

Reliability Methods: Determining small failure probabilities

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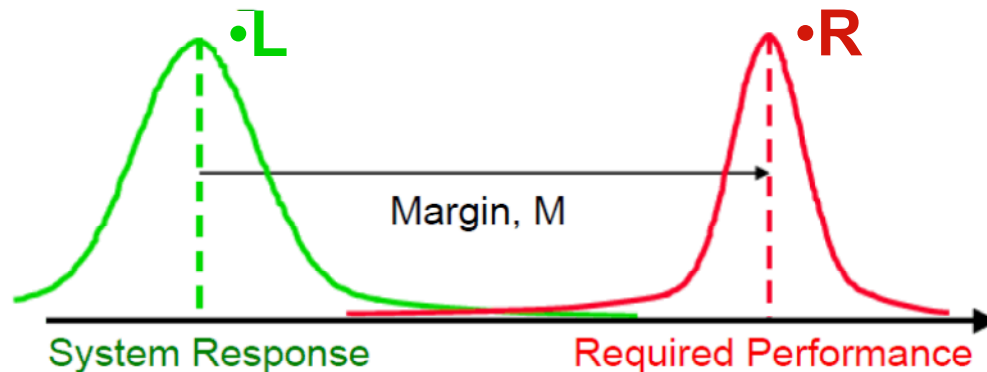


Lecture Outline

- **Safety Factor**
- **Mean Value Method**
- **FORM**
- **Advanced Mean Value**
- **SORM**
- **Limit State Approximations**
 - TANA
- **Probability Calculations**
 - Second order methods
 - Importance Sampling

Safety Factors

- Much of the early work on engineering reliability comes from the civil engineering field, concerned with reliability of structures
- In this lecture, the notation of L = load, R = resistance, we want $L < R$
- Nominal safety factor: $SF = R_{\text{nom}}/L_{\text{nom}}$, where R_{nominal} is usually a conservative value (e.g. 2-3 standard deviations below the mean) and L_{nominal} is also a conservative value (2-3 standard deviations above the mean)
- Problem: the nominal safety factor may not convey the true margin of safety in a design





Safety Factors

- Variety of approaches to improve a design
 - Increase the distance between the relative positions of the two curves: this reduces the probability of the overlapping area, and the probability of failure decreases
 - Reduce the dispersion of the two curves
 - Improve the shapes of the two curves



Probability of Failure

$$p_f = P(\text{failure}) = P(R < L)$$

$$p_f = \int_0^\infty \left[\int_0^l f_R(r) dr \right] f_L(l) dl$$

$$p_f = \int_0^\infty F_R(l) f_L(l) dl$$

In practice, this integration is hard to perform and doesn't always have an explicit form, except in some special cases



Probability of Failure

- **Special Case:** $R \sim N(\mu_R, \sigma_R)$, $L \sim N(\mu_L, \sigma_L)$
- **Define** $Z = R - L$

$$p_f = P(\text{failure}) = P(Z < 0)$$

$$p_f = \Phi \left[\frac{0 - (\mu_R - \mu_L)}{\sqrt{\sigma_R^2 + \sigma_L^2}} \right]$$

$$p_f = 1 - \Phi \left[\frac{(\mu_R - \mu_L)}{\sqrt{\sigma_R^2 + \sigma_L^2}} \right]$$

- **There are also modifications which treat multiple loads, or lognormal distributions**
- **This formulation allows more granularity: quantities such as the capacity reduction factor and load factor can be calculated (Haldar and Mahadevan)**



Reliability Analysis

- Assume that the probability of failure is based on a specific performance criterion which is a function of random variables, denoted X_i .

- The performance function is described by Z :

$$Z = g(X_1, X_2, X_3, \dots, X_n)$$

- The failure surface or limit state is defined as $Z = 0$. It is a boundary between safe and unsafe regions in a parameter space.
- Now we have a more general form of P_{failure}

$$p_f = P(\text{failure}) = P(Z < 0)$$

$$p_f = \int \dots \int_{g() < 0} f_X(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$



Reliability Analysis

- Note that the failure integral has the joint probability density function, f , for the random variables, and the integration is performed over the failure region

$$p_f = \int \dots \int_{g() < 0} f_X(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

- If the variables are independent, we can replace this with the product of the individual density functions
- In general, this is a multi-dimensional integral and is difficult to evaluate.
- People use approximations. If the limit state is a linear function of the inputs (or is approximated by one), first-order reliability methods (FORM) are used.
- If the nonlinear limit state is approximated by a second-order representation, second-order reliability methods (SORM) are used.



Mean Value Method (FOSM)

- Often called the First-Order Second-Moment (FOSM) method or the Mean Value FOSM method
- The FOSM method is based on a first-order Taylor series expansion of the performance function
- It is evaluated at the mean values of the random variables, and only uses means and covariances of the random variables
- The mean value method only requires one evaluation of the response function at the mean values of the inputs, plus n derivative values if one assumes the variables are independent $\rightarrow n+1$ evaluations in the simplest approach (CHEAP!)

$$\mu_g = g(\mu_x)$$

$$\sigma_g^2 = \sum_{i=1}^n \sum_{j=1}^n Cov(i, j) \frac{dg}{dx_i}(\mu_x) \frac{dg}{dx_j}(\mu_x)$$

$$\sigma_g^2 = \sum_{i=1}^n \left(\frac{dg}{dx_i}(\mu_x) \right)^2 Var(x_i)$$



Mean Value Method (FOSM)

- Introduce the idea of a safety index β (think of this as how far in “normal space” that your design is away from failure)”

$$\beta = \frac{\mu_g}{\sigma_g}$$

$$p_f = \Phi[-\beta] = 1 - \Phi[\beta]$$

- FOSM does not use distribution information when it is available
- When $g(x)$ is nonlinear, significant error may be introduced by neglecting higher order terms in the expansion
- The safety index fails to be constant under different problem formulations
- It can be very efficient. When $g(x)$ is linear and the input variables are normal, the mean value method gives exact results!



Mean Value Method (FOSM)

Some extensions/notation

$$\bar{z} \Rightarrow p, \beta$$

$$\beta_{cdf} = \frac{\mu_g - \bar{z}}{\sigma_g},$$

$$\beta_{ccdf} = \frac{\bar{z} - \mu_g}{\sigma_g}$$

$$\bar{p}, \bar{\beta} \Rightarrow z$$

$$z = \mu_g - \sigma_g \bar{\beta}_{cdf},$$

$$z = \mu_g + \sigma_g \bar{\beta}_{ccdf}$$

$$p_f = \Phi[-\beta] = 1 - \Phi[\beta]$$

p = probability of failure

β = reliability index

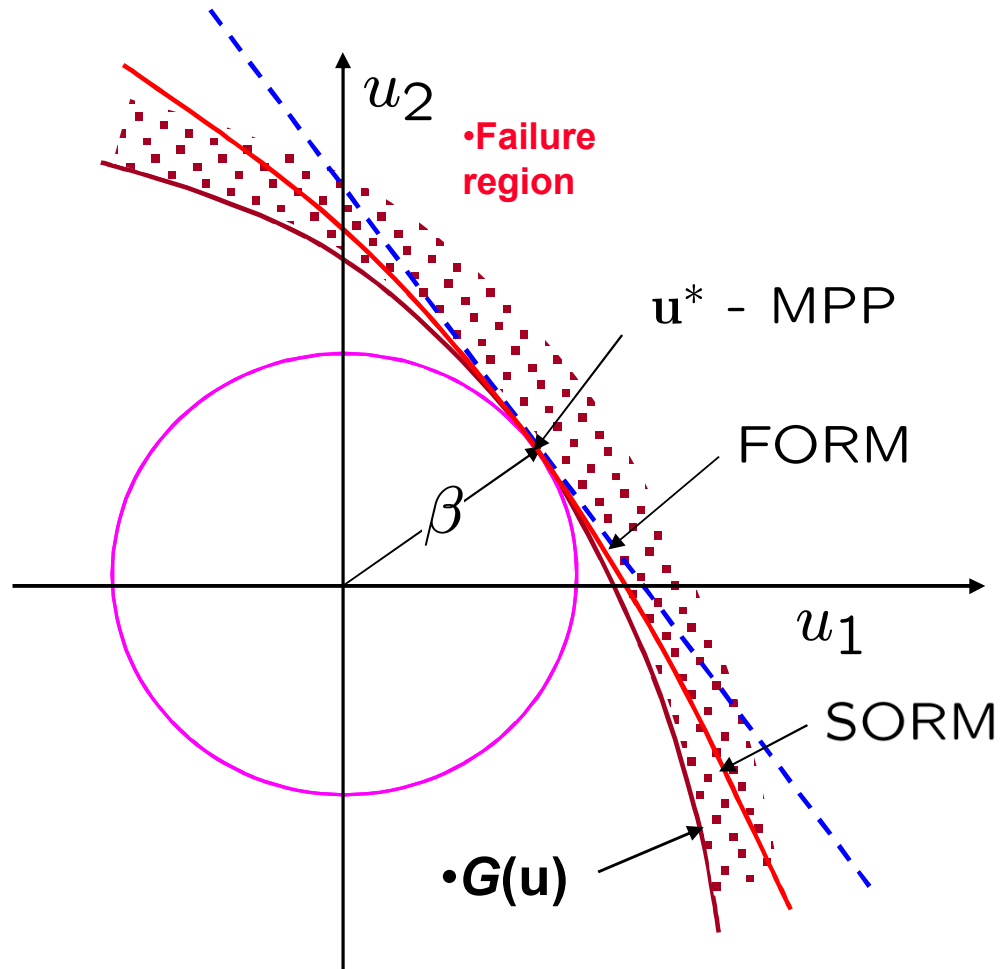
z = response level



Most Probable Point Methods

- Transform the uncertainty propagation problem into an optimization one: first transform all of the non-normal random variables into independent, unit normal variables. Then, find the point on the limit state surface with minimum distance to the origin.
- The point is called the Most Probable Point (MPP). The minimum distance, β , is called the safety index or reliability index.
- X is often called the original space, U is the transformed space.

MPP Search Methods





Uncertainty Transformations

- Want to go from correlated non-normals to uncorrelated standard normals (u)
- Several methods
 - Rosenblatt
 - Rackwitz-Fiesler
 - Chen-Lind
 - Wu-Wirshing
 - Nataf
- Rosenblatt: First transform a set of arbitrarily, correlated random variables $X_1 \dots X_n$ to uniform distributions, then transform to independent normals.
- Nataf: First transform to correlated normals (z), then to independent normals u . L is the Cholesky factor of the correlation matrix

$$U_1 = F_{X_1}(X_1)$$

$$U_2 = F_{X_2|X_1}(X_2 | x_1)$$

...

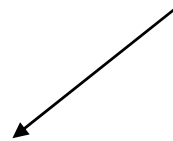
$$U_n = F_{X_n|X_1, X_2, \dots}(X_n | x_1, x_2, \dots, x_{n-1})$$

$$u_1 = \Phi^{-1}(U_1)$$

$$u_2 = \Phi^{-1}(U_2)$$

...

$$u_n = \Phi^{-1}(U_n)$$



$$\Phi(z_i) = F(x_i)$$

$$\mathbf{z} = \mathbf{L}\mathbf{u}$$

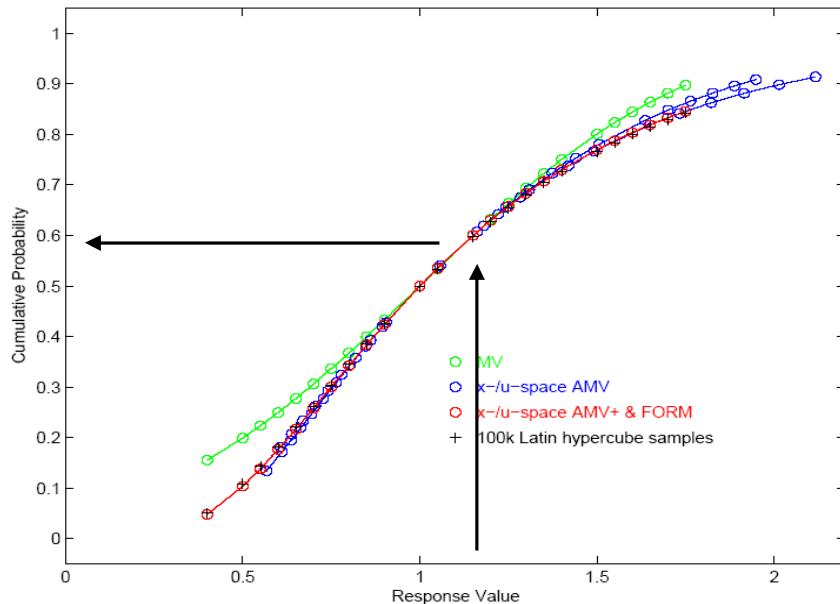


MPP Search Methods

Reliability Index Approach (RIA)

$$\begin{aligned} &\text{minimize} \quad \mathbf{u}^T \mathbf{u} \\ &\text{subject to} \quad G(\mathbf{u}) = \bar{z} \end{aligned}$$

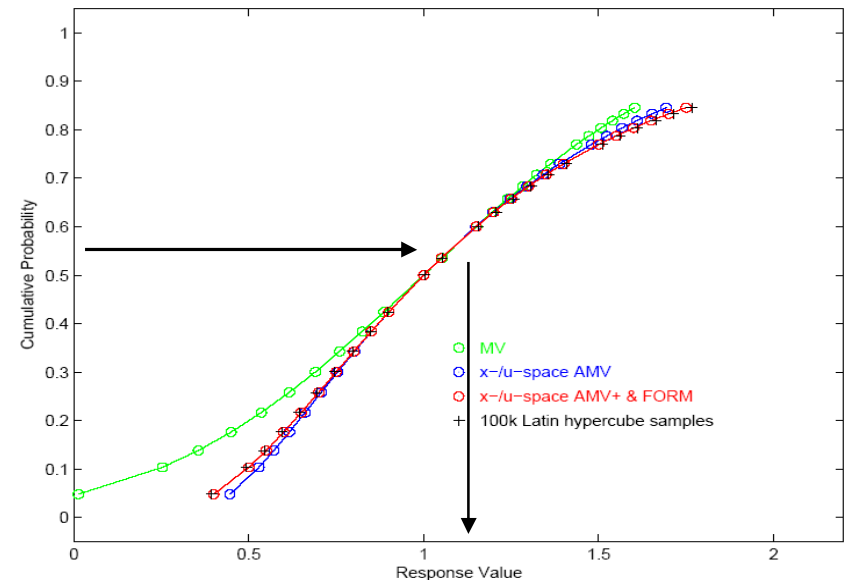
- Find min dist to G level curve
- Used for fwd map $z \rightarrow p/\beta$



Performance Measure Approach (PMA)

$$\begin{aligned} &\text{minimize} \quad \pm G(\mathbf{u}) \\ &\text{subject to} \quad \mathbf{u}^T \mathbf{u} = \bar{\beta}^2 \end{aligned}$$

- Find min G at β radius
- Used for inv map $p/\beta \rightarrow z$





Reliability Algorithm Variations: First-Order Methods

Limit state linearizations

AMV: $g(\mathbf{x}) = g(\mu_{\mathbf{x}}) + \nabla_x g(\mu_{\mathbf{x}})^T (\mathbf{x} - \mu_{\mathbf{x}})$

u-space AMV: $G(\mathbf{u}) = G(\mu_{\mathbf{u}}) + \nabla_u G(\mu_{\mathbf{u}})^T (\mathbf{u} - \mu_{\mathbf{u}})$

AMV+: $g(\mathbf{x}) = g(\mathbf{x}^*) + \nabla_x g(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*)$

u-space AMV+: $G(\mathbf{u}) = G(\mathbf{u}^*) + \nabla_u G(\mathbf{u}^*)^T (\mathbf{u} - \mathbf{u}^*)$

FORM: no linearization

Integrations

• 1st-
order:
$$\begin{cases} p(g \leq z) &= \Phi(-\beta_{cdf}) \\ p(g > z) &= \Phi(-\beta_{ccdf}) \end{cases}$$

MPP search algorithm

[HL-RF], Sequential Quadratic Prog. (SQP), Nonlinear Interior Point (NIP)

Warm starting

When: AMV+ iteration increment, $z/p/\beta$ level increment, or design variable change

What: linearization point & assoc. responses (AMV+) and MPP search initial guess

Reliability Algorithm Variations: Second-Order Methods

2nd-order local limit state approximations

- e.g., x-space AMV²⁺:

$$g(\mathbf{x}) \cong g(\mathbf{x}^*) + \nabla_{\mathbf{x}} g(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2} (\mathbf{x} - \mathbf{x}^*)^T \nabla_{\mathbf{x}}^2 g(\mathbf{x}^*) (\mathbf{x} - \mathbf{x}^*)$$

- Hessians may be full/FD/Quasi
- Quasi-Newton Hessians may be **BFGS** or **SR1**

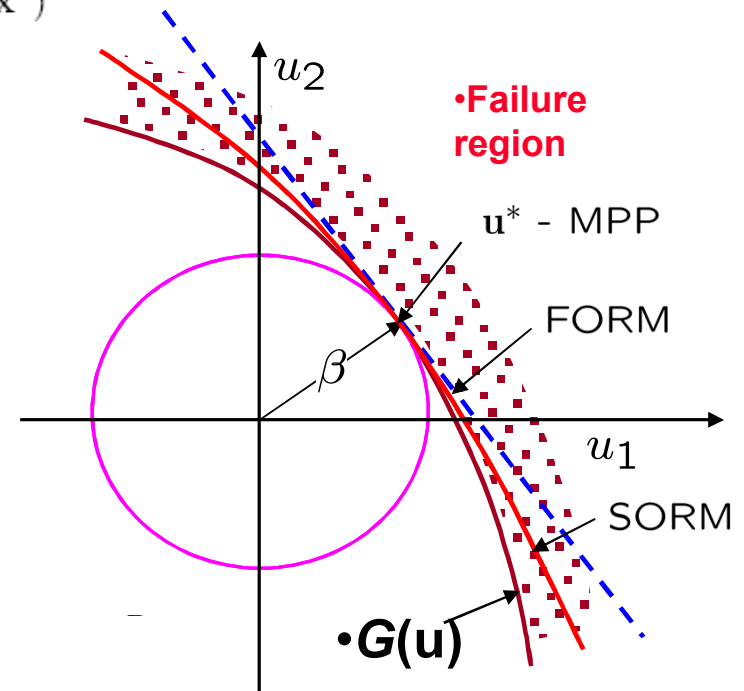
2nd-order integrations

$$p = \Phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1 + \beta \kappa_i}}$$

•curvature correction

Synergistic features:

- Hessian data needed for SORM integration can enable **more rapid MPP convergence**
- [QN] Hessian data accumulated during MPP search can enable **more accurate probability estimates**



Reliability Algorithm Variations: Second-Order Methods

Multipoint limit state approximations

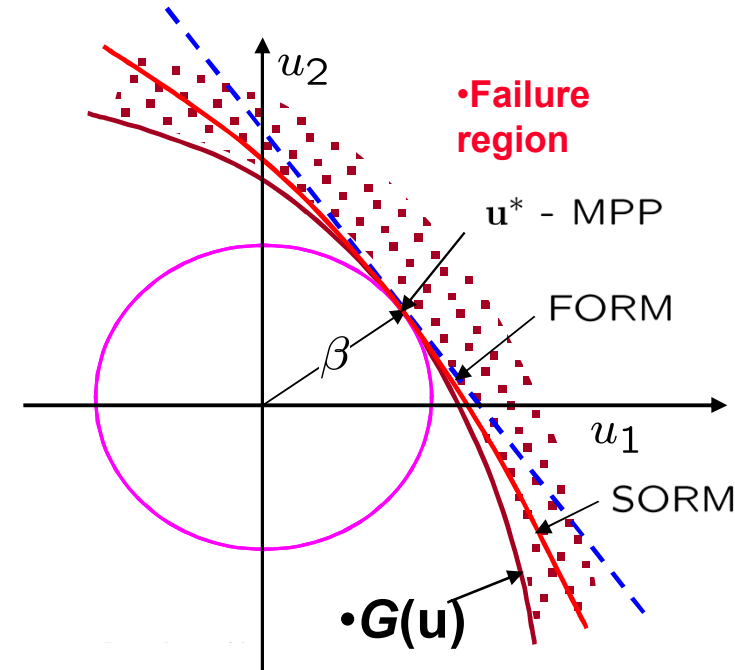
- e.g., TPEA, TANA:

$$g(\mathbf{x}) \cong g(\mathbf{x}_2) + \sum_{i=1}^n \frac{\partial g}{\partial x_i}(\mathbf{x}_2) \frac{x_{i,2}^{1-p_i}}{p_i} (x_i^{p_i} - x_{i,2}^{p_i}) + \frac{1}{2} \epsilon(\mathbf{x}) \sum_{i=1}^n (x_i^{p_i} - x_{i,2}^{p_i})^2$$

$$p_i = 1 + \ln \left[\frac{\frac{\partial g}{\partial x_i}(\mathbf{x}_1)}{\frac{\partial g}{\partial x_i}(\mathbf{x}_2)} \right] / \ln \left[\frac{x_{i,1}}{x_{i,2}} \right]$$

$$\epsilon(\mathbf{x}) = \frac{H}{\sum_{i=1}^n (x_i^{p_i} - x_{i,1}^{p_i})^2 + \sum_{i=1}^n (x_i^{p_i} - x_{i,2}^{p_i})^2}$$

$$H = 2 \left[g(\mathbf{x}_1) - g(\mathbf{x}_2) - \sum_{i=1}^n \frac{\partial g}{\partial x_i}(\mathbf{x}_2) \frac{x_{i,2}^{1-p_i}}{p_i} (x_{i,1}^{p_i} - x_{i,2}^{p_i}) \right]$$

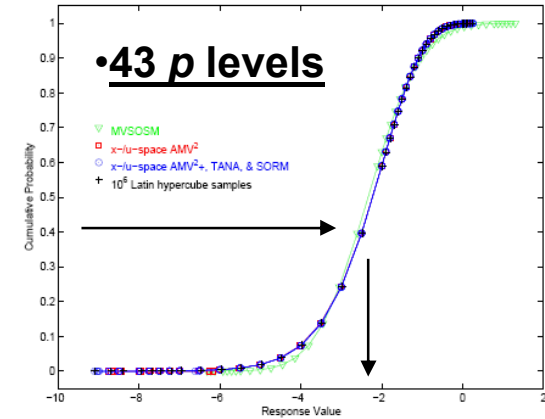
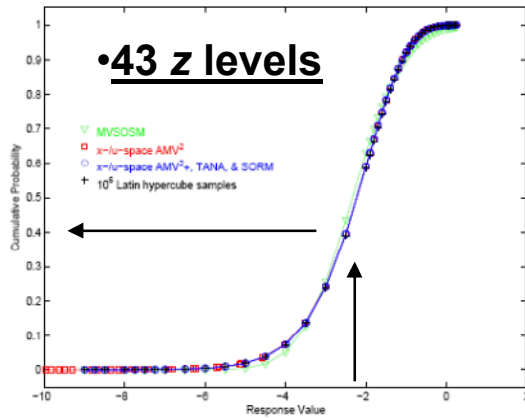


Importance Sampling

- Use of importance sampling to calculate prob of failure:
 - After MPP is identified, sample around MPP to estimate P_f more accurately

Reliability Algorithm Variations: Sample Results

Analytic benchmark test problems: lognormal ratio, **short column**, cantilever



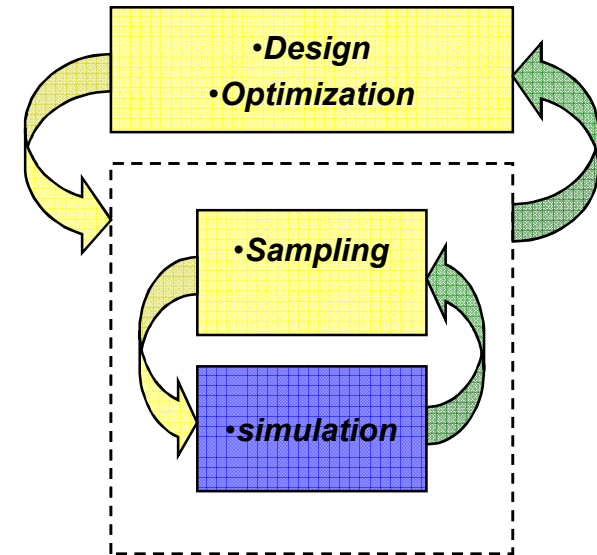
| RIA Approach | SQP Function Evaluations | NIP Function Evaluations | CDF p Error Norm | Target z Offset Norm |
|------------------|--------------------------|--------------------------|--------------------|------------------------|
| MVFOSM | 1 | 1 | 0.1548 | 0.0 |
| MVSOSM | 1 | 1 | 0.1127 | 0.0 |
| x-space AMV | 45 | 45 | 0.009275 | 18.28 |
| u-space AMV | 45 | 45 | 0.006408 | 18.81 |
| x-space AMV^2 | 45 | 45 | 0.002063 | 2.482 |
| u-space AMV^2 | 45 | 45 | 0.001410 | 2.031 |
| x-space $AMV+$ | 192 | 192 | 0.0 | 0.0 |
| u-space $AMV+$ | 207 | 207 | 0.0 | 0.0 |
| x-space AMV^2+ | 125 | 131 | 0.0 | 0.0 |
| u-space AMV^2+ | 122 | 130 | 0.0 | 0.0 |
| x-space TANA | 245 | 246 | 0.0 | 0.0 |
| u-space TANA | 296* | 278* | 6.982e-5 | 0.08014 |
| FORM | 626 | 176 | 0.0 | 0.0 |
| SORM | 669 | 219 | 0.0 | 0.0 |

| PMA Approach | SQP Function Evaluations | NIP Function Evaluations | CDF z Error Norm | Target p Offset Norm |
|------------------|--------------------------|--------------------------|--------------------|------------------------|
| MVFOSM | 1 | 1 | 7.454 | 0.0 |
| MVSOSM | 1 | 1 | 6.823 | 0.0 |
| x-space AMV | 45 | 45 | 0.9420 | 0.0 |
| u-space AMV | 45 | 45 | 0.5828 | 0.0 |
| x-space AMV^2 | 45 | 45 | 2.730 | 0.0 |
| u-space AMV^2 | 45 | 45 | 2.828 | 0.0 |
| x-space $AMV+$ | 171 | 179 | 0.0 | 0.0 |
| u-space $AMV+$ | 205 | 205 | 0.0 | 0.0 |
| x-space AMV^2+ | 135 | 142 | 0.0 | 0.0 |
| u-space AMV^2+ | 132 | 139 | 0.0 | 0.0 |
| x-space TANA | 293* | 272 | 0.04259 | 1.598e-4 |
| u-space TANA | 325* | 311* | 2.208 | 5.600e-4 |
| FORM | 720 | 192 | 0.0 | 0.0 |
| SORM | 535 | 191* | 2.410 | 6.522e-4 |

•Note: 2nd-order PMA with prescribed p level is harder problem → requires $\beta(p)$ update/inversion

Optimization under Uncertainty

- Design for reliability is a classic OUU problem, often called RBDO (reliability-based design optimization)
- Nice properties in that the reliability formulation itself generates quantities such as derivatives of performance function with respect to uncertain variables
- Variety of approaches (next page)
- Simplest case: think of a “nested” algorithm, with an optimization outer loop and sampling inner loop



RBDO Algorithms

Bi-level RBDO

- Constrain RIA $z \rightarrow p/\beta$ result
- Constrain PMA $p/\beta \rightarrow z$ result

$$\text{RIA RBDO} \left\{ \begin{array}{ll} \text{minimize} & f \\ \text{subject to} & \beta \geq \bar{\beta} \\ & \text{or } p \leq \bar{p} \end{array} \right.$$

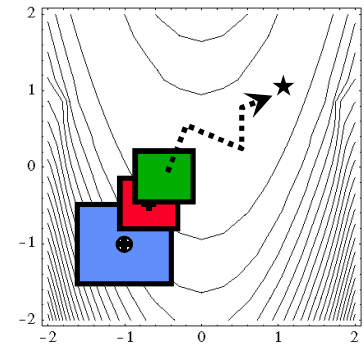
$$\text{PMA RBDO} \left\{ \begin{array}{ll} \text{minimize} & f \\ \text{subject to} & z \geq \bar{z} \end{array} \right.$$

Sequential/Surrogate-based RBDO:

- Break nesting: iterate between opt & UQ until target is met.
Trust-region surrogate-based approach is non-heuristic.

$$\begin{array}{ll} \text{minimize} & f(\mathbf{d}_0) + \nabla_d f(\mathbf{d}_0)^T (\mathbf{d} - \mathbf{d}_0) \\ \text{subject to} & \beta(\mathbf{d}_0) + \nabla_d \beta(\mathbf{d}_0)^T (\mathbf{d} - \mathbf{d}_0) \geq \bar{\beta} \\ & \|\mathbf{d} - \mathbf{d}_0\|_\infty \leq \Delta^k \end{array}$$

1st-order
(also 2nd-order w/ QN)



Unilevel RBDO:

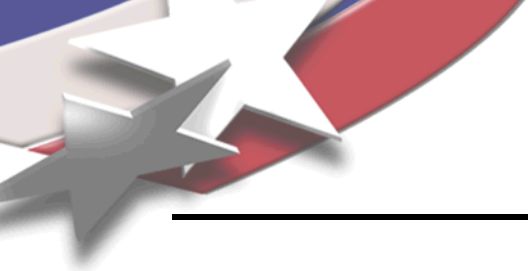
- All at once: apply KKT conditions of MPP search as equality constraints
 - Opt. increases in scale (\mathbf{d}, \mathbf{u})
 - Requires 2nd-order info for derivatives of 1st-order KKT

$$\begin{array}{ll} \min_{\mathbf{d}, \mathbf{u}_1, \dots, \mathbf{u}_{N_{hard}}} & f(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \\ \text{s. t.} & G_i^R(\mathbf{u}_i, \eta) = 0 \\ & \beta_{allowed} - \beta_i \geq 0 \\ & \|\mathbf{u}_i\| \|\nabla_{\mathbf{u}} G_i^R(\mathbf{u}_i, \eta)\| + \mathbf{u}_i^T \nabla_{\mathbf{u}} G_i^R(\mathbf{u}_i, \eta) = 0 \\ & \beta_i = \|\mathbf{u}_i\| \\ & \mathbf{d}^l \leq \mathbf{d} \leq \mathbf{d}^u \end{array} \quad \left. \vphantom{\begin{array}{l} \min \\ \text{s. t.} \end{array}} \right\} \text{KKT of MPP}$$



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- Bichon, B.J., Eldred, M.S., Swiler, L.P., Mahadevan, S., and McFarland, J.M., "Efficient Global Reliability Analysis for Nonlinear Implicit Performance Functions," *AIAA Journal*, Vol. 46, No. 10, October 2008, pp. 2459-2468.
- RIAC (Reliability Information Analysis Center): DoD site with useful information, guides on failure rates, accepted practices, etc.: <http://www.theriac.org/>



New Topic: Importance Sampling for Black-Box Simulators



Motivation

- USE CASE: We have DAKOTA users who take an initial set of Latin Hypercube samples, then would like to perform some additional samples to help refine a failure probability estimate
- They want to do this with relatively small number of samples: 100-200 initial samples and 100-200 samples from an importance sampling density.
- We developed a customized importance sampler where the importance sampling densities are constructed based on kernel density estimators.

Is importance sampling efficient and accurate for situations where we can only afford small numbers of samples?

Does importance sampling require the use of surrogate methods to generate a sufficient number of samples to increase the accuracy of the failure probability estimate?



Background

- Importance sampling is a method used to sample random variables from different densities than originally defined.
- These importance sampling densities are constructed to pick “important” values of input random variables to improve the estimation of a statistical response of interest, such as a mean or probability of failure.

$$E(r(X)) = \int r(x) f_X(x) dx$$

$$\hat{E}_n(r(X)) = \frac{1}{n} \sum_{i=1}^n r(x_i)$$

$$\mathbf{E}(r(X)) = \int r(x) \frac{f(x)}{h(x)} h(x) dx = \hat{\mathbf{E}}_h \left[r(X) \frac{f(X)}{h(X)} \right] = \frac{1}{n} \sum_{i=1}^n \frac{r(x_i) f(x_i)}{h(x_i)}, x_i \sim h(X)$$



Selection of importance density

- The variance of the importance sampling estimator is minimized when $h(x) \propto |r(x)f(x)|$.
- For black-box simulations that have multiple uncertain inputs which may come from a wide variety of random input distribution types, **we cannot generally assume that the importance sampling density will be normal or have a parametric form.**

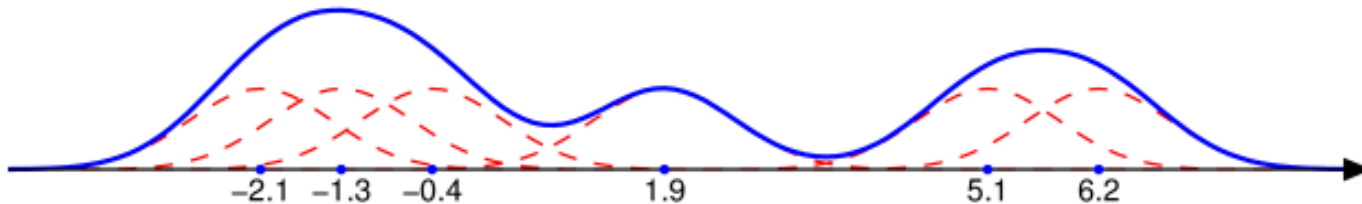
“The most difficult part in parametric importance sampling is choosing a suitable distribution family to start with. *There is no general recipe*, and *the issue remains largely a matter of art* in the literature. Most parametric distributions fail to include g (the optimal importance sampling density) as a member.”

- Nonparametric methods: Zhang (1996) demonstrates increased convergence but higher computational cost of nonparametric methods.

→ **Kernel density estimators**

Kernel density estimators

- Kernel Density Estimation (KDE) is a technique used to estimate the density of a random variable X given n independent samples X_1, \dots, X_n of it.

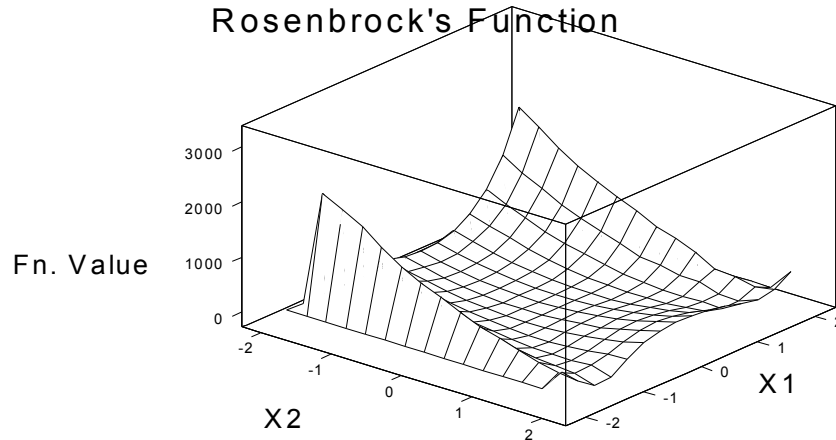


$$f_n^{\{KDE\}}(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right)$$

Used a Gaussian Kernel

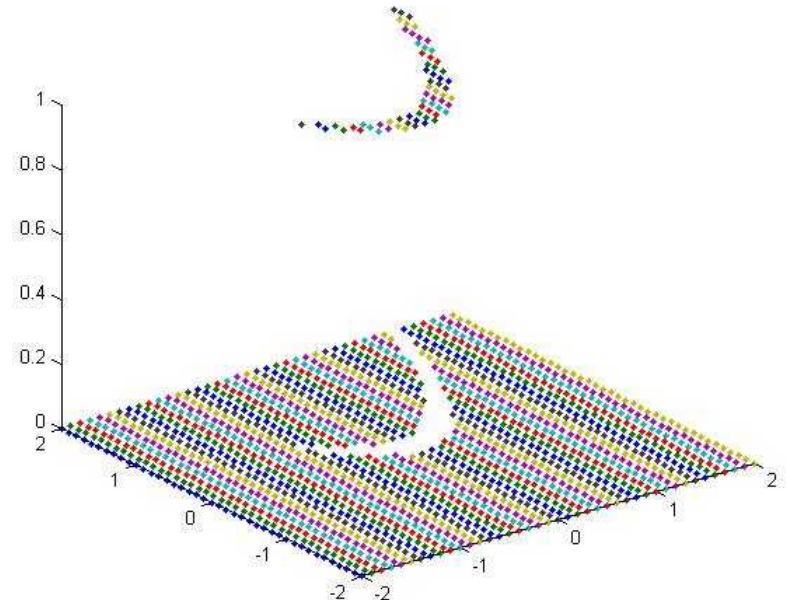
$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Rosenbrock test function



**Failure region defined as
(Rosenbrock function < 3)**

Probability of failure = 0.0383





Importance Sampling Test Results

- Rosenbrock function: X_1, X_2 both distributed as $U[-2,2]$
- Want to obtain estimate of failure probability ($\text{Rosenbrock} < 3$)
- For each combination of initial LHS/ IS points, we ran 100 replicates
- True value ≈ 0.0383

| Number of initial LHS samples | Number of Importance samples | LHS Mean Failure probability | LHS Std dev. Failure Probability | IS Mean Failure probability | IS Std dev. Failure Probability | Mean percentage of IS that “fail” |
|-------------------------------|------------------------------|------------------------------|----------------------------------|-----------------------------|---------------------------------|-----------------------------------|
| 100 | 200 | 0.039 | 0.01096 | 0.03798 | 0.01139 | 0.1283 |
| 50 | 100 | 0.0432 | 0.01429 | 0.03854 | 0.01665 | 0.1159 |
| 100 | 100 | 0.036 | 0.01241 | 0.03986 | 0.01567 | 0.133 |
| 200 | 200 | 0.03785 | 0.00955 | 0.03774 | 0.00823 | 0.1412 |
| 200 | 400 | 0.03702 | 0.00843 | 0.03868 | 0.00768 | 0.1432 |

- Note that failure probabilities not significantly different but we have increased percentage of points that fail by a factor of three
- Additional analysis showed that the accuracy of the failure estimates is not greatly improved IS due to the limited number of samples.
- The main benefit is that KDE IS estimators provide a quick way to generate more samples in the failure region.



Next steps

- We tested this approach on a variety of test problems, focusing on situations with small sample sizes to be representative of expensive computational simulations.
- We have looked at this approach for 5-D problems and a problem with a discontinuity in the response space: it worked fine in both cases
- Further investigation into scaling up to multiple dimensions
- Adding points “near” the response threshold can help
- Also investigated surrogate methods: need to have accurate surrogates
- This approach is reasonably robust and can produce failure probability estimates that are comparable to failure estimates produced by small numbers of LHS sample points.
- The main benefit we see by using this approach is that the kernel density estimators **provide a quick way to generate more samples in the failure region.** We found that importance sampling increased the number of samples in the failure region by a factor of 3 to 8 for our test cases.