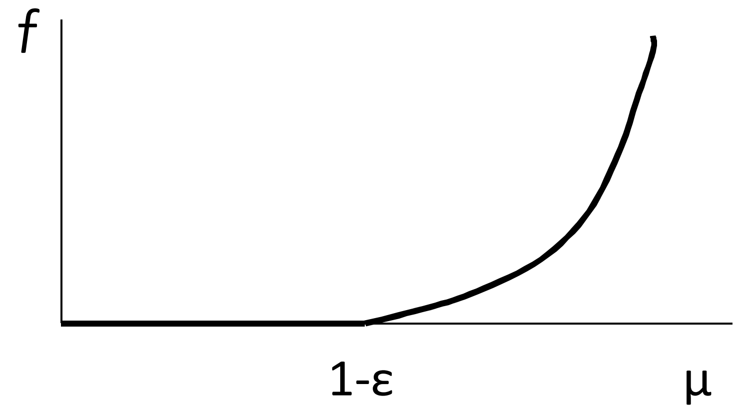
Clearly, this approach cannot guarantee that the optimal mesh will be untangled.

However, if the mesh is untangle-able, and a solution to the *global* optimization problem can be found, the resulting mesh will be untangled.

Intuition: as the objective function decreases, the Shape+Size of the mesh Improves, and the mesh may become untangled. In this method, untangling and improvement are seen to be consistent goals.

III. An Indirect Application of the Propositions

Define $$f(\mu) \equiv \left\{ \left| [1-\varepsilon] - \mu \right| - \left([1-\varepsilon] - \mu \right) \right\}^2$$ $0<\varepsilon<<1$.



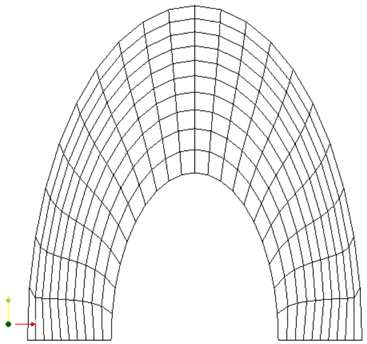
If $f=0$, then $\mu<1$, and so $\tau>0$.

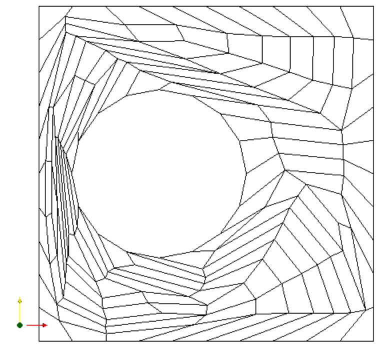
Minimize:

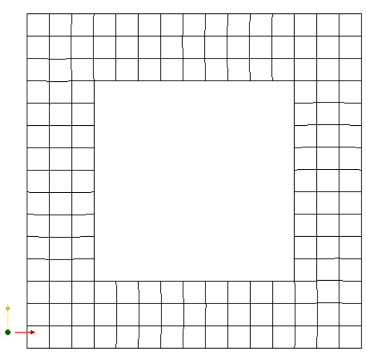
$$F = \sum_k f\left(\mu\left(T_k\right)\right)$$

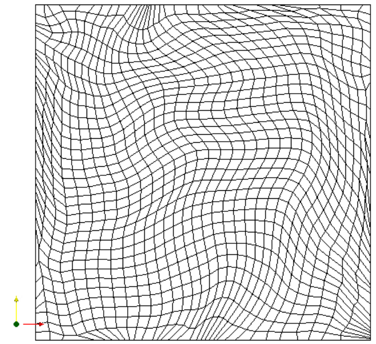
f penalizes locations in the mesh where the metric is greater than 1.
F=0 means mesh is untangled.

Optimal Meshes Created by f(μ_{RV})

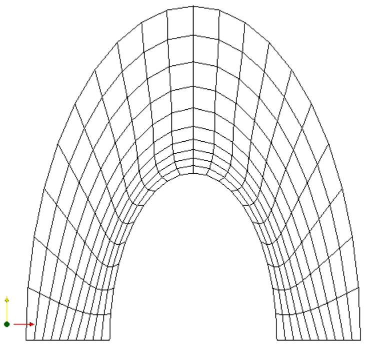


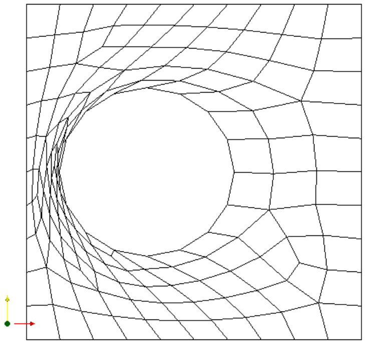


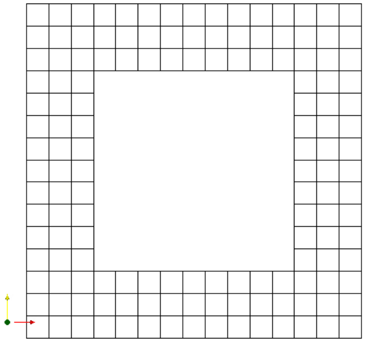


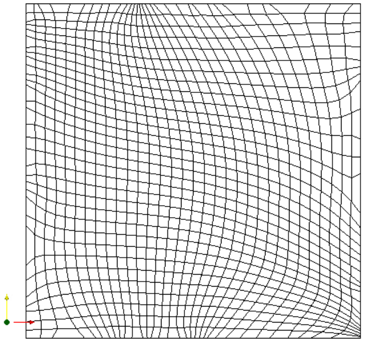


Optimal Meshes Created by f(μ_{SS})









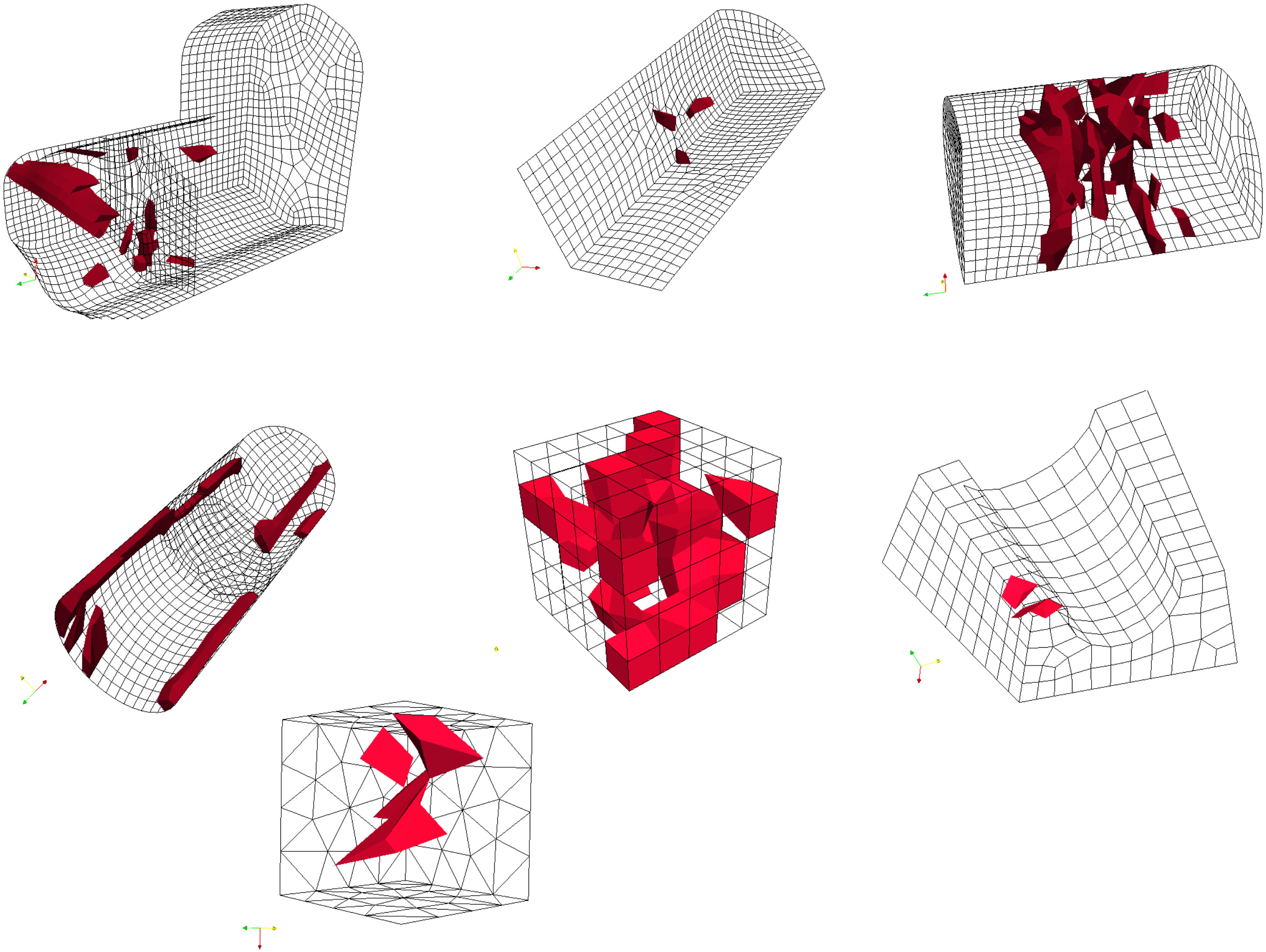
IV. Summary Table of Results

	Horseshoe	Shest	Hole-in-Square	Inv. Hole	#T
Initial Mesh	T	T	T	T	4
Laplace	T	U	T	U	2
Knupp 2001	U	U	U	U	0
Relative Vol	U	U	T	U	1
f(μ_{RV})	U	U	U	U	0
Shape+Size	U	U	T	U	1
f(μ_{SS})	U	U	T	U	1
	1	0	4	0	

Results show that the 2D test meshes are all untangle-able.

Since f(μ_{SS}) did not untangle the Hole-in-Square, a tangled *local* minimum must have been found. A better solver could help.

Method Extends to 3D Volume Meshes



Summary Table for 3D Results

	P1	P2	P3	P4	H5	HPT	T4	#T
Initial	T	T	T	T	T	T	T	7
Laplace	U	U	U	U	U	T	U	1
Knupp 01	U	U	T	T	U	U	U	2
Rel Vol	U	U	T	T	U	T	U	3
SS	U	U	U	U	U	T	U	1
f-RV	U	U	T	T	U	U	U	2
f-SS	U	U	T	U	U	T	U	2
	0	0	4	3	0	4	0	

Untangle failures again most likely due to failure to find global minimum.

Conclusions

1. In general, the simultaneous untangle & improve (SUI) methods appear superior to the pure untanglers because the shape quality is better.
2. The SUI methods can be applied to initially tangled meshes.
3. Maximum Value Method is an alternative to the two other methods for Simultaneous Mesh Untangling & Improvement.
4. MVM guarantees that the optimal mesh will be non-inverted provided (i) the mesh is untangle-able and (ii) the global minimum is found.
5. Maximum Value Method contains the *global* parameter ε, which is easy to choose. The Pseudo-Barrier method requires one to choose V-ref, which is a *local* parameter.
6. MVM also requires choosing the appropriate Target-Matrix (the Size factor in particular). For homogeneously-sized meshes, the Size factor is easy to choose, but could be difficult for heterogeneously-sized meshes.
7. f-SS would seem preferable to f-RV since the former can produce smooth meshes. MVM without use of f also worth considering.
8. Better solvers appear essential for robust MVM.