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Approximate Block Factorization Preconditioners for Primitive Variable Incompressible Resistive MHD

Finite Elements in Flow: March 23, 2011

Eric C. Cyr
Sandia National Laboratories

Collaborators:

John N. Shadid, Ray S. Tuminaro, Roger P. Pawlowski
Sandia National Laboratories

Luis Chacón
Oak Ridge National Laboratory

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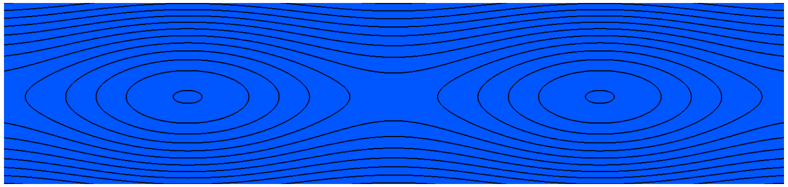
Magnetohydrodynamics



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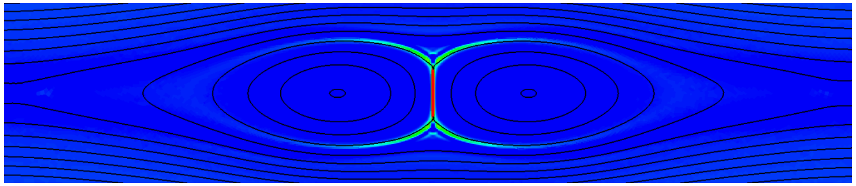

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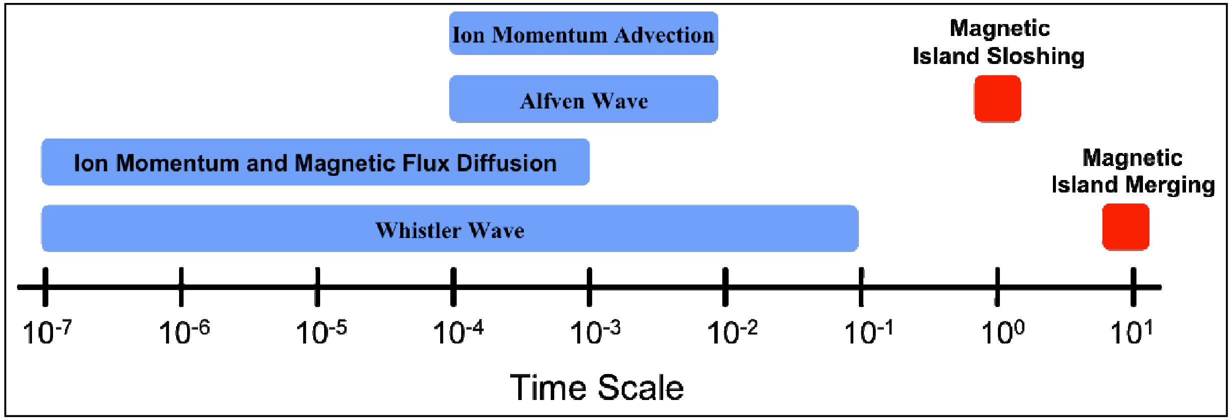
Magnetic island formation/coalescence

➔ Can be modeled by Extended MHD

Timescales for Magnetohydrodynamics



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Magnetohydrodynamics

MHD times scales difficult for explicit, operator-split, and semi-implicit integration

- Fast modes prohibit explicit simulation
  - × Stability restrictions imply small time steps
  - × For long time integration accuracy becomes problematic
- Interacting time scales make semi-implicit and operator split methods challenging

Stable long time scale integration can be enabled by implicit time stepping

- Must solve linear system: Newton's Method (one iteration)

$$\text{Solve } \mathbf{J}p_k = -F(x_k) \text{ where } \mathbf{J} = \partial F/\partial x$$

$$x_{k+1} = x_k + p_k$$

Our approach is to solve using preconditioned Newton-Krylov methods

- Effective preconditioning is key to parallel scalability

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Preconditioning

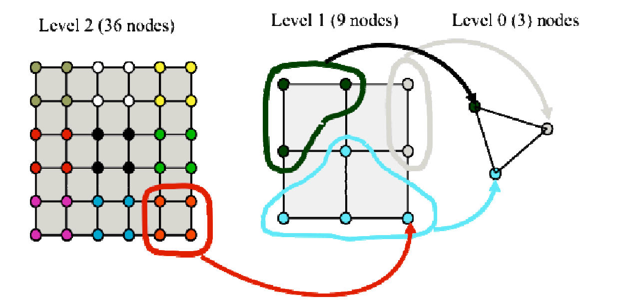
Three flavors of preconditioning

1. Domain Decomposition

   - ILU Factorization on each processor (with overlap)

2. Multilevel methods: ML Library (Tuminaro, Sala, Hu, Siefert, Gee)

   - Smoothed Aggregation
   - Aggressive Coarsening (AggC):
     - 3-Level method
     - Multiple unknowns per node
     - Aggregation rate chosen to fix coarse grid size
     - ILU solvers to fix aggressive aggregation



Visualization of effect of partition of matrix graph on mesh

3. Block Preconditioners

Aggregation based Multigrid:
- Vanek, Mandel, Brezina, 1996
- Vanek, Brezina, Mandel, 2001
- Sala, Formaggia, 2001

Block Preconditioning

For preconditioning system first do Block Factorization

$$\begin{bmatrix} F & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ BF^{-1} & I \end{bmatrix} \begin{bmatrix} F & B^T \\ 0 & S \end{bmatrix}$$

where Schur complement is

$$S = C - BF^{-1}B^T$$

Benefits

- Segregates matrix into individual physics
- Localizes coupling in Schur complement
- Enables Jacobi/GS and non-pivoting ILU type prec./smoothers for C = 0

Challenges

- Approximating inverse of Schur complement

Incompressible MHD: 2D Vector Potential Formulation

Magnetohydrodynamics (MHD) equations couple fluid flow to Maxwell's equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla p + \nabla \cdot \left(-\frac{1}{\mu_0} \mathbf{B} \otimes \mathbf{B} + \frac{1}{2\mu_0} \|\mathbf{B}\|^2 \mathbf{I} \right) = f$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial A_z}{\partial t} + \mathbf{u} \cdot \nabla A_z - \frac{\eta}{\mu_0} \nabla^2 A_z = -E_z^0$$

$$\text{where} \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{A} = (0, 0, A_z)$$

Discretized using a stabilized finite element formulation

Incompressible MHD: Discrete Formulation

Stabilized finite element method in residual form

Momentum	$\mathbf{F}_{m,i} = \int_{\Omega} \Phi \mathbf{R}_{m,i} d\Omega + \sum_e \int_{\Omega_e} \hat{\tau}_m (\mathbf{v} \cdot \nabla \Phi) \mathbf{R}_{m,i} d\Omega$
Total Mass	$F_P = \int_{\Omega} \Phi R_P d\Omega + \sum_e \int_{\Omega_e} \hat{\tau}_m \nabla \Phi \cdot \mathbf{R}_m d\Omega$
Z-Vector Potential	$F_{A_z} = \int_{\Omega} \Phi R_{A_z} d\Omega + \sum_e \int_{\Omega_e} \hat{\tau}_{A_z} (\mathbf{v} \cdot \nabla \Phi) R_{A_z} d\Omega$

Structure of discretized Incompressible MHD system is

$$\mathcal{J}\mathbf{x} = \begin{bmatrix} F & B^T & Z \\ B & C & 0 \\ Y & 0 & D \end{bmatrix} \begin{bmatrix} u \\ p \\ A \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ e \end{bmatrix}$$

Matrices F and D are transient convection operators, C is stabilization matrix

Nested Schur Complements

Block LU factorization gives

$$\begin{bmatrix} F & B^T & Z \\ B & C & 0 \\ Y & 0 & D \end{bmatrix} = \begin{bmatrix} I & & \\ BF^{-1} & I & \\ YF^{-1} & -YF^{-1}B^TS^{-1} & I \end{bmatrix} \begin{bmatrix} F & B^T & Z \\ & S & -BF^{-1}Z \\ & & P \end{bmatrix}$$

where

$$S = C - BF^{-1}B^T$$
$$P = D - YF^{-1}(I + B^TS^{-1}BF^{-1})Z$$

- 3x3 system leads to nested Schur complements
- Nesting is independent of ordering (C^{-1} doesn't exist!)
- How is P approximated?

SIMPLE Motivated Preconditioner

$$\mathcal{M} = \begin{bmatrix} F & B^T & Z \\ & S_{Neu} & -BF^{-1}Z \\ & & P_{Neu} \end{bmatrix}$$

where

$$\begin{aligned} F_{Neu} &= \mathbf{AbsRowSum}(F) \\ S_{Neu} &= C - BF_{Neu}^{-1}B^T \\ P_{Neu} &= D - YF_{Neu}^{-1}\,\mathbf{AbsRowSum}(I \\ &\qquad + B^T\mathbf{AbsRowSum}(S_{Neu})^{-1}BF_{Neu}^{-1})Z \end{aligned}$$

Issues
- SIMPLEC Approximation has issues with large CFL
- Not scalable for fixed timesteps

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Splitting for MHD

Using approximation $1 + A + B = (1 + A)(1 + B) - AB$
$\approx (1 + A)(1 + B)$

$$\begin{bmatrix} F & B^T & Z \\ B & C & \\ Y & & D \end{bmatrix} = \begin{bmatrix} F & & \\ & I & \\ & & D \end{bmatrix} \left(\begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix} + \begin{bmatrix} & F^{-1}B^T & \\ B & C - I & \\ & & 0 \end{bmatrix} + \begin{bmatrix} & & F^{-1}Z \\ & 0 & \\ D^{-1}Y & & \end{bmatrix} \right)$$

$$\approx \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix} = \begin{bmatrix} F & B^T & Z \\ B & C & \\ Y & \boxed{YF^{-1}B^T} & D \end{bmatrix}$$

or

$$\approx \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix} = \begin{bmatrix} F & B^T & Z \\ B & C & \boxed{BF^{-1}Z} \\ Y & & D \end{bmatrix}$$

Splitting for MHD

Does splitting make good preconditioner?

$$\mathcal{M} = \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix}$$

1. Structurally small perturbation

$$\mathcal{M} = \begin{bmatrix} F & B^T & Z \\ B & C & \\ Y & \boxed{YF^{-1}B^T} & D \end{bmatrix}$$

2. Favorable spectrum

$$\mathcal{J}\mathcal{M}^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ K_u & K_p & (I - YF^{-1}B^T S^{-1} B F^{-1} Z P^{-1}) \end{bmatrix}$$

Challenges of splitting: Requires action of two 2x2 inverses

1. Naver-Stokes system
2. Magnetics-Velocity system

Approximating Velocity/Magnetics Coupling

$$\begin{bmatrix} F & Z \\ Y & D \end{bmatrix} = \begin{bmatrix} I & \\ YF^{-1} & I \end{bmatrix} \begin{bmatrix} F & Z \\ & P \end{bmatrix}$$

$$\text{where } \; P = D - YF^{-1}Z$$

Strong form commuting condition

$$\nabla A_z \cdot \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla - \; \nu_F \nabla^2 \right) \approx \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla - \; \nu_M \nabla^2 \right) \nabla A_z \cdot$$

motivates discrete commuting

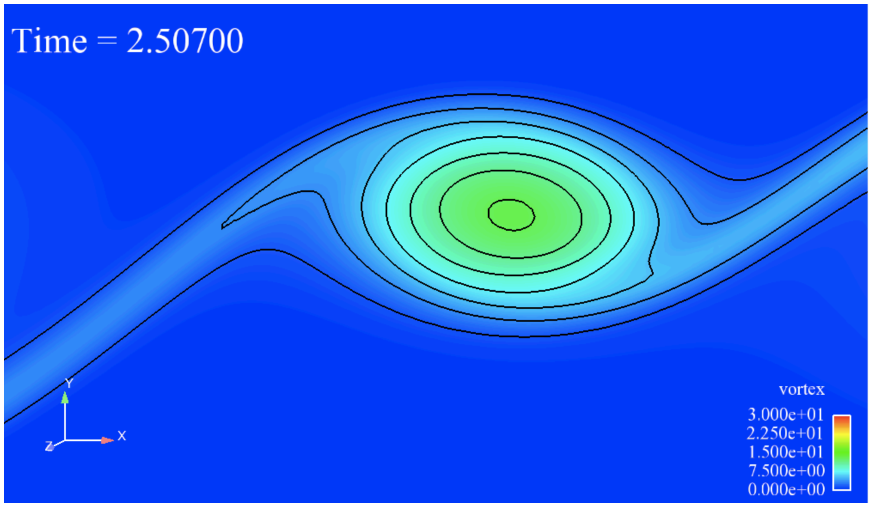
$$YQ_u^{-1}F \approx DQ_a^{-1}Y$$

which gives an approximate Schur complement

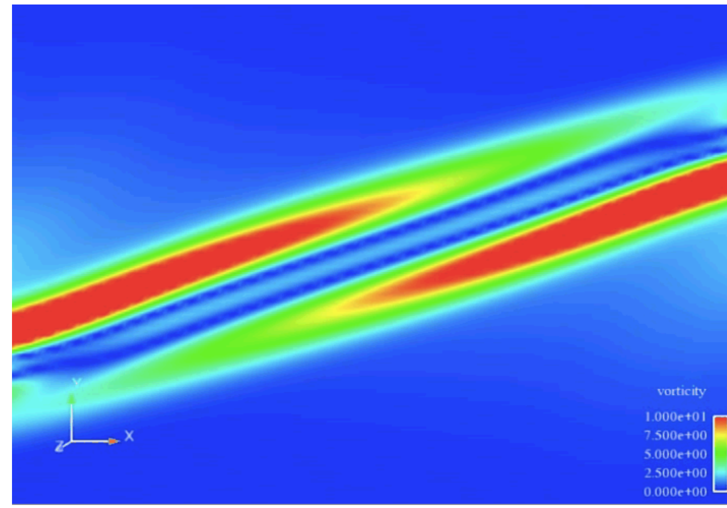
$$D - YF^{-1}Z \approx$$

$$Q_a D^{-1} \left(DQ_a^{-1}D - YQ_u^{-1}Z \right)$$

Results: Kelvin-Helmholtz



Roll up from shear instability with no magnetic field



Shear instability stabilized by a x-direction magnetic field

Results details

- Reynolds number: 10^3
- Lundquist number: 10^4
- Inflow velocity: +1 upper left, -1 lower right
- Run on 1, 4, 16, 64, and 256 processors (80,000 unks/core)

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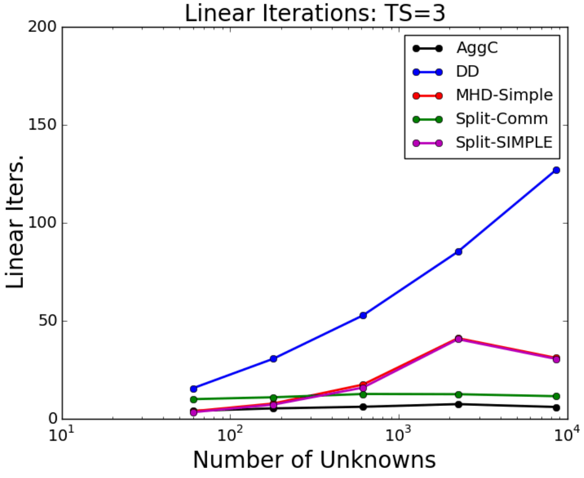
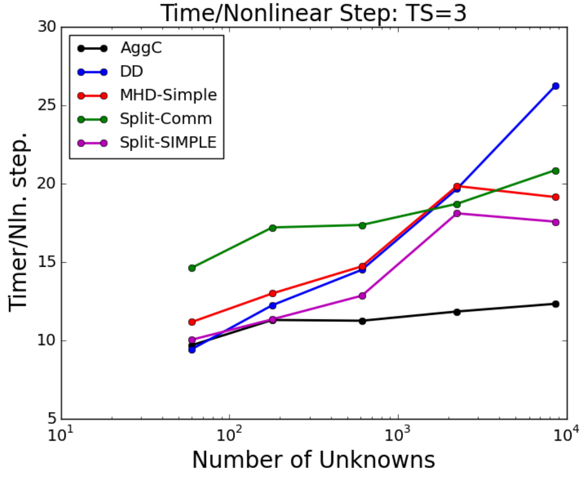
Results: Kelvin-Helmholtz





Take home:

1. Split preconditioner has flat iterations with weak scaling (so does aggressive coarsening)
2. SIMPLE based preconditioners performance suffers with increased CFL
3. Run times are for unoptimized code

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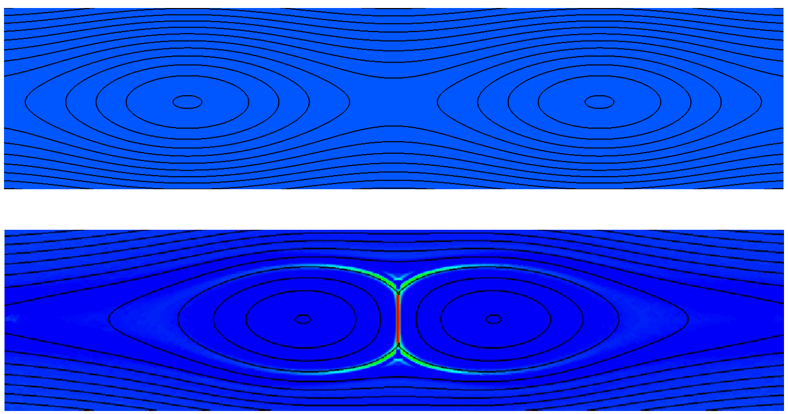
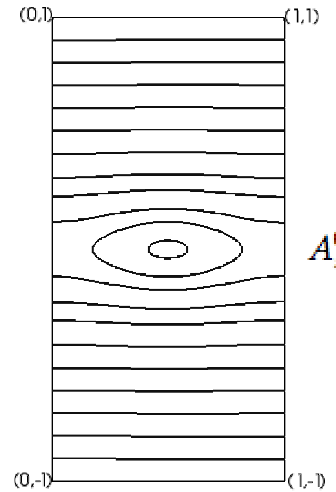
Results: Island Coalescence





Simulation on half domain

- Symmetry BC
- Perturbed Harris-Sheet

$$A_z^0(x,y,0) = \delta \; ln \left[\cosh\left(\frac{y}{\delta}\right) + \epsilon \; \cos\left(\frac{x}{\delta}\right)\right]$$

Results details

- Reynolds number and Reynolds number: 10^4
- Starting time right before reconnection: 5.75s
- Results averaged over 45 uniform timesteps
- Run on 1, 4, 16, 64, 256, and 1024 processors (130000 unks/core)

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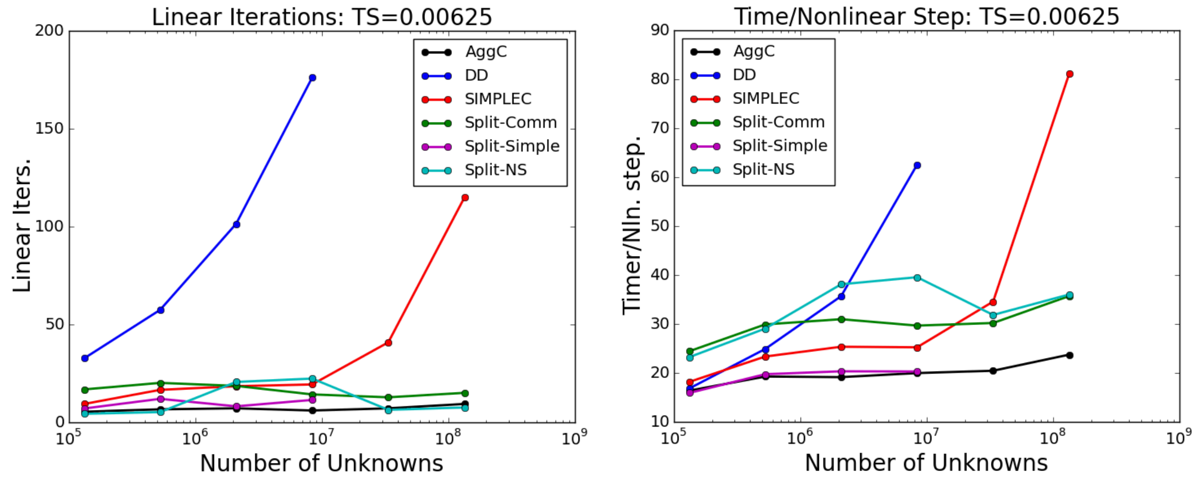
Results: Island Coalescence

ts = 0.00625

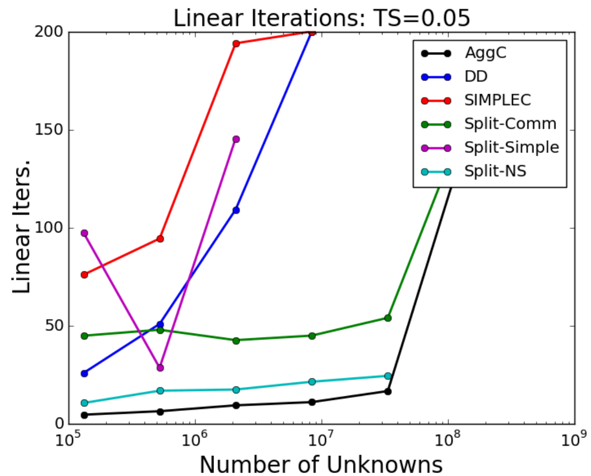


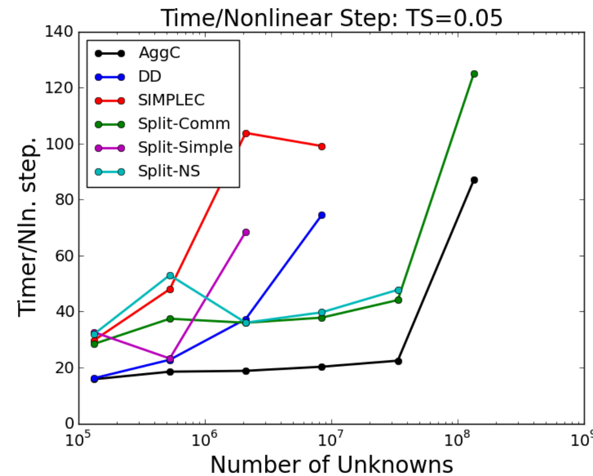
Take home:

1. Split preconditioner has flat iterations with weak scaling (so does aggressive coarsening)
2. SIMPLE based preconditioners performance suffers with increased CFL
3. Run times are for unoptimized code

Results: Island Coalescence

ts = 0.05





Take home:

1. Split preconditioner still has flat iterations with weak scaling (so does aggressive coarsening)
2. SIMPLE based preconditioners performance suffers with increased CFL
3. Problem! For this large time step multigrid solvers breaking down on magnetics Schur complement

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Conclusions

Demonstrated block factorization preconditioners MHD

- Performance not optimized, however results are encouraging
- 3x3 block system has nested Schur complements
- Uses operator splitting approach
  - Separates fluid and magnetics couplings
  - Preconditioner is (structurally) small perturbation of original operator
  - Requires approximating inverse action of two 2x2 operators
  - Weak scaling for fixed timestep indicates scalability with respect to mesh size
- Explored usage of SIMPLEC preconditioner
  - Strong dependence on CFL number
  - Extreme approximations using diagonal approximations

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