Stable and Efficient Galerkin Reduced Order Models for Non-Linear Fluid Flow

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CFD modeling of unsteady 3D flows is expensive!

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A **Reduced Order Model (ROM)** is a surrogate numerical model that aims to capture the essential dynamics of a full model but with far fewer dofs.

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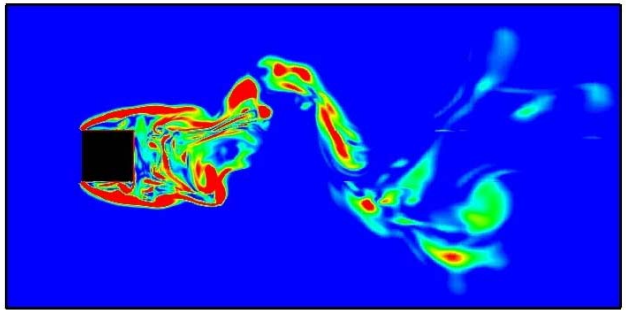

Applications in Fluid Dynamics:

- Predictive modeling across a parameter space (e.g., aeroelastic flutter analysis).
- System modeling for active flow control.
- Long-time unsteady flow analysis, e.g., fatigue of a wind turbine blade under variable wind conditions.

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Motivation for Numerical Analysis of ROMs

Use of ROMs in predictive applications raises questions about their stability & convergence.

- Projection ROM approach is an alternative discretization of the governing PDEs.
- Desired numerical properties of a ROM discretization:
  - **Consistency** (with continuous PDEs):
  - **Stability**:
  - **Convergence**: requires consistency and stability.

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This talk focuses on how to construct a Galerkin ROM that is **stable** *a priori*

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Outline

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Model Reduction Approach

High-Fidelity CFD Simulations:

- Snapshot 1
- Snapshot 2
- ⋮
- Snapshot K

Step 1

Fluid Modal Decomposition (POD):

$$\mathbf{u}_M = \sum_{k=1}^{M} a_k(t)\boldsymbol{\phi}_k(\mathbf{x})$$

Step 2

Galerkin Projection of Fluid PDEs:

$$(\boldsymbol{\phi}_j, \dot{\mathbf{u}}_M + \nabla \cdot \mathbf{F}(\mathbf{u}_M)) = 0$$

"Small" ROM ODE System:

$$\dot{a}_k = f(a_1, \ldots, a_M)$$

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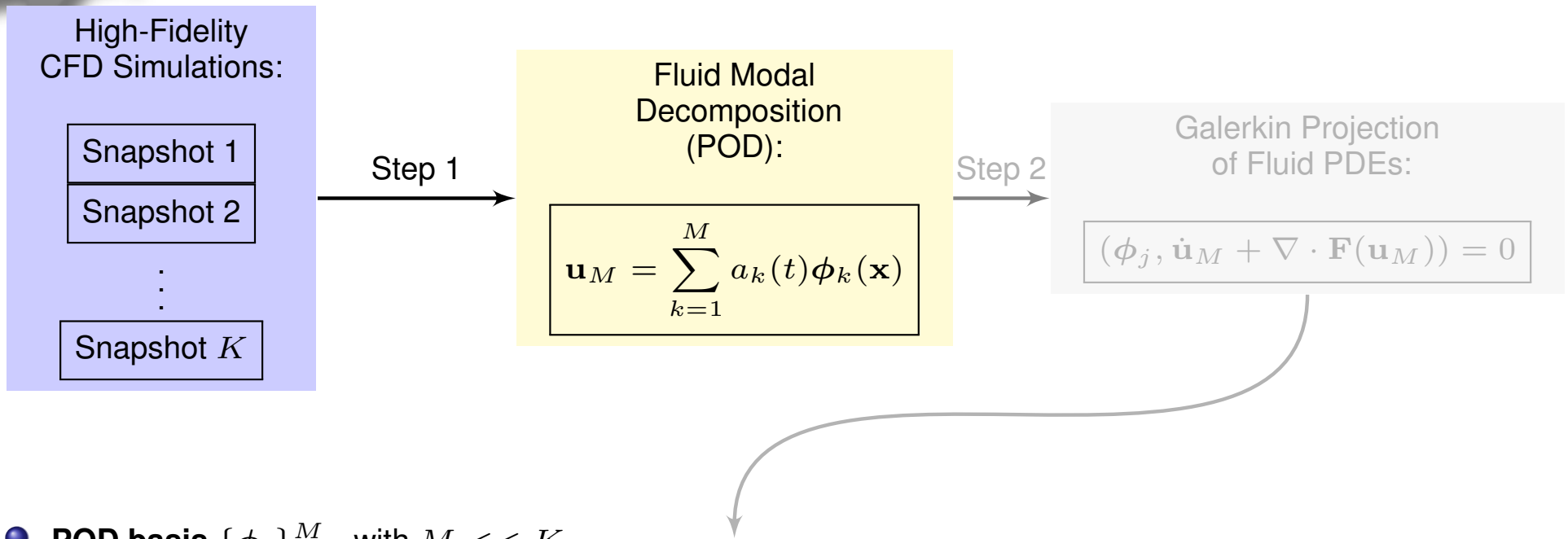
Step 1: Constructing the Modes



- **POD basis** $\{\boldsymbol{\phi}_i\}_{i=1}^{M}$ with $M << K$ maximizes the energy in the projection of snapshots onto span$\{\boldsymbol{\phi}_i\}$.
- **POD eigenvalue problem**:

$$\mathbf{R}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$$

where $\mathbf{R}\boldsymbol{\phi} \equiv \langle \mathbf{u}^k(\mathbf{u}^k, \boldsymbol{\phi})\rangle$.

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Step 2: Galerkin Projection

High-Fidelity CFD Simulations:

Snapshot 1

Snapshot 2

⋮

Snapshot K

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"Small" ROM ODE System:

$$\dot{a}_k = f(a_1, ..., a_M)$$

- Galerkin projection of ***continuous*** equations in ***continuous*** inner product onto reduced basis modes $\{\phi_i\}_{i=1}^{M}$.

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Stability Definitions

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Linearized Compressible Euler Equations:
$\frac{dE}{dt} \leq 0$
Non-increasing energy

$\longleftarrow$ duality $\longrightarrow$

Compressible Navier-Stokes Equations:
$\frac{d}{dt} \int_\Omega \rho \eta d\Omega \geq 0$
Clausius-Duhem Inequality
Non-decreasing entropy

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- Analyzed using the **Energy Method**: Uses an equation for the evolution of numerical solution "energy" (or "entropy") to determine stability.

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3D Linearized Compressible Euler Equations

- Useful for aero-elasticity, aero-acoustics, flow instability analysis.
- Linearization of full compressible Euler equations:

$$\mathbf{q}^T(\mathbf{x},t) \equiv \begin{pmatrix} u_1 & u_2 & u_3 & \zeta & p \end{pmatrix} \equiv \underbrace{\bar{\mathbf{q}}^T(\mathbf{x})}_{\text{mean}} + \underbrace{\mathbf{q}'^T(\mathbf{x},t)}_{\text{fluctuation}} \in \mathbb{R}^5$$

$$\Rightarrow \mathbf{q}'_{,t} + \mathbf{A}_i \mathbf{q}'_{,i} + \mathbf{C}\mathbf{q}' = \mathbf{0}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} \bar{u}_1 & 0 & 0 & 0 & \bar{\zeta} \\ 0 & \bar{u}_1 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_1 & 0 & 0 \\ -\bar{\zeta} & 0 & 0 & \bar{u}_1 & 0 \\ \gamma\bar{p} & 0 & 0 & 0 & \bar{u}_1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \bar{u}_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_2 & 0 & 0 & \bar{\zeta} \\ 0 & 0 & \bar{u}_2 & 0 & 0 \\ 0 & -\bar{\zeta} & 0 & \bar{u}_2 & 0 \\ 0 & \gamma\bar{p} & 0 & 0 & \bar{u}_2 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} \bar{u}_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_3 & 0 & \bar{\zeta} \\ 0 & 0 & -\bar{\zeta} & \bar{u}_3 & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{u}_3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \bar{u}_{1,1} & \partial\bar{u}_{1,2} & \bar{u}_{1,3} & \bar{p}_{,1} & 0 \\ \bar{u}_{2,1} & \bar{u}_{2,2} & \bar{u}_{2,3} & \bar{p}_{,2} & 0 \\ \bar{u}_{3,1} & \bar{u}_{3,2} & \bar{u}_{3,3} & \bar{p}_{,3} & 0 \\ \bar{\zeta}_{,1} & \bar{\zeta}_{,2} & \bar{\zeta}_{,3} & -\nabla\cdot\bar{\mathbf{u}} & 0 \\ \bar{p}_{,1} & \bar{p}_{,2} & \bar{p}_{,3} & \gamma\nabla\cdot\bar{\mathbf{u}} & 0 \end{pmatrix}$$

Symmetrized Compressible Euler Equations & Symmetry Inner Product

Energy stability of the Galerkin ROM can be proven* following a "symmetrization" of the linearized compressible Euler equations.

- Linearized hyperbolic compressible Euler system is "symmetrizable".
- Pre-multiply equations by symmetric positive definite matrix:

$$\mathbf{H} = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & \alpha^2\gamma\bar{\rho}^2\bar{p} & \bar{\rho}\alpha^2 \\ 0 & 0 & 0 & \bar{\rho}\alpha^2 & \frac{1+\alpha^2}{\gamma\bar{p}} \end{pmatrix} \Rightarrow \boxed{\mathbf{H}\mathbf{q}'_{,t} + \mathbf{H}\mathbf{A}_i\,\mathbf{q}'_{,i} + \mathbf{H}\mathbf{C}\mathbf{q}' = \mathbf{0}}$$

- $\mathbf{H}$ is called the "symmetrizer" of the system: $\mathbf{H}\mathbf{A}_i$ are all symmetric.
- Define the "symmetry" inner product and "symmetry" norm:

$$(\mathbf{q}'^{(1)}, \mathbf{q}'^{(2)})_{(\mathbf{H},\Omega)} \equiv \int_\Omega [\mathbf{q}'^{(1)}]^T \mathbf{H}\mathbf{q}'^{(2)} d\Omega, \qquad ||\mathbf{q}'||_{(\mathbf{H},\Omega)} \equiv (\mathbf{q}', \mathbf{q}')_{(\mathbf{H},\Omega)}$$

* M.F. Barone, D.J. Segalman, H. Thornquist, I. Kalashnikova. "Galerkin Reduced Order Models for Compressible Flow with Structural Interaction". *AIAA Paper No. 2008-0612*, 46th AIAA Aerospace Science Meeting and Exhibit, Reno, NV (Jan. 2008); [2-4].

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A Stable Galerkin ROM

- Stability analysis dictates that we use the symmetry inner product to compute the POD modes and perform the Galerkin projection.

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- Energy estimate: $||\mathbf{q}'_M(\mathbf{x},t)||_{(\mathbf{H},\Omega)} \leq e^{\beta t}||\mathbf{q}'_M(\mathbf{x},0)||_{(\mathbf{H},\Omega)}$.

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Practical Implication:
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- **Stability-Preserving Discrete Implementation:**
  - Define snapshots and POD modes using piecewise smooth finite elements.
  - Apply Gauss quadrature rules of sufficient accuracy to compute exactly inner products.
  - Fairly general, works for any nodal mesh that can be represented using finite elements.

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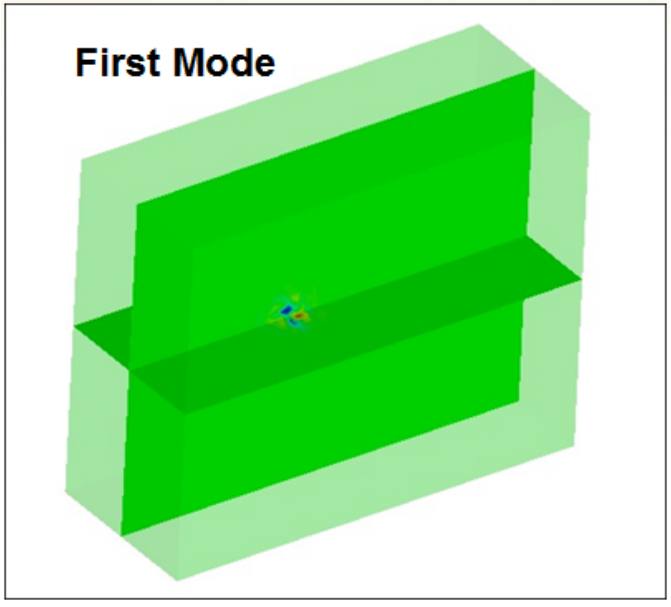
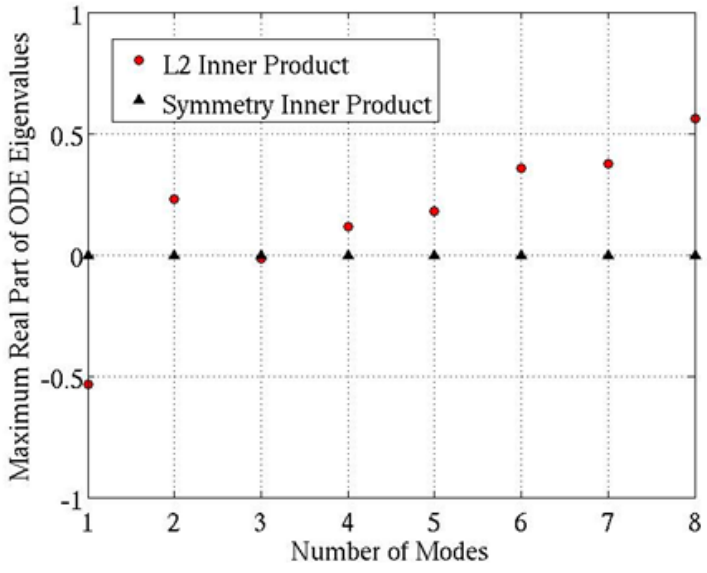
A computer code was written that reads in the snapshot data written by AERO-F*, assembles the necessary finite element representation of the snapshots, computes the numerical quadrature for evaluation of the inner products, and projects the equations onto the modes.

* AERO-F is an arbitrary Lagrangian–Eulerian code that can be used for high-fidelity aeroelastic analysis (Lieu, Farhat *et al.*).

Numerical Study 1: Purely Random Basis





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- Each mode is a random disturbance field that decays to 0 at the domain boundaries.
- Model problem for modes dominated by numerical error: extreme case of “bad” modes.

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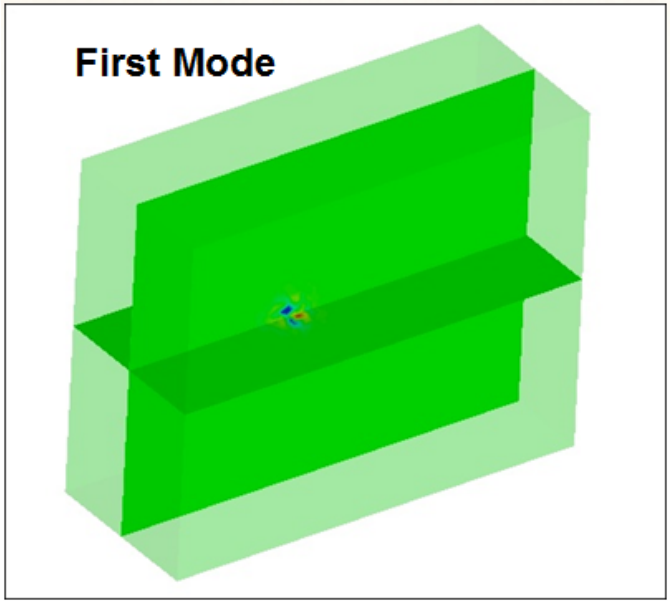

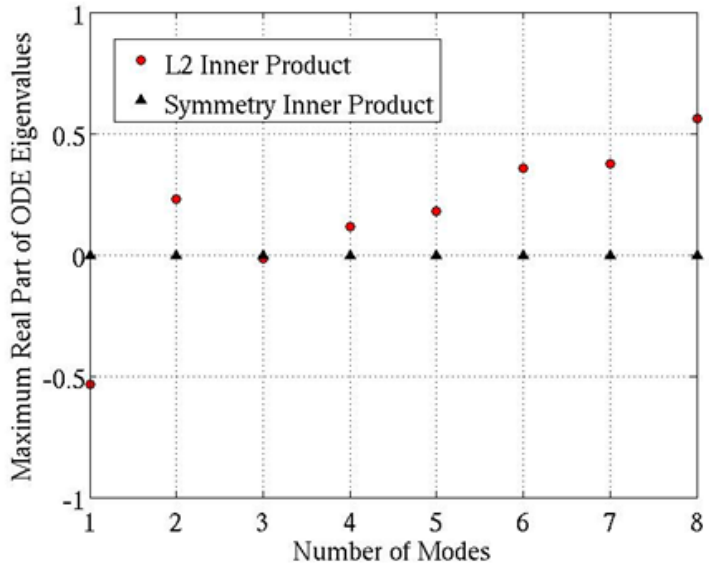

- Uniform base flow: physically stable to any linear disturbance.
- Each mode is a random disturbance field that decays to 0 at the domain boundaries.
- Model problem for modes dominated by numerical error: extreme case of “bad” modes.
- To test *a posteriori* the **stability** of a ROM dynamical system $\dot{\mathbf{a}}_M = \mathbf{K}\mathbf{a}_M$, check the Lyapunov condition:

$$\max_i \mathcal{R}\{\lambda_i(\mathbf{K})\} \leq 0?$$

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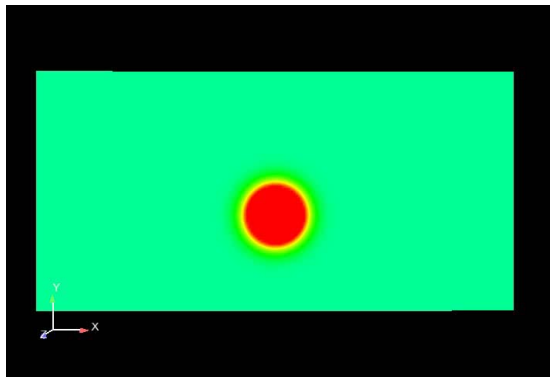
Numerical Study 2: 2D Pressure Pulse

- Reflection of cylindrical Gaussian pressure pulse in uniform base flow, $M_\infty = 0.25$.



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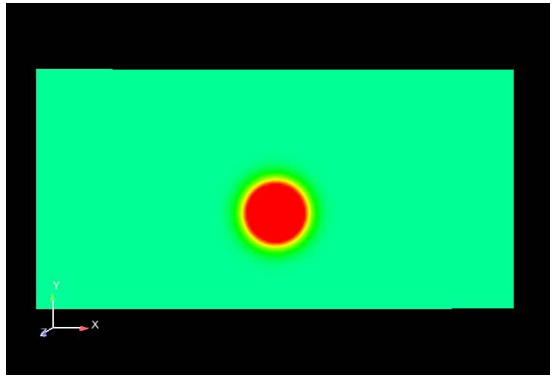
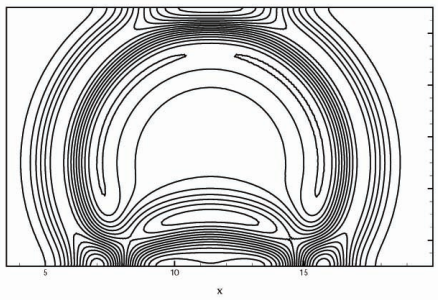
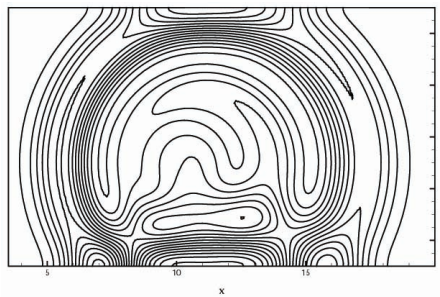


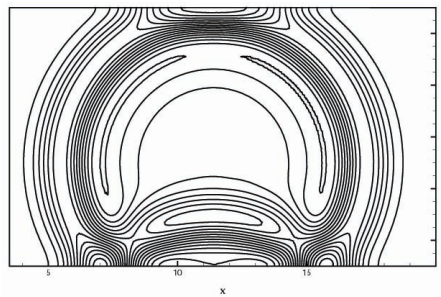
- Reflection of cylindrical Gaussian pressure pulse in uniform base flow, $M_\infty = 0.25$.
- Good qualitative agreement between CFD solution and 6 mode symmetry ROM (with BCs) on large scale.
- Excellent agreement between CFD solution and 14 mode symmetry ROM (with BCs).
- Symmetry ROM (with BCs) is stable – vs. L^2 ROM, which experienced instability when more than 6 or 7 modes were used.



CFD



6 mode ROM



14 mode ROM

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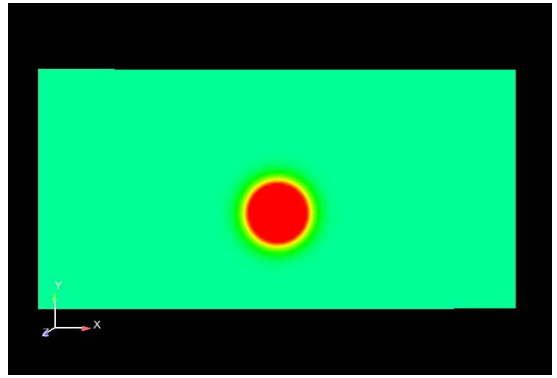
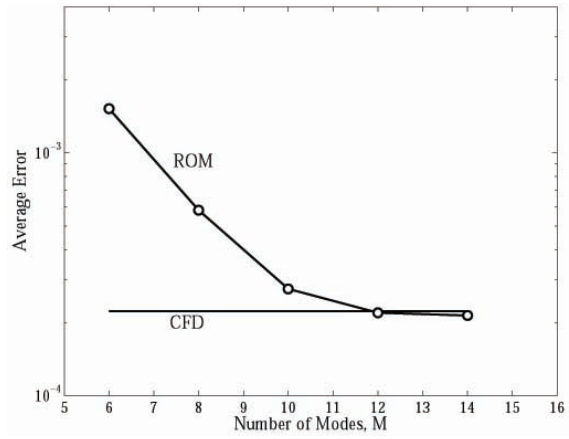
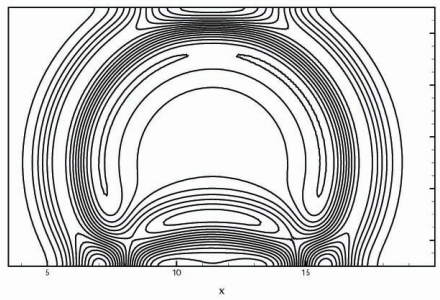
- Symmetry ROM with BCs is convergent (*a priori* [2] and *a posteriori*).







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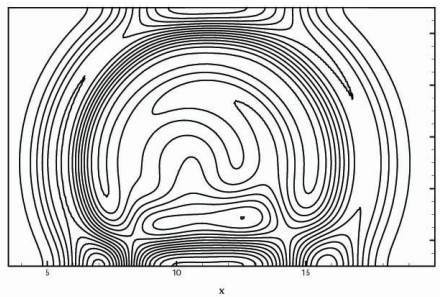

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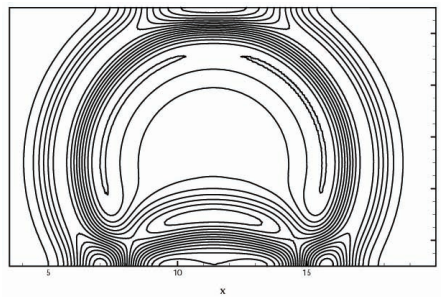

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Full 3D Compressible Navier-Stokes Equations

- Required to describe satisfactorily compressible flows at transonic, supersonic and hypersonic Mach numbers where non-linear effects are significant.
- High accuracy simulations (DNS, LES) are required to capture correctly viscous and nonlinear effects (e.g., boundary layers, shocks, turbulence).
- Full compressible Navier-Stokes equations in the conservation variables:

$$\mathbf{U}^T(\mathbf{x},t) \equiv \begin{pmatrix} \rho & \rho u_1 & \rho u_2 & \rho u_3 & \rho e \end{pmatrix} \in \mathbb{R}^5$$

$$\Rightarrow \mathbf{U}_{,t} + \mathbf{A}_i \mathbf{U}_{,i} - (\mathbf{K}_{ij}\mathbf{U}_{,j})_{,i} = \mathbf{0}$$

where

$$\mathbf{F}_{i,i} = \mathbf{F}_{i,\mathbf{U}}\mathbf{U}_{,i} \equiv \mathbf{A}_i \mathbf{U}_{,i}, \qquad \mathbf{F}_i^{\nu} \equiv \mathbf{K}_{ij}^{\nu}\mathbf{U}_{,j}, \qquad \mathbf{F}_i^{h} \equiv \mathbf{K}_{ij}^{h}\mathbf{U}_{,j}$$

$$\mathbf{K}_{ij} \equiv \mathbf{K}_{ij}^{\nu} + \mathbf{K}_{ij}^{h}.$$

with

$$\underbrace{\mathbf{F}_i = u_i \mathbf{U} + p \begin{pmatrix} 0 \\ \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ u_i \end{pmatrix}}_{\text{Euler flux}}, \qquad \underbrace{\mathbf{F}_i^{\nu} = \begin{pmatrix} 0 \\ \tau_{1i} \\ \tau_{2i} \\ \tau_{3i} \\ \tau_{ij}u_j \end{pmatrix}}_{\text{viscous flux}}, \qquad \underbrace{\mathbf{F}_i^{h} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -q_i \end{pmatrix}}_{\text{heat flux}},$$

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Entropy Variables & Entropy Stability

Entropy stability of the Galerkin ROM can be proven following a "symmetrization" of the compressible Navier-Stokes equations.

- Parabolic-hyperbolic compressible Navier-Stokes system is "symmetrizable".

Theorem (Mock) [6]

A parabolic-hyperbolic system of conservation laws possesses a (convex) **generalized entropy function** $H(\mathbf{U})$ and becomes symmetric under the change of variables

$$\mathbf{V}^T = H_{,\mathbf{U}}$$

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- Examples of entropy functions:
  - *Scalar conservation law (e.g., Burgers' equation):* $H(\mathbf{U}) = \frac{u^2}{2}$.
  - *Shallow water equations:* $H(\mathbf{U}) = \frac{1}{2}(gh^2 + |\mathbf{u}|^2 h)$ [8].
  - *Compressible Euler equations:* $H(\mathbf{U}) = Kp\rho^{-\gamma}$ [6].
  - *Compressible Navier-Stokes equations:* $H(\mathbf{U}) = -\rho s$ [6, 9].

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  - *Compressible Euler equations:* $H(\mathbf{U}) = Kp\rho^{-\gamma}$ [6].
  - ***Compressible Navier-Stokes equations:*** $H(\mathbf{U}) = -\rho s$ [6, 9].

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Symmetrized Compressible Navier-Stokes Equations

- Compressible Navier-Stokes equations in the entropy variables:

$$\mathbf{V}^T(\mathbf{x},t) \equiv \left(\; -U_5 + \rho_1(\gamma+1-s), \quad U_2, \quad U_3, \quad U_4, \quad -U_1 \; \right) \in \mathbb{R}^5$$

$$s = \ln\left[\frac{(\gamma-1)\rho_1}{U_1^\gamma}\right], \qquad \rho_1 = U_5 - \frac{1}{2U_1}(U_2^2+U_3^2+U_4^2)$$

$$\Rightarrow \mathbf{A}_0\mathbf{V}_{,t} + \tilde{\mathbf{A}}_i\mathbf{V}_{,i} - (\tilde{\mathbf{K}}_{ij}\mathbf{V}_{,j})_{,i} = \mathbf{0}$$

where

$$\mathbf{A}_0 \equiv \mathbf{U}_{,\mathbf{V}}, \qquad \tilde{\mathbf{A}}_i \equiv \mathbf{A}_i\mathbf{A}_0, \qquad \tilde{\mathbf{K}}_{ij} \equiv \mathbf{K}_{ij}\mathbf{A}_0,$$

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- Equations in entropy variables are a **symmetric parabolic** system:
  - The matrices $\mathbf{A}_0$ and $\tilde{\mathbf{A}}_i$ are symmetric.
  - The matrix $\tilde{\mathbf{K}} \equiv \begin{pmatrix} \tilde{\mathbf{K}}_{11} & \tilde{\mathbf{K}}_{12} & \tilde{\mathbf{K}}_{13} \\ \tilde{\mathbf{K}}_{21} & \tilde{\mathbf{K}}_{22} & \tilde{\mathbf{K}}_{23} \\ \tilde{\mathbf{K}}_{31} & \tilde{\mathbf{K}}_{32} & \tilde{\mathbf{K}}_{33} \end{pmatrix}$ is symmetric positive semi-definite.

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Numerical schemes for the compressible Navier-Stokes in the physical entropy variables were studied extensively by Hughes *et al.* [9]

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- Substitute modal decomposition $\mathbf{V}_M = \sum_{k=1}^{M} a_k(t)\boldsymbol{\phi}_k(\mathbf{x})$ to obtain an $M \times M$ ***non*-linear** dynamical system of the form

$$\sum_{n=1}^{M} (\boldsymbol{\phi}_m, [\mathbf{A}_0]_M \boldsymbol{\phi}_n)\, \dot{a}_n = -\left(\boldsymbol{\phi}_m, [\tilde{\mathbf{A}}_i]_M \mathbf{V}_{M,i}\right) - (\boldsymbol{\phi}_{m,i}, [\tilde{\mathbf{K}}_{ij}]_M \mathbf{V}_{M,j})$$

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Efficiency: Interpolation of Non-Linear Terms

- Discrete ***non*-linear** ROM system is of the form:

$$\sum_{n=1}^{M} \left(\boldsymbol{\phi}_m, [\mathbf{f}_0(\mathbf{V}_M)]_n\right) \dot{a}_n = - \left(\boldsymbol{\phi}_m, \mathbf{f}_1(\mathbf{V}_M)\right) - \sum_{i=1}^{3} \left(\boldsymbol{\phi}_{m,i}, \mathbf{f}_{i+1}(\mathbf{V}_M)\right)$$

where

$$\mathbf{f}_i(\mathbf{V}_M) \equiv \mathbf{f}_i\left(\sum_{m=1}^{M} a_m(t)\boldsymbol{\phi}_m(\mathbf{x})\right), \qquad i = 0, \ldots, 4.$$

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- To recover efficiency, interpolate* non-linear terms:

$$\mathbf{f}_i(\mathbf{V}_M) \approx \sum_{m=1}^{M} \mathbf{f}_i\left(\sum_{n=1}^{M} a_n(t)\phi_n(\mathbf{x}_m^{\mathbf{f}_i})\right)\psi_m^{\mathbf{f_i}}, \qquad i = 0, \ldots, 4$$

$\mathbf{x}_m^{\mathbf{f}_i}$ = interpolation points for $\mathbf{f}_i$, $\psi_m^{\mathbf{f}_i}$ = "cardinal functions" computed for $\mathbf{f}_i$

* Computed via the "best points" interpolation procedure of Peraire, Nguyen *et al.* [5].

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- ROM ODE system with interpolation:

$$\mathbf{M}\dot{\mathbf{a}}_M + \sum_{i=1}^{4} \mathbf{G}^{\mathbf{f}_i}\mathbf{f}_i(\mathbf{D}^{\mathbf{f}_i}\mathbf{a}_M) = \mathbf{0}$$

where $\mathbf{M}$, $\mathbf{G}^{\mathbf{f}_i}$, $\mathbf{D}^{\mathbf{f}_i}$ are pre-computed in the offline stage of ROM.

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Preliminary Numerical Study 1: Viscous Burgers Equation

$$u_t + \left(\frac{u^2}{2}\right)_x = \mu u_{xx}, \qquad -1 < x < 3, \quad 0 < t < T,$$
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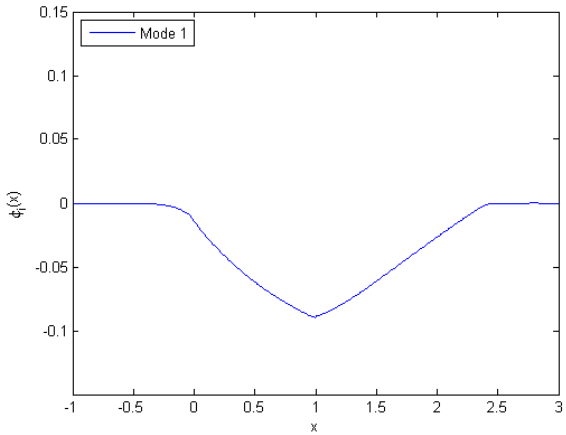

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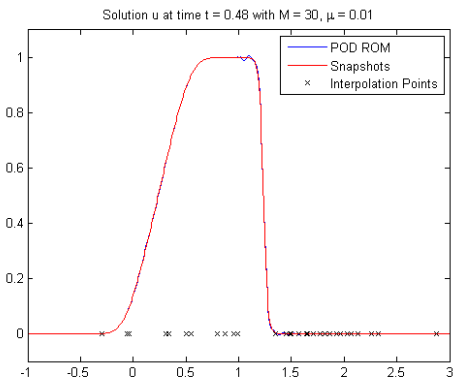
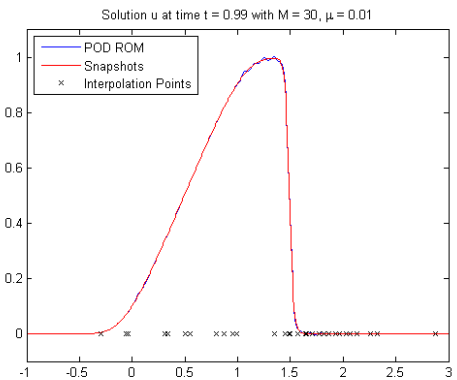


$t = 0.48$



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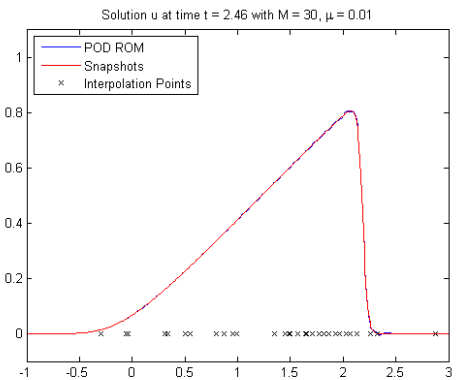

$t = 2.46$

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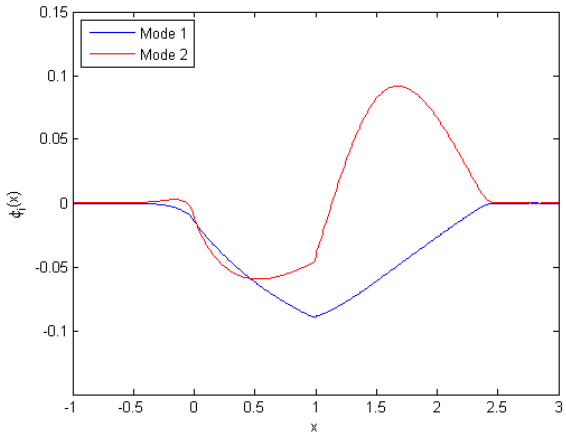

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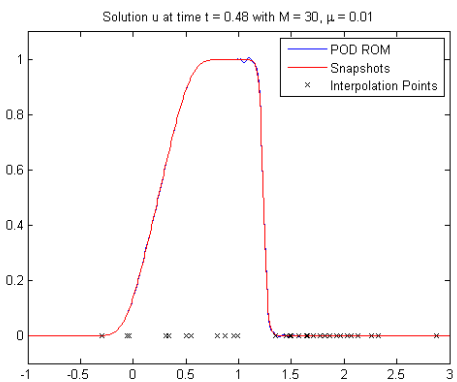

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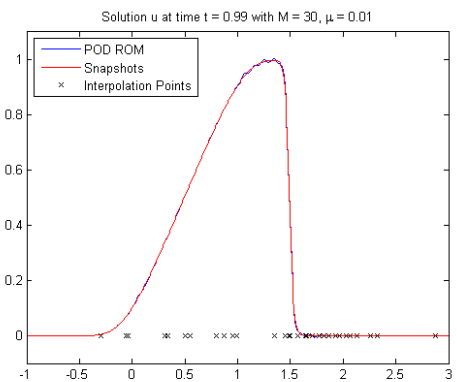

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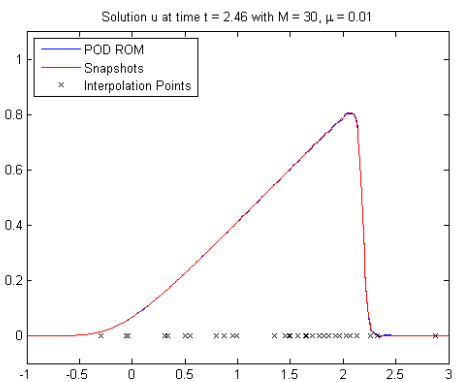

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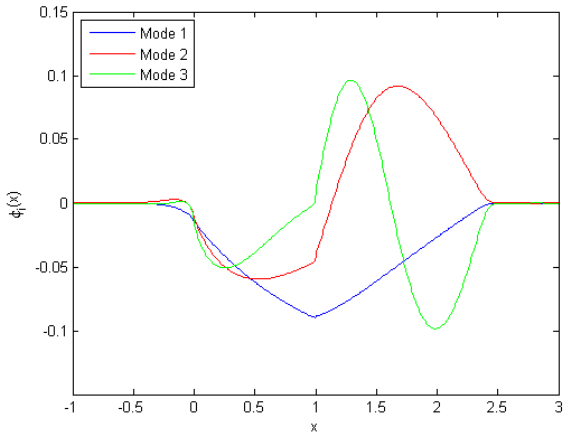

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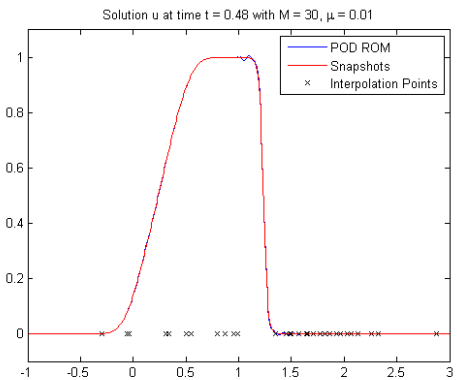

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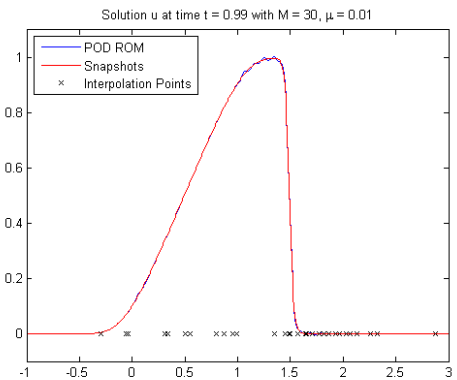

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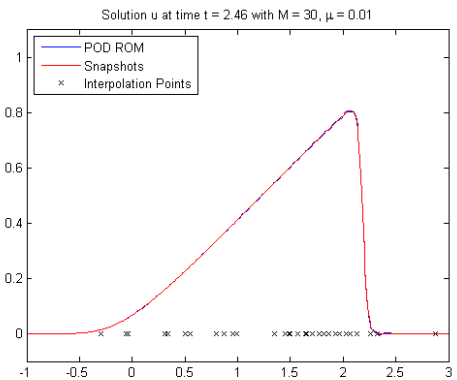

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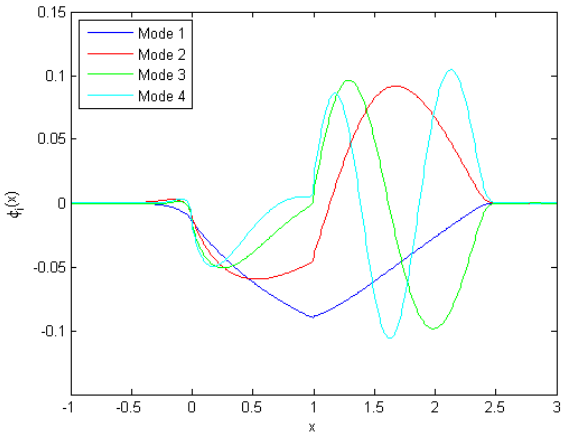

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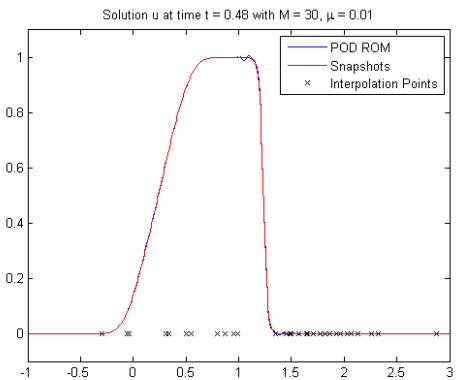

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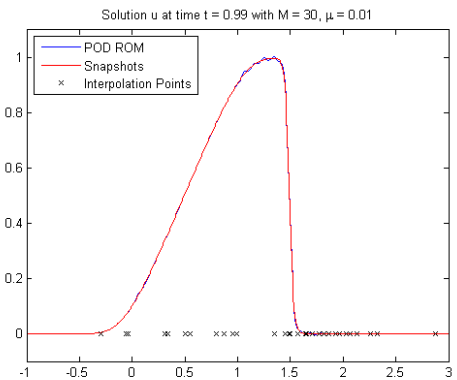

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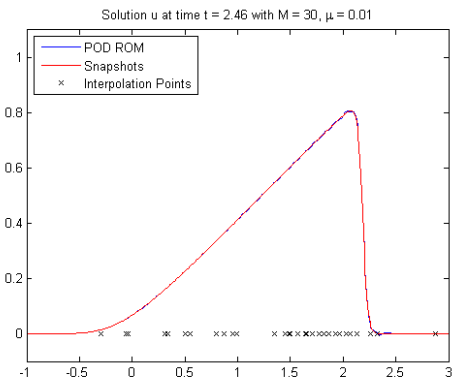

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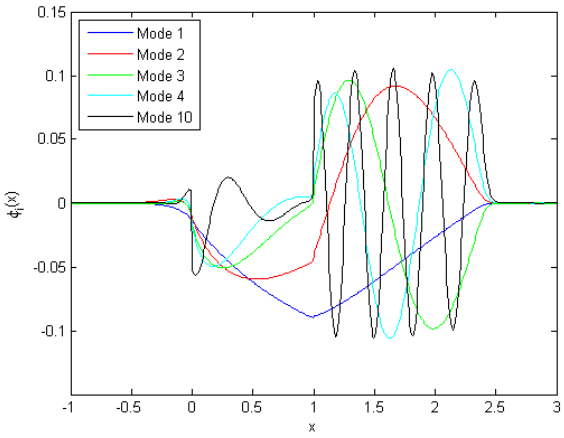

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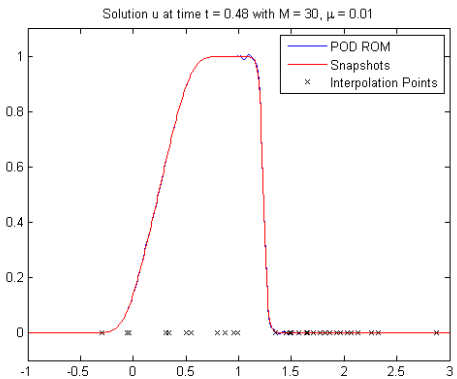
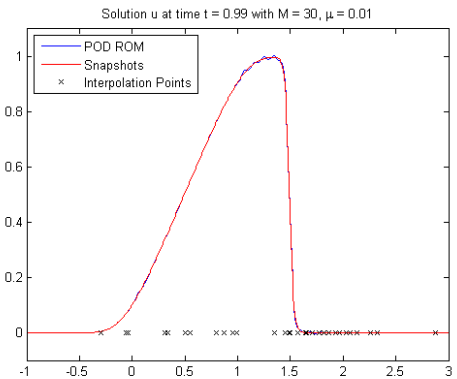


$t = 0.48$



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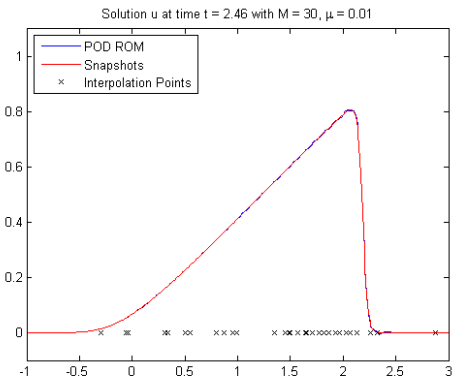
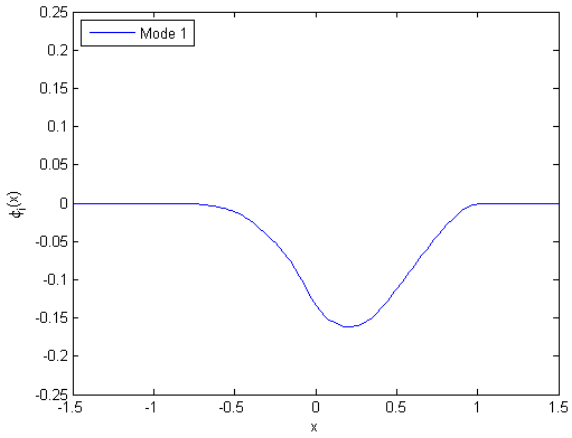


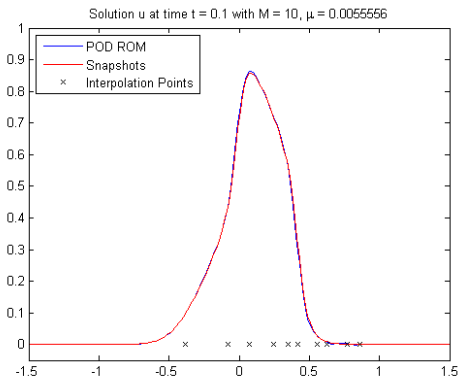
$t = 2.46$

Preliminary Numerical Study 2: Buckley-Leverett Equation

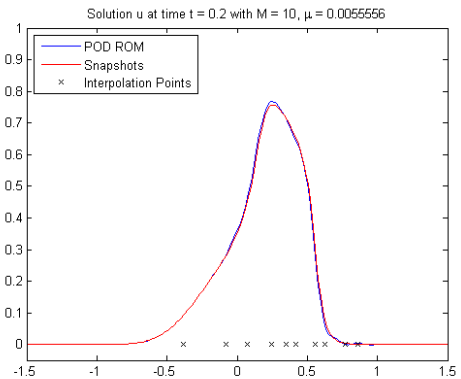
$$u_t + \left(\frac{u^2}{u^2 + (1-u)^2} \right)_x = \mu u_{xx}, \quad -1.5 < x < 1.5,$$
$$u(-1.5, t) = u(1.5, t) = 0,$$



- Used to model two-phase flow in porous media.
- Highly non-linear, non-convex flux.
- Gaussian initial condition: $u(x, 0) = e^{-16x^2}$.
- Results shown for $\mu = 0.05$, $M = 10$ modes, computed from $K = 50$ snapshots of ENO-LLF "high fidelity" finite volume solution.



$t = 0.1$



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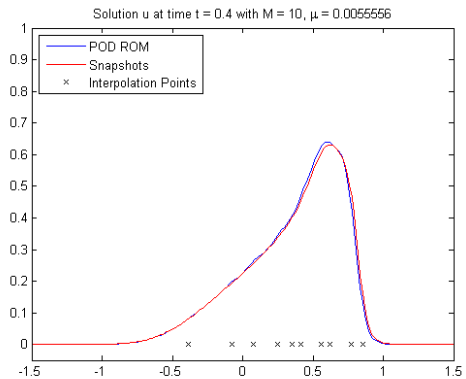

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Preliminary Numerical Study 2: Buckley-Leverett Equation

$$u_t + \left(\frac{u^2}{u^2 + (1-u)^2} \right)_x = \mu u_{xx}, \quad -1.5 < x < 1.5,$$
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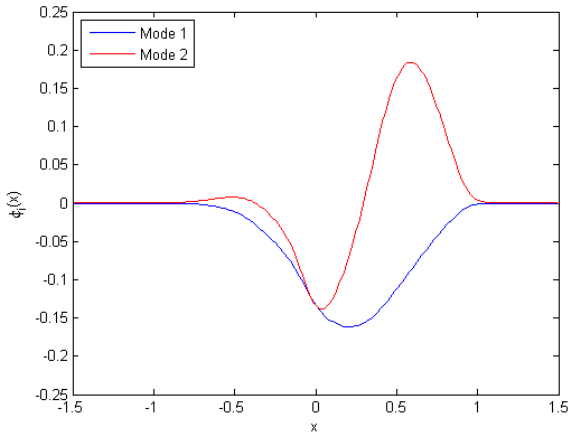

- Used to model two-phase flow in porous media.
- Highly non-linear, non-convex flux.
- Gaussian initial condition: $u(x, 0) = e^{-16x^2}$.
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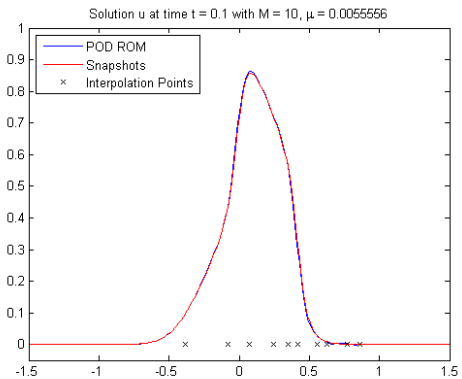

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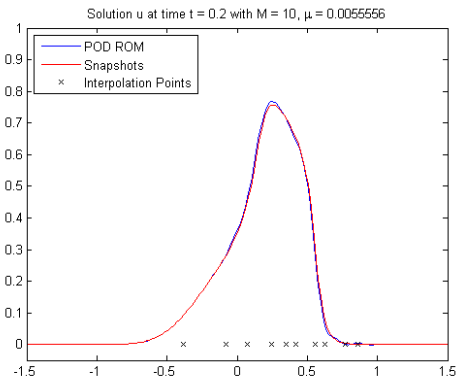

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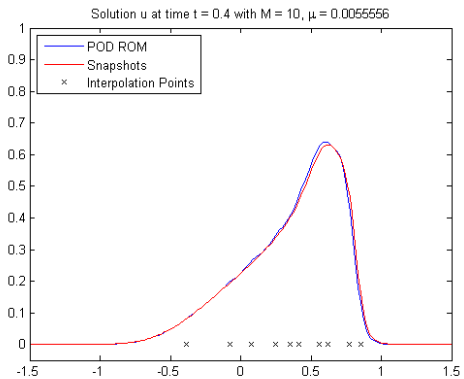

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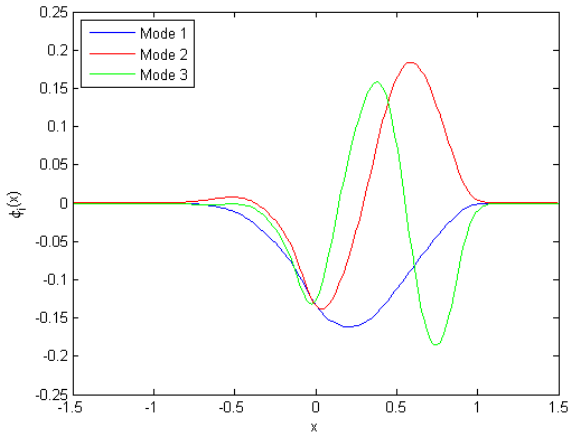

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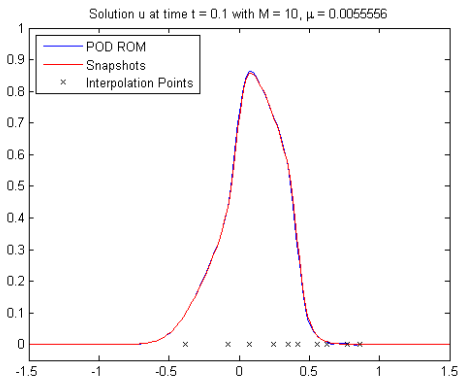

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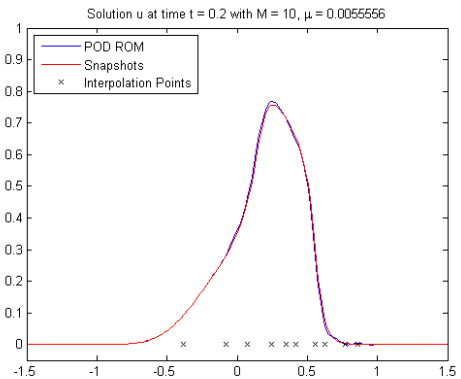

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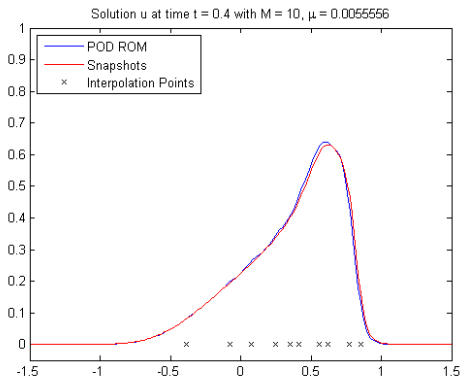

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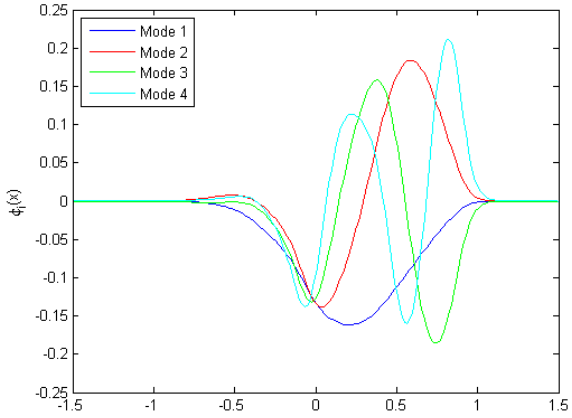

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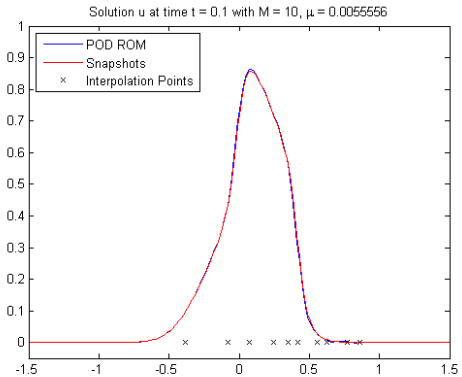

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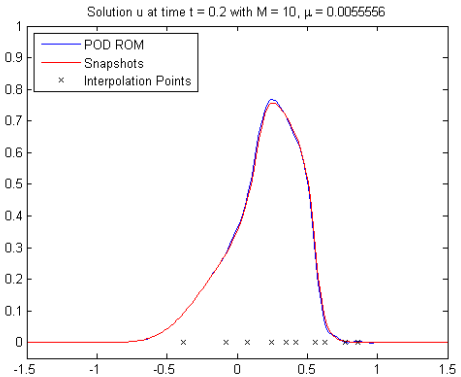

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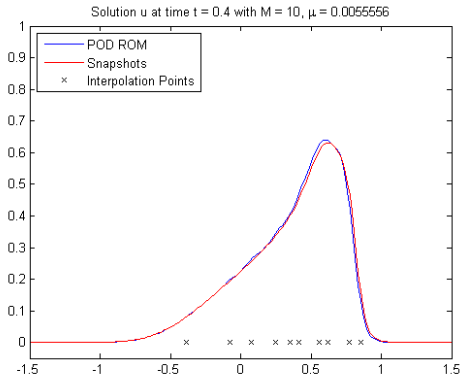

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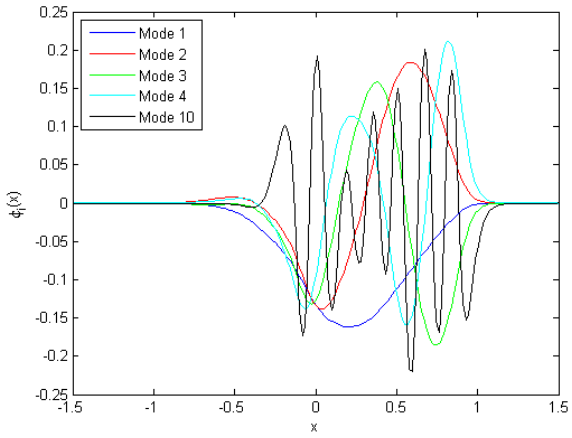

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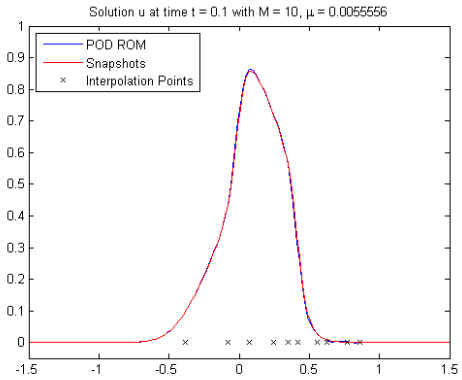
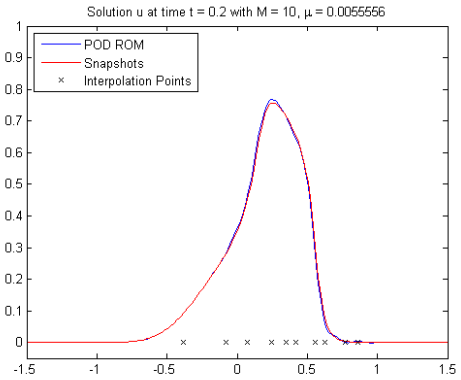
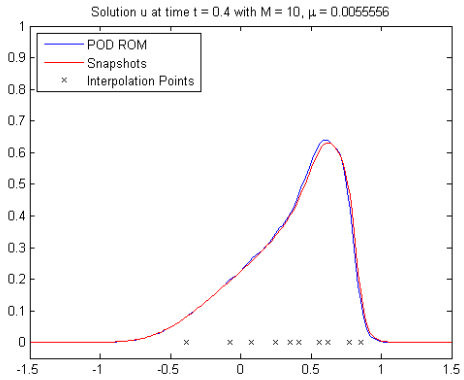


$t = 0.1$



$t = 0.2$



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Summary & Future Work

This Paper: Extension stable Galerkin ROM based on the continuous projection method previously developed [2-4] for ***linearized*** compressible flow equations to ***non-linear*** equations.

- Implement the entropy stable compressible Navier-Stokes ROM formulated in this paper; compare to other non-linear model reduction techniques (e.g., discrete Galerkin projection approach).
- Extend model reduction technique to allow incorporation of stabilization and shock-capturing operators [9].
- Extend model reduction technique to allow incorporation of turbulence models (LES, RANS-LES).
- Explore robustness of ROM with respect to parameter changes (reduced basis interpolation techniques [7]).
- Investigate the viability of the POD basis for non-linear problems: are there "better" bases to employ (e.g., balanced POD)? (entropy stability result is **basis independent**!)

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References (www.stanford.edu/∼irinak)

[1] **I. Kalashnikova**, M.F. Barone. "Efficient Non-Linear POD Reduced Order Models with Stable Penalty Enforcement of Boundary Conditions". Submitted to *Int. J. Numer. Meth. Engng.* (under review).

[2] **I. Kalashnikova**, M.F. Barone. "On the Stability and Convergence of a Galerkin Reduced Order Model (ROM) of Compressible Flow with Solid Wall and Far-Field Boundary Treatment". *Int. J. Numer. Meth. Engng.* **83** (2010) 1345-1375.

[3] M.F. Barone, **I. Kalashnikova**, D.J. Segalman, H. Thornquist. "Stable Galerkin Reduced Order Models for Linearized Compressible Flow". *J. Comput. Phys.* **288** (2009) 1932-1946.

[4] M.F. Barone, D.J. Segalman, H. Thornquist, **I. Kalashnikova**. "Galerkin Reduced Order Models for Compressible Flow with Structural Interaction". *AIAA Paper No. 2008-0612*, 46th AIAA Aerospace Science Meeting and Exhibit, Reno, NV (Jan. 2008).

[5] N.C. Nguyen, J. Peraire. "An efficient reduced-order modeling approach for non-linear parametrized partial differential equations". *Int. J. Numer. Meth. Engng.* **76**:27–55 (2008).

[6] A. Harten. "On the symmetric form of systems of conservations laws with entropy". *J. Comput. Phys.* **49**: 151–164 (1983).

[7] D. Amsallem, C. Farhat. "An Interpolation Method for Adapting Reduced-Order Models and Application to Aeroelasticity". *AIAA Journal*, **46** (2008) 1803–1813.

[8] S.W. Bova, G.F. Carey. "An entropy variable formulation and applications for the two-dimensional shallow water equations". *Int. J. Numer. Meth. Fluids* **23** 29–46 (1996).

[9] F. Shakib, T.J.R. Hughes, Z. Johan. "A new finite element formulation for computational fluid dynamics: X. The compressible Euler and Navier-Stokes equations". *Comput. Meth. Appl. Mech. Engng.* **89** 141–219 (1991).

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Thank you! Questions?

ikalash@sandia.gov

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Symmetrizing Matrix $\mathbf{A}_0$

- Introduce the notation:

$$\begin{array}{lll}
\bar{\gamma} = \gamma - 1, & k_1 = \frac{1}{2V_5}(V_2^2 + V_3^2 + V_4^2), & k_2 = k_1 - \gamma, \\
k_3 = k_1^2 - 2\gamma k_1 + \gamma, & k_4 = k_2 - \bar{\gamma}, & k_5 = k_2^2 - \bar{\gamma}(k_1 + k_2), \\
c_1 = \bar{\gamma} V_5 - V_2^2, & d_1 = -V_2 V_3, & e_1 = V_2 V_5, \\
c_2 = \bar{\gamma} V_5 - V_3^2, & d_2 = -V_2 V_4, & e_2 = V_3 V_5, \\
c_3 = \bar{\gamma} V_5 - V_4^2, & d_3 = -V_3 V_4, & e_3 = V_4 V_5.
\end{array}$$

$$\rho_1 = \left[\frac{\gamma - 1}{(-V_5)^\gamma}\right]^{1/(\gamma-1)} \exp\left(\frac{-s}{\gamma - 1}\right).$$

- Inverse transformation $\mathbf{V} \to \mathbf{U}$:

$$\mathbf{U}^T = \rho_1 \left(\; -V_5, \quad V_2, \quad V_3, \quad V_4, \quad 1 - \tfrac{1}{2V_5}(V_2^2 + V_3^2 + V_4^2) \; \right)$$

- Symmetrizing matrix $\mathbf{A}_0$:

$$\mathbf{A}_0 = \mathbf{U}_{,\mathbf{V}} = \frac{\rho_1}{\bar{\gamma} V_5} \begin{pmatrix} -V_5^2 & e_1 & e_2 & e_3 & V_5(1 - k_1) \\ & c_1 & d_1 & d_2 & V_2 k_2 \\ & & c_2 & d_3 & V_3 k_2 \\ & & & c_3 & V_4 k_2 \\ \text{symm.} & & & & -k_3 \end{pmatrix}$$

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Jacobians of Symmetrized Euler Fluxes $\tilde{\mathbf{A}}_i$

$$\tilde{\mathbf{A}}_1 = \tilde{\mathbf{F}}_{1,\mathbf{v}} = \frac{\rho_1}{\bar{\gamma} V_5^2} \begin{pmatrix} e_1 V_5 & c_1 V_5 & d_1 V_5 & d_2 V_5 & k_2 e_1 \\ & -(c_1 + 2\bar{\gamma} V_5) V_2 & -c_1 V_3 & -c_1 V_4 & c_1 k_2 + \bar{\gamma} V_2^2 \\ & & -c_2 V_2 & -d_1 V_4 & k_4 d_1 \\ & & & -c_3 V_2 & k_4 d_2 \\ \text{symm.} & & & & k_5 V_2 \end{pmatrix},$$

$$\tilde{\mathbf{A}}_2 = \tilde{\mathbf{F}}_{2,\mathbf{v}} = \frac{\rho_1}{\bar{\gamma} V_5^2} \begin{pmatrix} e_2 V_5 & d_1 V_5 & c_2 V_5 & d_3 V_5 & k_2 e_2 \\ & -c_1 V_3 & -c_2 V_2 & -d_1 V_4 & k_4 d_1 \\ & & -(c_2 + 2\bar{\gamma} V_5) V_3 & -c_2 V_4 & c_2 k_2 + \bar{\gamma} V_3^2 \\ & & & -c_3 V_3 & k_4 d_3 \\ \text{symm.} & & & & k_5 V_3 \end{pmatrix},$$

$$\tilde{\mathbf{A}}_3 = \tilde{\mathbf{F}}_{3,\mathbf{v}} = \frac{\rho_1}{\bar{\gamma} V_5^2} \begin{pmatrix} e_3 V_5 & d_2 V_5 7 d_3 V_5 & c_3 V_5 & k_2 e_3 & \\ & -c_1 V_4 & -d_2 V_3 & -c_3 V_2 & k_4 d_2 \\ & & -c_2 V_4 & -c_3 V_3 & k_4 d_3 \\ & & & -(c_3 + 2\bar{\gamma} V_5) V_4 & c_3 k_2 + \bar{\gamma} V_4^2 \\ \text{symm.} & & & & k_5 V_4 \end{pmatrix}.$$

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Symmetrized Viscous and Heat Fluxes $\tilde{\mathbf{K}}_{ij}$

$$\tilde{\mathbf{K}}_{11} = \frac{1}{V_5^3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -(\gamma-2\mu)V_5^2 & 0 & 0 & (\lambda+2\mu)e_1 \\ 0 & 0 & -\mu V_5^2 & 0 & \mu e_2 \\ 0 & 0 & 0 & -\mu V_5^2 & \mu e_3 \\ 0 & (\lambda+2\mu)e_1 & \mu e_2 & \mu e_3 & -\left[(\lambda+2\mu)V_2^2+\mu(V_3^2+V_4^2)-\frac{\gamma\mu V_5}{Pr}\right] \end{pmatrix},$$

$$\tilde{\mathbf{K}}_{12} = \frac{1}{V_5^3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda V_5^2 & 0 & \lambda e_2 \\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu e_2 & \lambda e_1 & 0 & (\lambda+\mu)d_1 \end{pmatrix},$$

$$\tilde{\mathbf{K}}_{13} = \frac{1}{V_5^3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda V_5^2 & \lambda e_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1 \\ 0 & \mu e_3 & 0 & \lambda e_1 & (\lambda+\mu)d_2 \end{pmatrix},$$

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Symmetrized Viscous and Heat Fluxes $\tilde{\mathbf{K}}_{ij}$ (Continued)

$$\tilde{\mathbf{K}}_{22} = \frac{1}{V_5^3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1 \\ 0 & 0 & -(\lambda+2\mu)V_5^2 & 0 & (\lambda+2\mu)e_2 \\ 0 & 0 & 0 & -\mu V_5^2 & \mu e_3 \\ 0 & \mu e_1 & (\lambda+2\mu)e_2 & \mu e_3 & -\left[(\lambda+2\mu)V_3^2+\mu(V_2^2+V_4^2)-\frac{\gamma\mu V_5}{Pr}\right] \end{pmatrix},$$

$$\tilde{\mathbf{K}}_{23} = \frac{1}{V_5^3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda V_5^2 & \lambda e_3 \\ 0 & 0 & -\mu V_5^2 & 0 & \mu e_2 \\ 0 & 0 & \mu e_3 & \lambda e_2 & (\lambda+\mu)d_3 \end{pmatrix},$$

$$\tilde{\mathbf{K}}_{33} = \frac{1}{V_5^3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1 \\ 0 & 0 & -\mu V_5^2 & 0 & \mu e_2 \\ 0 & 0 & 0-(\lambda+2\mu)V_5^2 & (\lambda+2\mu)e_3 & \\ 0 & \mu e_1 & \mu e_2 & (\lambda+2\mu)e_3 & -\left[(\lambda+2\mu)V_4^2+\mu(V_2^2+V_3^2)-\frac{\gamma\mu V_5}{Pr}\right] \end{pmatrix},$$

$$\tilde{\mathbf{K}}_{21} = \tilde{\mathbf{K}}_{12}^T, \qquad \tilde{\mathbf{K}}_{31} = \tilde{\mathbf{K}}_{13}^T, \qquad \tilde{\mathbf{K}}_{32} = \tilde{\mathbf{K}}_{23}^T.$$

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Semi-Discrete ROM System Matrices

- Semi-discrete ROM ODE system following interpolation:

$$\mathbf{M}\dot{\mathbf{a}}_M + \sum_{i=1}^{4} \mathbf{G}^{\mathbf{f}_i}\,\mathbf{f}_i(\mathbf{D}^{\mathbf{f}_i}\,\mathbf{a}_M) = \mathbf{0},$$

- $\mathbf{G}^{\mathbf{f}_i}$ matrices:

$$\mathbf{G}^{\mathbf{f}_1}_{l,[5(m-1)+1:5m]} = \int_\Omega (\phi_l^1 \psi_m^{f_1^1}, \phi_l^2 \psi_m^{f_1^2}, \phi_l^3 \psi_m^{f_1^3}, \phi_l^4 \psi_m^{f_1^4}, \phi_l^5 \psi_m^{f_1^5}) d\Omega,$$

$$\mathbf{G}^{\mathbf{f}_{i+1}}_{l,[5(m-1)+1:5m]} = \int_\Omega (\phi_{l,i}^1 \psi_m^{f_{i+1}^1}, \phi_{l,i}^2 \psi_m^{f_{i+1}^2}, \phi_{l,i}^3 \psi_m^{f_{i+1}^3}, \phi_{l,i}^4 \psi_m^{f_{i+1}^4}, \phi_{l,i}^5 \psi_m^{f_{i+1}^5}) d\Omega, \, i = 1, 2, 3$$

  for $l, m = 1, \ldots M$.
- Mass matrix $\mathbf{M}$: for $k = 1, \ldots, M$,

$$\mathbf{M}_{[1:M],k} = \mathbf{G}^{[\mathbf{f}_0]_k}\,[\mathbf{f}_0]_k \left(\mathbf{D}^{[\mathbf{f}_0]_k}\,\mathbf{a}_M\right),$$

- $\mathbf{D}^{\mathbf{f}_i}$ matrices:

$$\mathbf{D}^{\mathbf{f}_i} \equiv \begin{pmatrix} \phi_1\left(\mathbf{x}_1^{\mathbf{f}_i}\right) & \ldots & \phi_M\left(\mathbf{x}_1^{\mathbf{f}_i}\right) \\ \vdots & \ddots & \vdots \\ \phi_1\left(\mathbf{x}_M^{\mathbf{f}_i}\right) & \ldots & \phi_M\left(\mathbf{x}_M^{\mathbf{f}_i}\right) \end{pmatrix}$$

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