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Calculating Damping from Ring-Down Using Hilbert Transform and Curve Fitting

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ABSTRACT: A cantilever beam is released from an initial condition. The velocity at the tip is recorded using a laser Doppler vibrometer. The ring-down time history is analyzed using Hilbert transform, which gives the natural frequency and damping through time-differentiation. An important issue with the Hilbert transform is vulnerability to noise. Thus, curve fitting is often necessary to suppress noise. Linear curve fitting gives very good results for linear beams with low damping. For nonlinear beams with higher damping, polynomial curve fitting captures the time variations. The method was used for estimating quality factors of a few shim metals and PZT bimorphs.

1 INTRODUCTION

Damping plays an important role in structural vibration. For example, damping determines how much energy is dissipated as heat when a PZT bimorph is used as a vibration energy harvester. This paper presents a method for obtaining damping and oscillation frequency in a single-degree-of-freedom (SDOF) oscillator from its free-decaying ring-down. The natural frequency and damping could be obtained by measuring the time lapse and the logarithmic decrement between consecutive peaks of the trace of free-decay time history (Inman 2001). In practical response histories, however, the peaks may not be captured exactly. For example, a peak may appear higher than its preceding peak because of nonlinearity, sampling, noise, and various other conditions, resulting in erroneous negative damping. The method presented in this paper is based on the Hilbert transform. This method is more robust and considerably easier to apply than previous methods. For example, the method in Feldman (1997) is much more involved and appears susceptible to experimental noise, especially because of double time differentiation of the test data. The method discussed here overcomes the noise problem by replacing single time differentiation with least-squares curve fitting. In the past, a similar method has been applied to a MEMS oscillator with squeeze-film damping (Sumali 2007). The method in this paper extends that previous method by capturing nonlinearity with curve fitting. In both applications, the ring-down responses were measured with a laser Doppler vibrometer, which avoided contact and loading.

The method was used for measuring damping of linear cantilever beams made of sheets of a few different metals. A purpose of the tests was to select the metal with the lowest damping for use as shim material in the fabrication of piezoelectric bimorph beams for generating voltage from vibration. Another purpose of the tests was to get an idea of how much damping (hence energy dissipated as heat) was typical of the PZT bimorphs. For estimating damping of the slightly nonlinear bimorph beams with rather high damping, the method uses polynomial curve fitting instead of linear curve fitting. The resulting natural frequency and damping become functions of time or amplitude.

2 THEORY

2.1 Linear analysis

For an underdamped linear SDOF oscillator, the free decaying velocity response to an initial condition is

$$v(t) = A(t)\cos(\omega_d t + \phi_0). \tag{1}$$

where ω_d is the oscillation frequency, and ϕ_0 is initial phase. $A(t)$ is the decay envelope. As a function of time,

$$A(t) = A_0 \exp(-\zeta\omega_n t), \tag{2}$$

where A_0 is the initial amplitude, ζ is the damping ratio and ω_n is the natural frequency. To obtain the decay envelope from the free response signal, Hilbert transformation augments the above real response signal $v(t)$ into a complex form, called the analytic representation, defined as

$$V(t) = v(t) + \mathrm{j}\tilde{v}(t), \tag{3}$$

where the imaginary part $\tilde{v}(t)$ is generated by the Hilbert transform, and $\mathrm{j} = \sqrt{-1}$. The magnitude of the above complex form is the envelope of the real signal, i.e.

$$|V(t)| = A(t). \tag{4}$$

The logarithm of Eq. (2) represents a straight line. Thus, the decay rate $\zeta\omega_n$ of Eq. (2) can be obtained from the decay envelope $|V(t)|$ measured at $t = t_0, t_1, \ldots, t_{N-1}$, N = number of data points, by finding the least-squares solution to

$$\begin{Bmatrix} \ln|V(t_0)| \\ \ln|V(t_1)| \\ \vdots \\ \ln|V(t_{N-1})| \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} C \\ \ln(A_0) \end{Bmatrix}. \tag{5}$$

Thus, the above linear regression analysis gives the decay rate

$$\zeta\omega_n = -C. \tag{6}$$

The analytic representation in Eq. (3) is a rotating phasor. The length $|V(t)|$ decays at the above rate $\zeta\omega_n$. The phase ϕ grows with time. For an oscillation with a frequency ω_d,

$$\phi(t) = \phi_0 + \omega_d t. \tag{7}$$

That phase can be obtained from the analytic form of the measured signal (Eq. (3)) by

$$\phi(t) = \tan^{-1}\left(\tilde{v}(t) / v(t)\right). \tag{8}$$

In principle, the damped oscillation frequency ω_d can be obtained numerically by time-differentiation of the phase in Eq. (7),

$$\omega_d = d\phi(t)/dt. \tag{9}$$

However, in many cases noise will overwhelm the result. A better method to obtain the oscillation frequency is by finding the least-squares solution to

$$\begin{Bmatrix} \phi(t_0) \\ \phi(t_1) \\ \vdots \\ \phi(t_{N-1}) \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} \omega_d \\ \phi_0 \end{Bmatrix}. \tag{10}$$

The above linear regression analysis gives the damped oscillation frequency ω_d. To obtain the undamped natural frequency, use C from Eq. (6) and the relationship

$$\omega_d = \omega_n \sqrt{1-\zeta^2}, \tag{11}$$

which give

$$\omega_n = \sqrt{\omega_d^2 + C^2}. \tag{12}$$

With the natural frequency ω_n known, the damping ratio ζ can be obtained from Eq. (6).

$$\zeta = -C/\omega_n. \tag{13}$$

2.2 Nonlinear analysis

If the response is nonlinear, the damped oscillation frequency in Eq. (9) may be time-varying. Time-differentiation of the phase in Eq. (7) may give the frequency as a function of time. However, practical measured signals usually have noisy phase. Time differentiation will give a frequency function that is dominated by noise. To avoid that problem, we propose an approach where the phase is first curve-fit into a polynomial of sufficient degree P:

$$\begin{Bmatrix} \hat{\phi}(t_0) \\ \hat{\phi}(t_1) \\ \vdots \\ \hat{\phi}(t_{N-1}) \end{Bmatrix} = \begin{bmatrix} t_0^P & \cdots & t_0 & 1 \\ t_1^P & \cdots & t_1 & 1 \\ \vdots & \cdots & \vdots & 1 \\ t_{N-1}^P & \cdots & t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} b_P \\ \vdots \\ b_1 \\ b_0 \end{Bmatrix}. \tag{14}$$

The polynomial coefficients b_P, $p = 0, ..., P$, can be obtained by polynomial regression analysis similar to Eq. (10). Then, according to Eq. (7), the time-varying damped oscillation frequency can be estimated as the time-differential of Eq. (14), i.e.

$$\hat{\omega}_d(t) = \frac{d\hat{\phi}(t)}{dt} = \begin{bmatrix} Pt_0^{\ P-1} & \cdots & 1 & 0 \\ Pt_1^{\ P-1} & \cdots & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ Pt_{N-1}^{\ P-1} & \cdots & 1 & 0 \end{bmatrix} \begin{Bmatrix} b_P \\ \vdots \\ b_1 \\ b_0 \end{Bmatrix}. \quad (15)$$

Because the response is nonlinear, the logarithm of the decay envelope does not trace a straight line with time. Instead of the linear function in Eq. (5), the logarithmic decay envelope can be better estimated with a cubic function with the coefficients from the least-squares solution to the equation

$$\begin{Bmatrix} \ln|V(t_0)| \\ \ln|V(t_1)| \\ \vdots \\ \ln|V(t_{N-1})| \end{Bmatrix} = \begin{bmatrix} t_0^{\ 3} & t_0^{\ 2} & t_0 & 1 \\ t_1^{\ 3} & t_1^{\ 2} & t_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ t_{N-1}^{\ 3} & t_{N-1}^{\ 2} & t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} c_3 \\ c_2 \\ c_1 \\ \ln(A_0) \end{Bmatrix}. \quad (16)$$

Now the above cubic regression analysis gives a nonlinearly decaying envelope

$$\hat{A}(t) = A_0 \exp\left(-c_1 t - c_2 t^2 - c_3 t^3\right), \quad (17)$$

which can be written as Eq. (2) if

$$\zeta(t)\omega_n(t) \equiv C(t) = \left(c_1 + c_2 t + c_3 t^2\right). \quad (18)$$

The time-varying natural frequency $\omega_\mathrm{n}(t)$ can be computed as in Eq. (12),

$$\omega_\mathrm{n}(t) = \sqrt{\left(\omega_\mathrm{d}(t)\right)^2 + \left(C(t)\right)^2}\ . \quad (19)$$

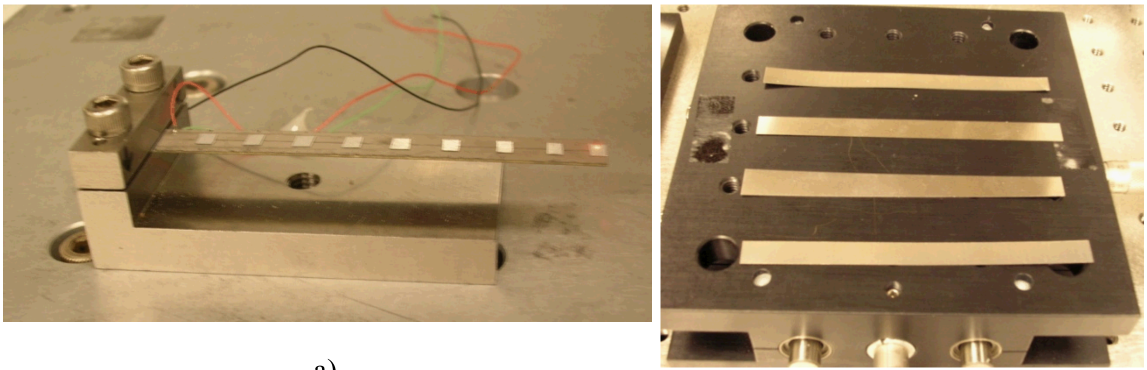
And the time-varying damping ratio $\zeta(t)$ can be computed as in Eq. (13),

$$\zeta(t) = -C(t)/\omega_\mathrm{n}(t). \quad (20)$$

3 EXPERIMENT

3.1 Setup

In the experiment, a beam made of sheet metal is clamped to form a cantilever (Fig. 1a). The beam was initially deflected by pressing on the tip, and then released suddenly. A laser Doppler vibrometer (LDV) measured the velocity at the tip of the cantilever. Several metals were tested for their damping values. Figure 1b shows four 102mm long samples. Also tested were bimorph beams made of two layers of PZT with a thin metal shim sandwiched in between, as shown in Fig. 1a. (In this test, the cables shown in the figure were not connected. The square patches were thin metal foils for improving reflectivity for the LDV laser beam.)

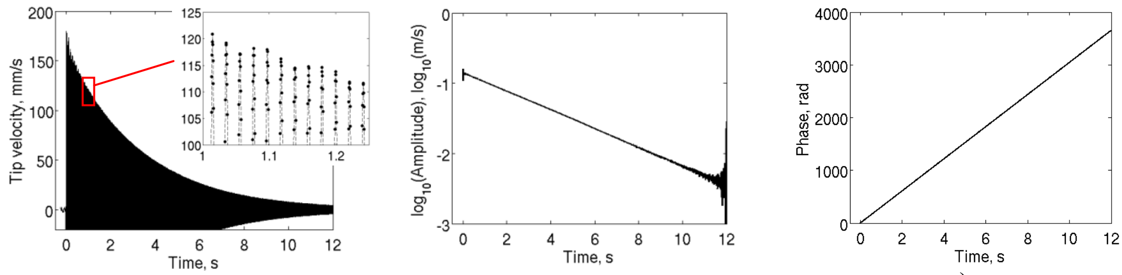


a) b)

Figure 1 : a) Cantilever with LDV laser beam at the tip. b) Thin beams of four different metals.

3.2 Processing of Linear Data

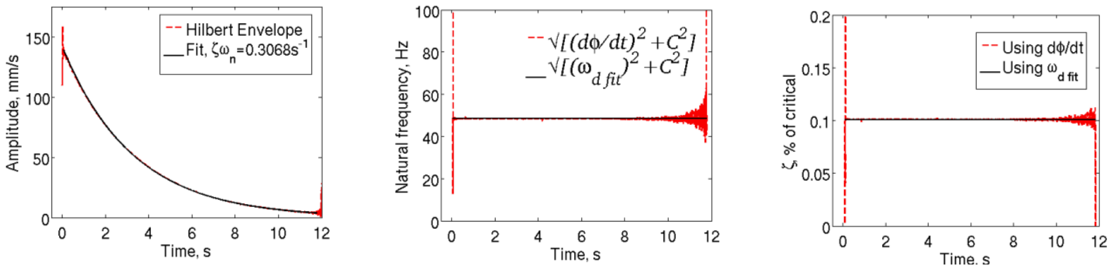
Tip velocity measurement of a monel cantilever resulted in a typical time-decaying velocity versus time curve in Fig. 2a. Fourier transform of the velocity signal (not shown here) showed that the ring-down response was linear and practically single-mode. Responses at higher modal frequencies and harmonics were over two orders of magnitude lower than the fundamental. The cantilever response is dominated by a single-degree-of-freedom system because the static shape of the initial condition closely resembles the first mode shape. Figure 2b shows the decaying amplitude (envelope) of the velocity, from Eq. (4). The log scale of the y-axis shows a straight line because the decay was linear. Figure 2c shows the phase of the velocity, from Eq. (8).



a) b) c)

Figure 2: Time-decaying velocity of monel cantilever tip versus time:
a) History, b) Log amplitude, c) Phase.

Figure 3a shows the decay rate from the Hilbert-transform amplitude of the measured signal (dashed line with noise at beginning and end), compared to $A_0 \exp(-\zeta\omega_n t)$ with the parameters from the curve fitting in Eq. (5) (solid line). The two lines practically overlay, except for very short times at the beginning and the end. Figure 3b shows the (undamped) natural frequency from Eq. (12). The dashed line shows the result of using ω_d from numerical time-differentiation of the phase (Eq. (9)). The noise at the beginning of the measurement is large, but lasts for a very short time. The noise after 10s is understandable since the measured velocity was very small. The solid line shows the result of using ω_d from the curve fitting in Eq. (10), which appears to be very robust against noise. Figure 3c shows the damping ratio ζ from Eq. (14). The dashed line is obtained using ω_d from numerical time-differentiation of the phase. The solid line uses ω_d from curve fitting.

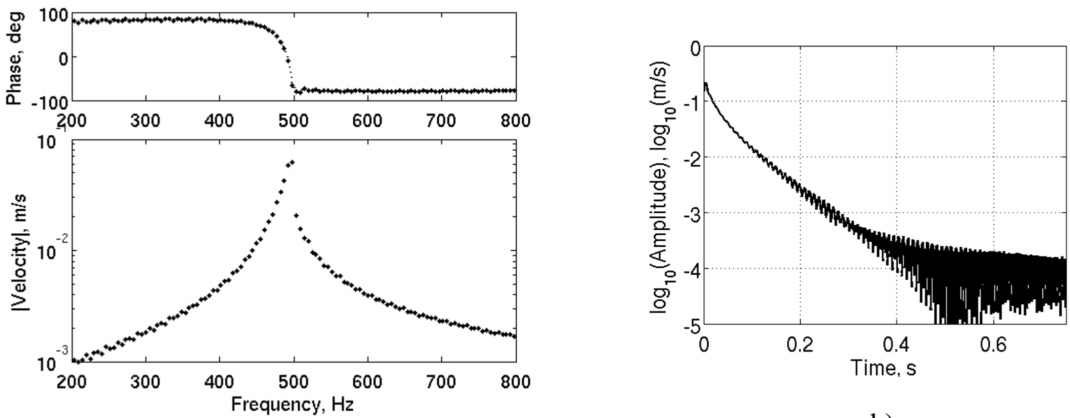


a) b) c)

Figure 3: Linear fit results: a) Envelope, b) Frequency, c) Damping.

3.3 *Processing of Nonlinear Data*

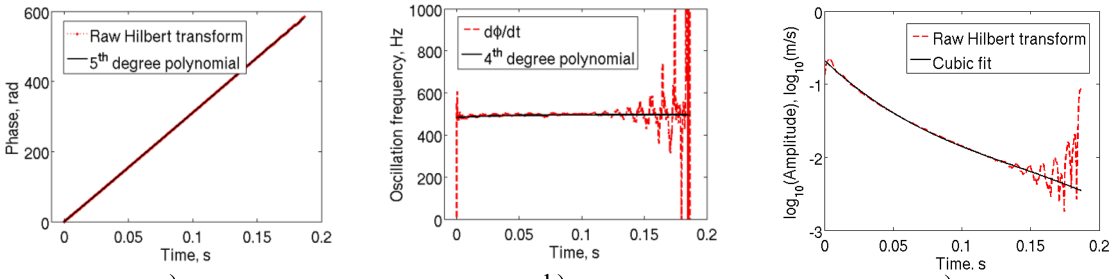
Data from the bimorphs were considerably more difficult to process than the linear data discussed above. The following are factors that complicate the computation of natural frequency and damping by any method. The first and most important problem is nonlinearity. Figure 4a shows that the FFT of the response tilts to the right, indicating nonlinearity. That means that the natural frequency and/or damping are not constant as in linear systems. The second problem is high decay rate $\zeta\omega_n$. Figure 4b shows that a bimorph's response died out much faster than a metal beam's (Fig. 2b). The high high decay rate $\zeta\omega_n$ shortens the usable part of the envelope by a factor of 40 compared to the envelope of the monel data (Fig.2b). Being several times thicker, the bimorphs were much stiffer than the metal cantilevers of similar length, hence higher ω_n. Also, the bimorphs have much higher damping because they have several surfaces bonded with adhesives, hence higher ζ. The third problem is noise, as obvious from Fig. 4b. Any time-differentiation of the signal would result in even more overwhelming noise.



a) b)

Figure 4 : Time-decaying velocity of PZT bimorph cantilever tip:
a) Magnitude and phase versus frequency; b) Log amplitude versus time.

Figure 5a shows the phase as obtained by Hilbert transform (dots), and as curve-fit (solid line, from Eq. (14)). Figure 5b shows that time-differentiation of the phase from the Hilbert transform (dashed curve) is very noisy. On the other hand, the algebraic time-differentiation of the curve-fit phase (solid curve, from Eq. (15)) eliminates the noise. Likewise, the amplitude from Hilbert transform becomes very noisy after 0.15s (Fig. 5c, dashed curve); but the curve-fit amplitude (solid curve, from Eq. (16)) eliminates the noise.

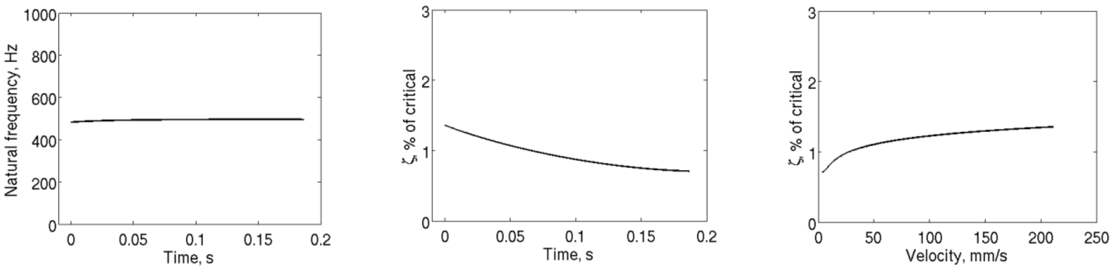


a) b) c)

Figure 5 : Nonlinear fit results: a) Phase, b) Oscillation frequency, c) Amplitude.

For the nonlinear response, the natural frequency and damping are functions of time. The natural frequency from Eq. (19) is shown in Fig. 6a. Figure 6b shows that the damping computed from Eq. (20) decreases with time. Because the amplitude decreases with time, the damping increases with the amplitude, as shown in Fig. 6c.

Three PZT bimorphs were tested and analyzed using the above procedure. Although the damping ratios are not constant, in the next section the damping ratio of each bimorph is given as one number which is the average damping ratio. The quality factor Q is defined as $1/(2\zeta)$.



a) b) c)

Figure 6 : Nonlinear analysis results : a) Natural frequency, b) Damping, c) Damping versus amplitude.

4 RESULT AND COMMENTS

The testing and data processing described above were applied to beams made of brass, bronze, monel, and tantalum sheet metals, a PZT sheet, and three PZT bimorphs. The results are shown in Table 1. From the test samples, the order of sheet metals from low to high damping is: brass from Piezo Systems™, bronze, brass from sheet stock, and monel. All the bimorphs have considerably higher damping than both the metal beams and the PZT402 sheet. Repeating the tests did not result in significantly different results, except for the in-house PZT 5A bimorph which may have undergone some change during the test.

5 CONCLUSIONS

The method presented here has enabled the computation of natural frequency and damping of an SDOF system from its free ring-down history. Noise is a major problem. Curve fitting helps suppress the noise from time-differentiation. The method produced very good results for the linear beams. The nonlinear bimorph beams with higher damping gave very noisy data. In that case, curve fitting is very important. The test results give an idea of how much damping (hence energy dissipated as heat) is typical of PZT bimorphs used for generating voltage from vibration. They also help in selecting the material to be used as the shim metal in fabricating the bimorphs.

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Table 1: Quality factors of tested beams.

Material	Test run #	$Q = 1/(2\zeta)$	f_n, Hz
Brass from sheet stock		580	36.3
Brass from Piezo Systems™	1	640	62.18
Brass from Piezo Systems™	2	640	62.18
Bronze	1	625	37.8
Bronze	2	624	37.8
Monel	1	498	48.4
Monel	2	492	48.5
Monel	3	495	48.4
PZT402 sheet (in-house)	1	357	60.5
PZT402 sheet (in-house)	2	351	60.5
PZT 5A bimorph (in-house)	1	80	377
PZT 5A bimorph (in-house)	2	47	378
PZT 5A bimorph from Piezo Systems™	1	96	499.8
PZT 5A bimorph from Piezo Systems™	2	96	500.7

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