SAND2013-7079C

An Optimal Lower Bound for Monotonicity Testing over Hypergrids

C. Seshadhri
(Sandia National Labs, Livermore)

Joint work with Deeparnab Chakrabarty
(Microsoft Research, Bangalore)

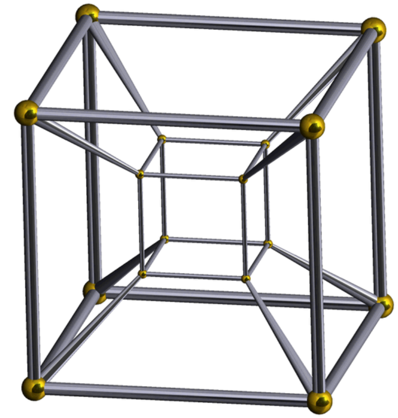
Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000."

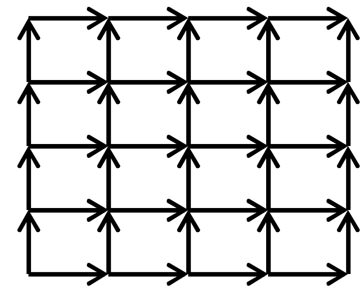
The domain

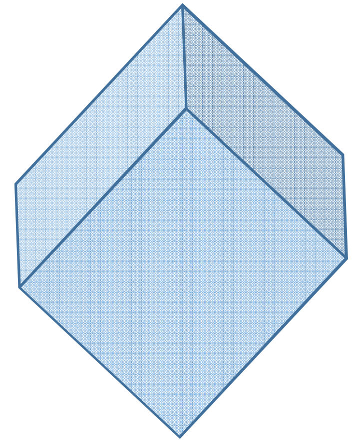
- $f: D \to \boldsymbol{N}$

- **Boolean Hypercube:** $\{0,1\}^d$
- **Hypergrid:** $[n]^d$
  (includes line, tesseract, etc.)

$x \prec y$ iff $x_i \le y_i$ for all i

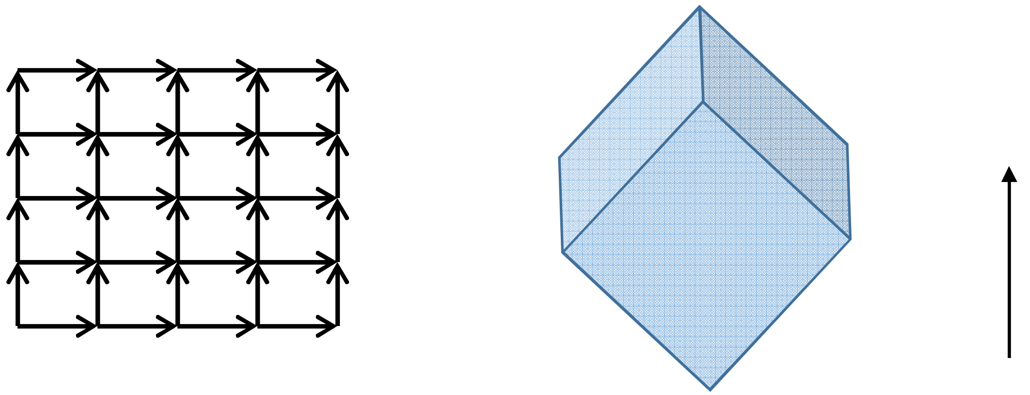






Monotonicity

- $f : [n]^d \to \boldsymbol{N}$
- **Monotonicity**: $f(x) \leq f(y)$ whenever $x \prec y$

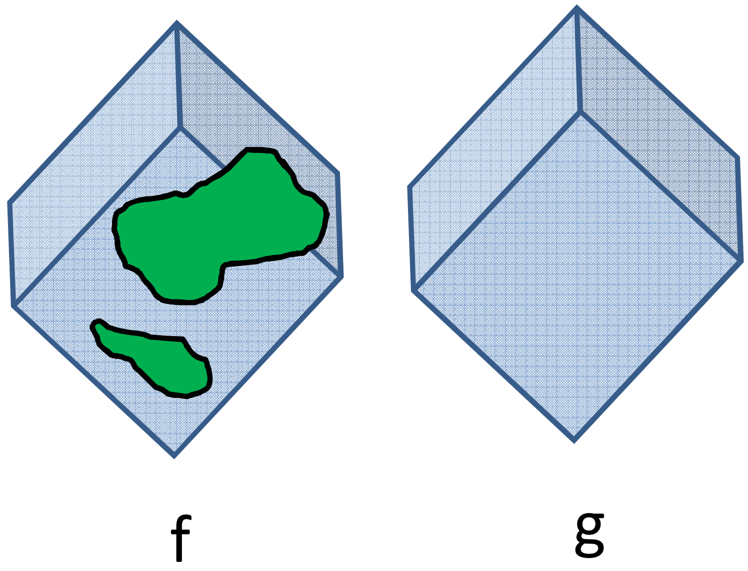


- **Why monotonicity?**: Classic property of interest

Monotonicity Testing

$$dist(f,g) = \frac{|f-g|_0}{|D|}$$

$$dist(f,P) = \min_{g \in P} dist(f,g)$$



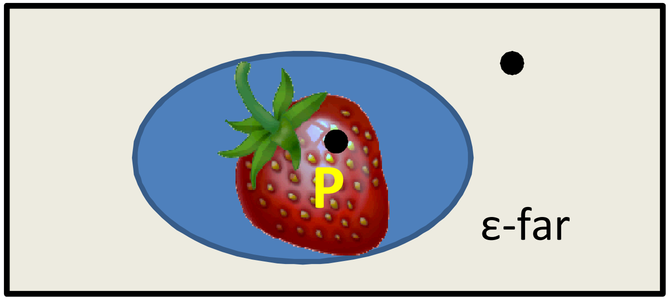
f is ϵ-far from P means $dist(f,P) > \epsilon$

Parameter ϵ, query access to f

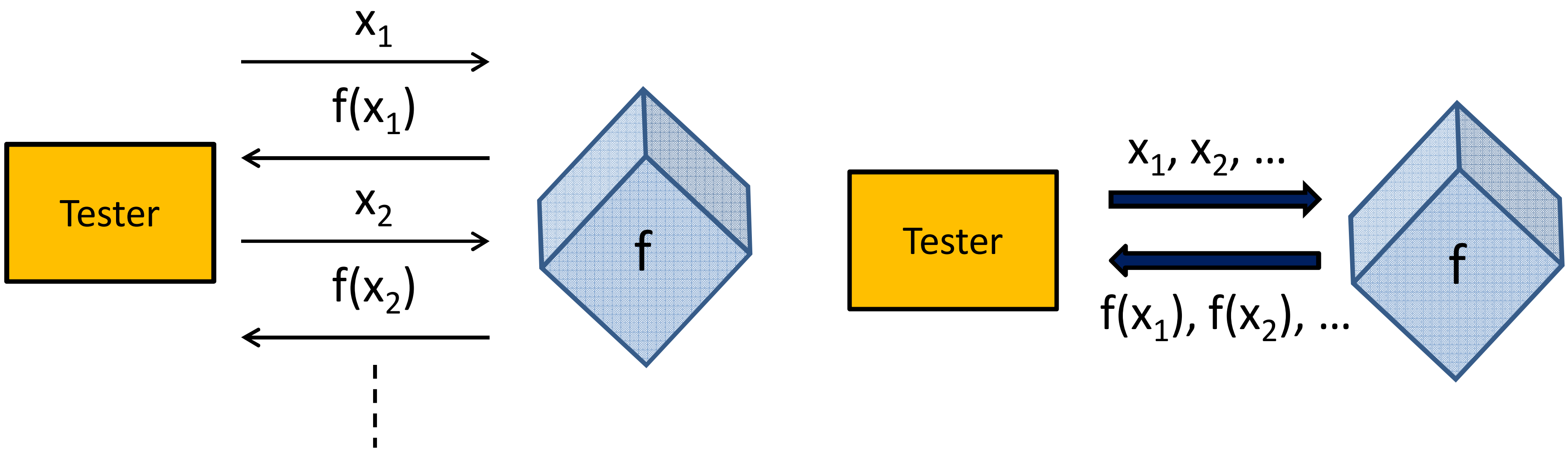
Property tester is randomized algo:

ACCEPT if f has P

REJECT if f is ϵ-far from P

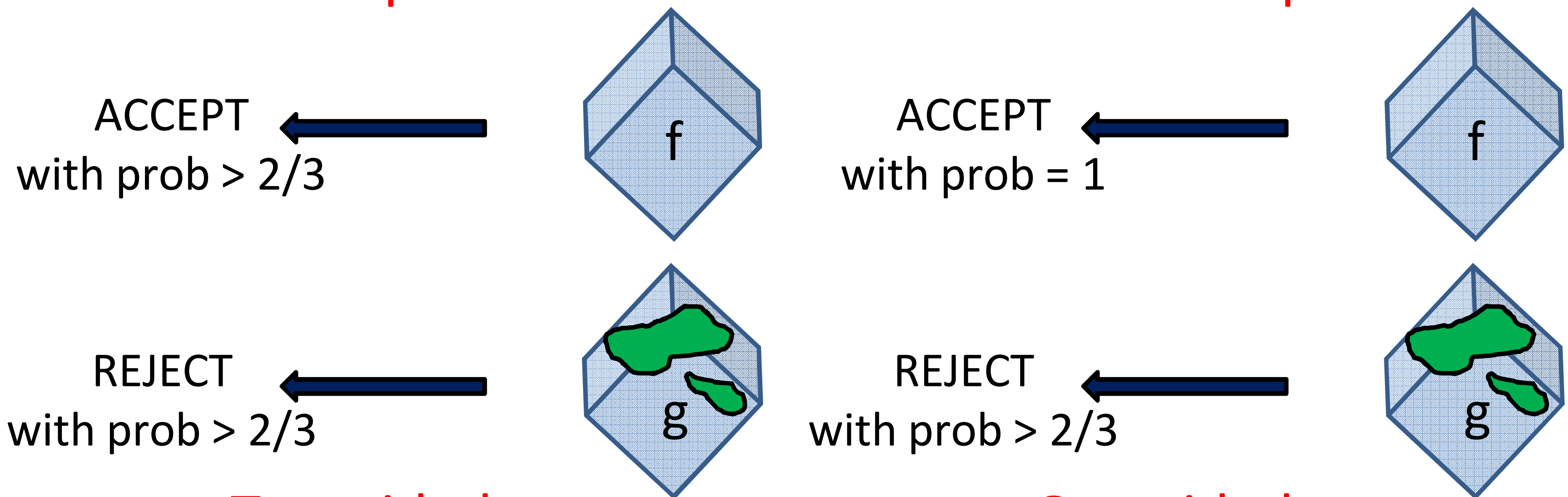


The variants



Adaptive

Non-adaptive



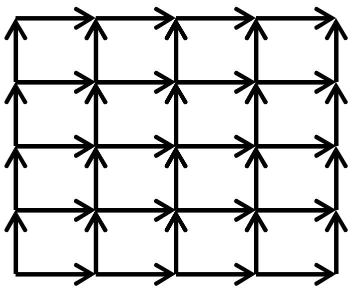
Two-sided error

One-sided error

Goldreich,Goldwasser,Lehman,Ron,Samorodnitsky	$O(\epsilon^{-1}d)$ for Boolean range	1998
Ergun,Kannan,Kumar,Rubinfeld,Viswanathan	$O(\epsilon^{-1}\log n)$ for line	1998
Dodis,Goldreich,Lehman,Raskhodnikova,Ron, Samorodnitsky	$O(\epsilon^{-1}d^2)$ on hypercube $O(\epsilon^{-1}d^2\log^2 n)$ on the hypergrid	1999
Lehman, Ron	Conjectural attack: $O(\epsilon^{-1}d)$ for hypercube	2001
Fischer,Lehman,Newman,Raskhodnikova,Rubinfeld	$\Omega(\sqrt{d})$ lower bnd, *nonadaptive*, Boolean	2002
Halevy, Kushilevitz & Ailon, Chazelle	$O(\epsilon^{-1}2^d\log n)$ on hypergrid	2008
Briet, Chakraborty, Garcia, Matsilah	Disproved Lehman-Ron $\Omega(d)$ lower bnd, *nonadaptive*	2010
Blais, Brody, Matulef	$\Omega(d)$ lower bnd, *adaptive*	2010
Chakrabarty, S.	$O(\epsilon^{-1}n\log k)$ for hypergrid	2013
Blais, Raskhodnikova, Yaroslavtsev	$\Omega(n\log k)$ *nonadaptive* hypergrid	2013

Lower bounds for monotonicity testing

- [Chakrabarty-S. 13]. $f : [n]^d \rightarrow \boldsymbol{N}$. There is a non-adaptive one-sided monotonicity tester that makes $O(\epsilon^{-1} d \log n)$ queries.



- This paper: Any adaptive, two-sided monotonicity tester requires $\Omega(\epsilon^{-1} d \log n - \epsilon^{-1} \log \epsilon^{-1})$ queries

- Best of both worlds! Closes chapter (ok, not quite) on a long line of work

Previously…

- [Blais-Brody-Matulef 11]. $f: \{0,1\}^d \rightarrow [\sqrt{d}]$.

Adaptive, two-sided lower bound of $\Omega(d)$

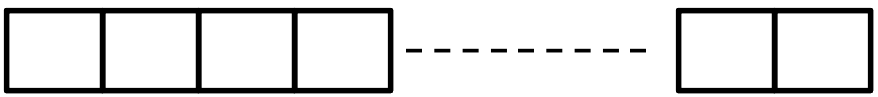
- [Blais-Raskhodnikova-Yaroslavtsev 13]. $f: [n]^d \rightarrow [nd]$.

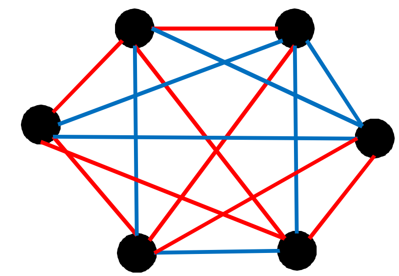
Non-adaptive, two-sided lower bound of $\Omega(d \log n)$

- Use communication complexity lower bounds for Set-Disjointness and Augmented-Index
  - Get nice bounds on ranges

- We get away with a simple self-contained proof
  - (And I do mean simple)

Standing on shoulders

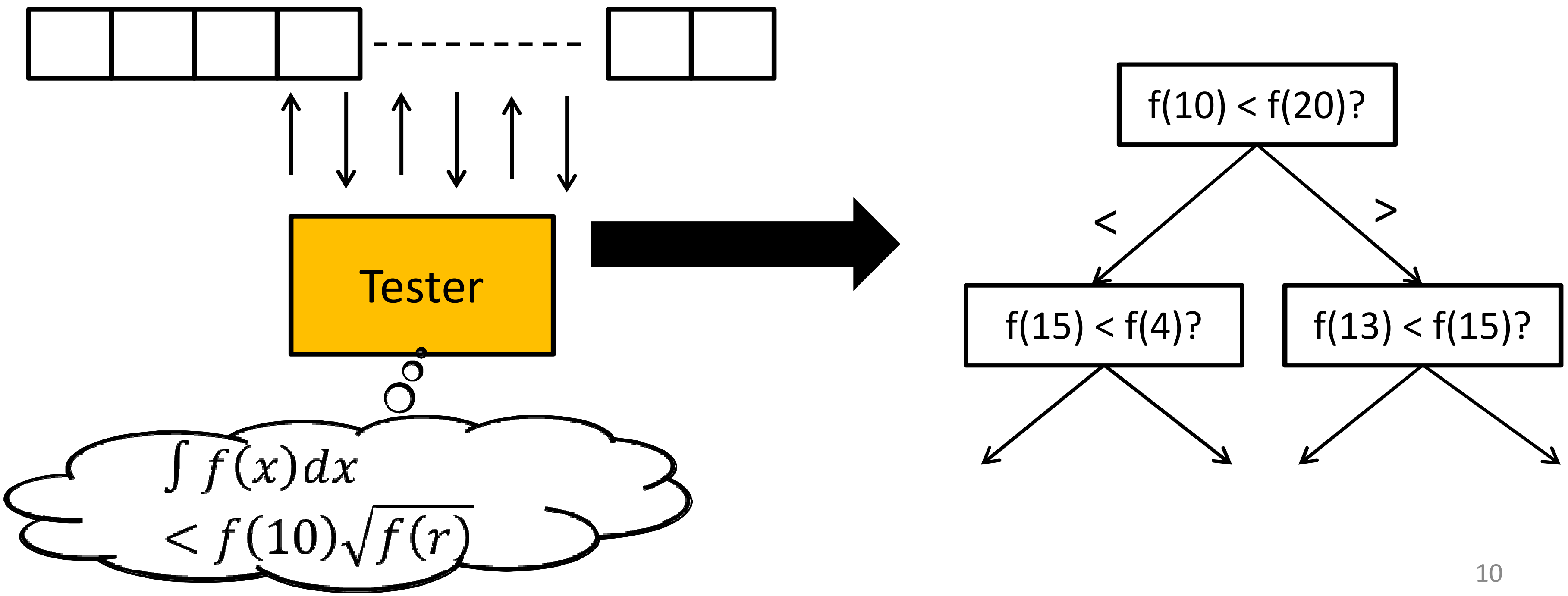




- [Fischer 04]. $f : [n] \to \boldsymbol{N}$. Adaptive l.b. of $\Omega(\log n)$
- Main argument: any monotonicity tester making t queries can be converted to non-adaptive tester making t queries
- [Ergun et al 99] Non-adaptive l.b. of $\Omega(\log n)$

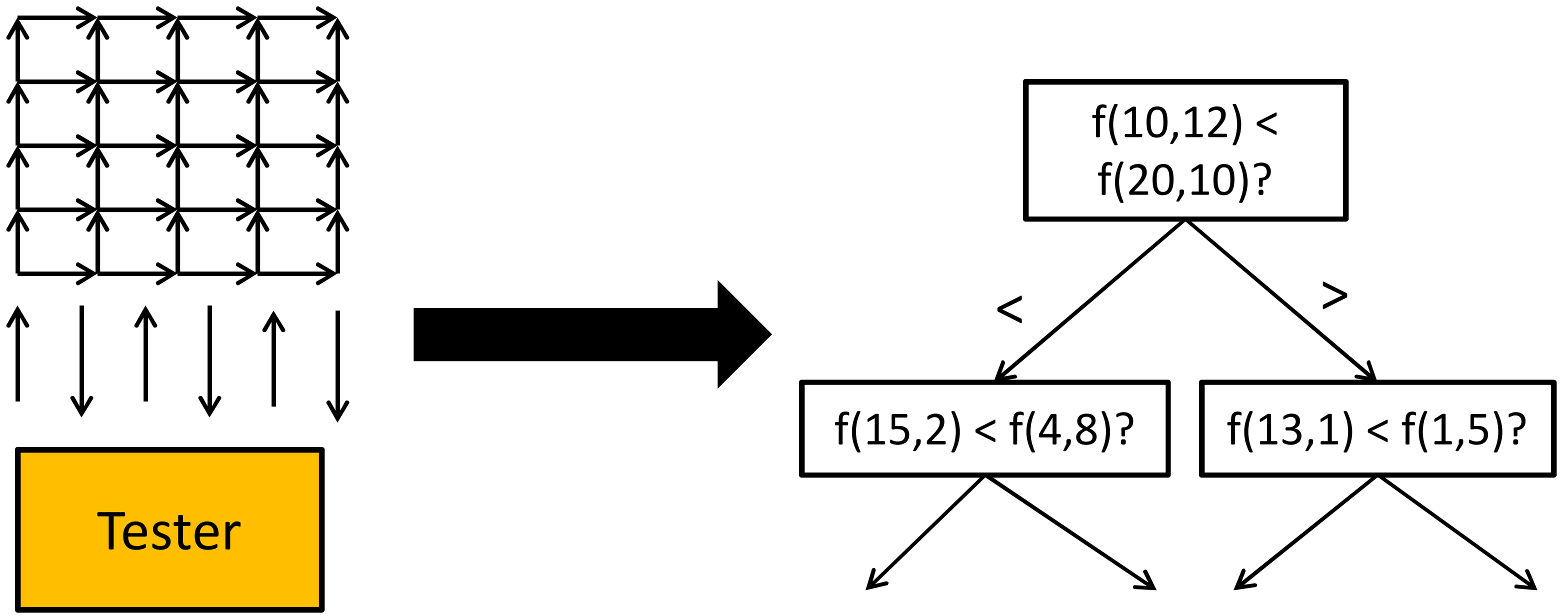
Looking closely

- [Fischer 04]. $f: [n] \to \boldsymbol{N}$. Adaptive l.b. of $\Omega(\log n)$
- Main argument: any monotonicity tester making t queries can be converted to comparison based tester making t queries
  - Comparison based tester is non-adaptive for [n]

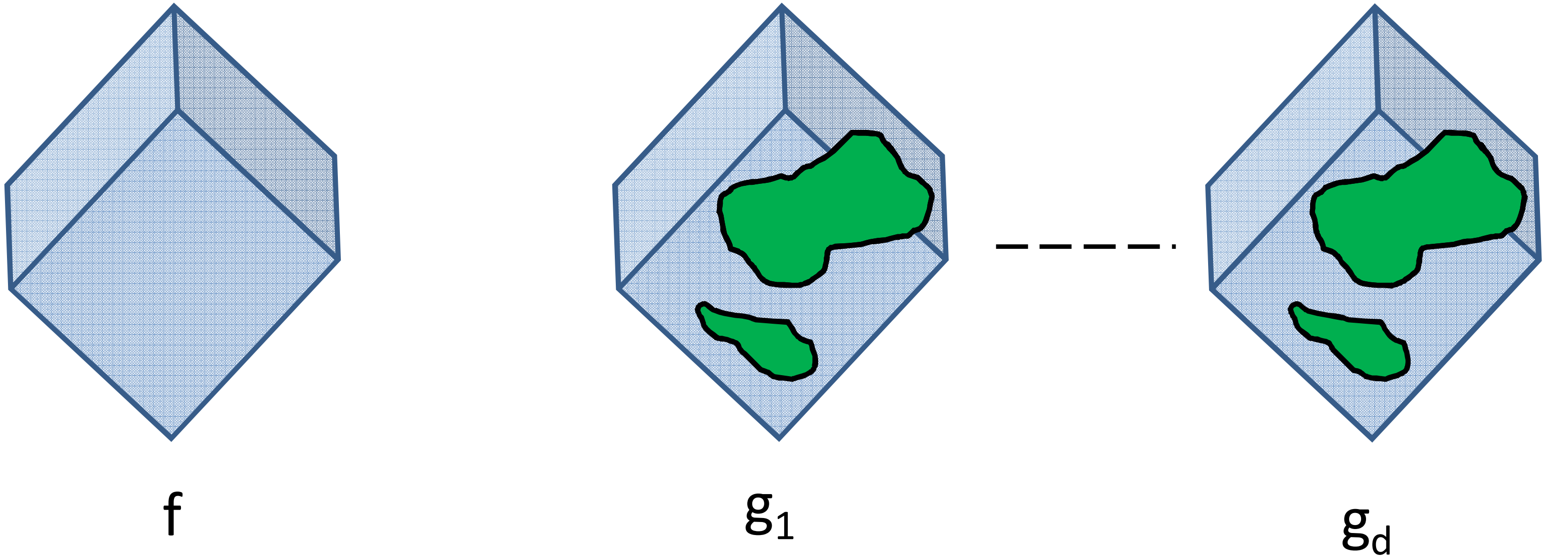


A simple observation

- [We observe] $f: \boldsymbol{D} \rightarrow \boldsymbol{N}$. Any monotonicity tester making t queries can be converted to comparison based tester making t queries
- Just rewrite Fischer's proof, replace [n] by **D**.

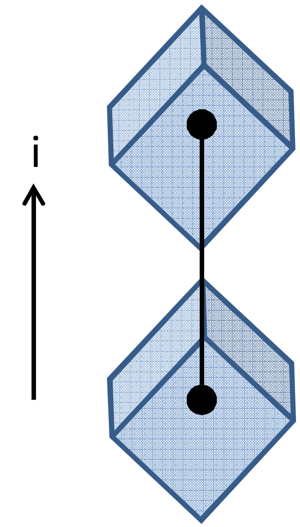


Just comparison-based now



- $f, g_i : \{0,1\}^d \rightarrow \boldsymbol{N}$
- Tester gets f w.p. ½ or u.a.r. g_i w.p. 1/2
- Deterministic comparison tree must tell if it is f or some g with error < 1/4
- Will prove that depth of tree must be $\Omega(d)$

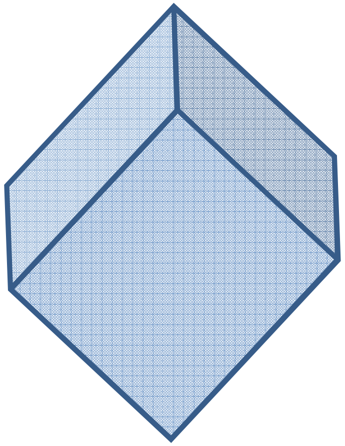
The functions



- $f(x_1, x_2, \ldots, x_d) = 2x_1 + 2^2 x_2 + 2^3 x_3 + \ \ldots$
  - Clearly monotone
- $g_i(x_1, x_2, \ldots, x_d) = 2x_1 + 2^2 x_2 \ \ldots - 2^i x_i + 2^{i+1} x_{i+1} \ldots$
  - Clearly far from monotone
- x,y where largest bit of difference is i: $x_i = 0$, $y_i = 1$
- Then $g_i(x) > g_i(y)$, but $f(x) < f(y)$ and $g_j(x) < g_j(y)$ (for $j \neq r$)
- Any comparison can only "catch" one g_i
- Need d/2 comparisons to catch d/2 such g_i's

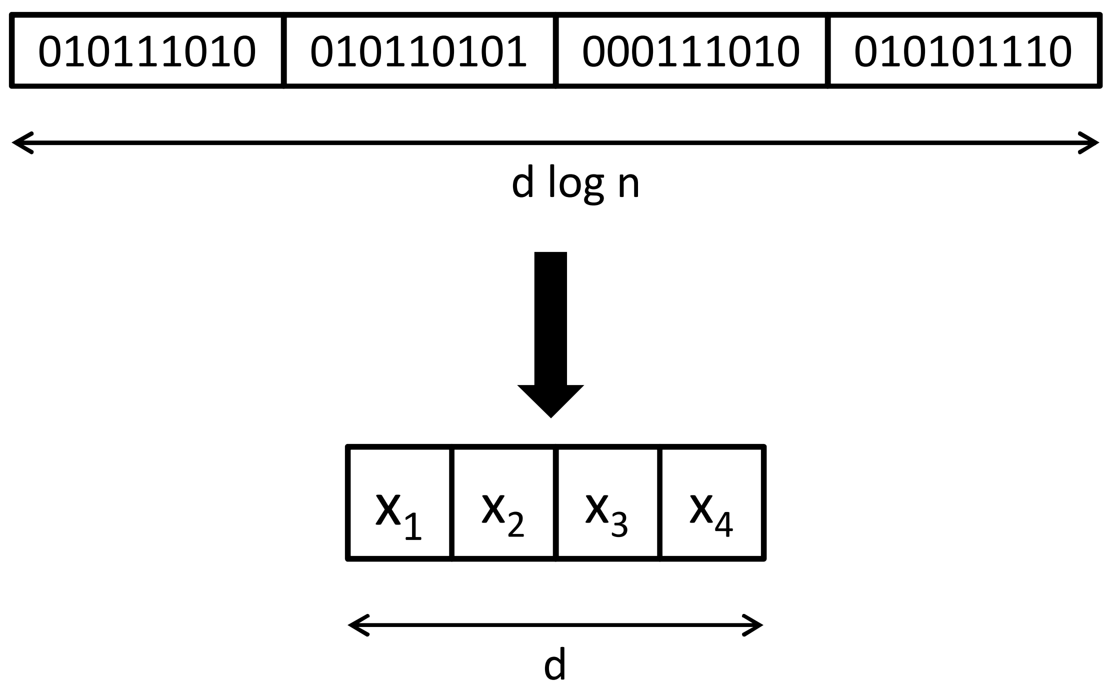
That's it

- $f:\{0,1\}^d \rightarrow \boldsymbol{N}$
- Any adaptive, two-sided tester for monotonicity requires $\Omega(d)$ queries



Hypercube to hypergrid

- Convert $f: \{0,1\}^{d \log n} \rightarrow \boldsymbol{N}$ to $f: [n]^d \rightarrow \boldsymbol{N}$



- Our functions are monotone or far from monotone in both worlds

The full theorem

- $f: [n]^d \to \boldsymbol{N}$
- Any adaptive, two-sided tester for monotonicity requires $\Omega(d \log n)$ queries
- Getting $\Omega(\epsilon^{-1} d \log n\ )$ requires some care
  - But same idea

- Comparison-based testers are **really** easy to argue about
  - Don't have to worry about testers being clever

The range issue

- $f:[n]^d \to \boldsymbol{N}$ vs $f:[n]^d \to [k]$
- Reduction to comparison-based requires humungous range for Ramsey's theorem
- But range plays role in monotonicity testing

- [CS 13] $f:\{0,1\}^d \to \{0,1\}$. Tester with $d^{7/8}$ queries

- [BBM 11] When range > $\sqrt{d}$, l.b. of $\Omega(d)$
- [BRY13] Hypergrid setting. When range > nd, l.b. of $\Omega(d \log n)$ (but non-adaptive)

General posets

- $f: D \to \boldsymbol{N}$
- [Fischer et al 02] There exists D, such that non-adaptive lower bound of $\Omega(D^{1/\log\log D})$
- No D known where adaptive lower bound better than $\Omega(\log D)$

- But we only need to prove comparison-based lower bounds. Surely, something can be done?