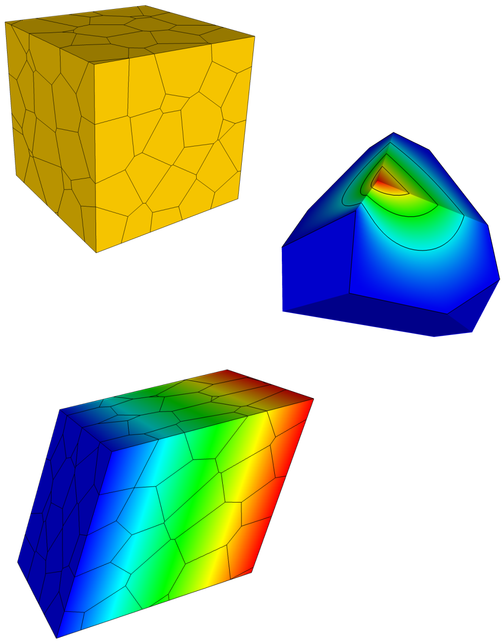
SAND2013-5954C

A Displacement-Based Finite-Element Formulation for General Polyhedra using Harmonic Shape Functions



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Albuquerque, NM

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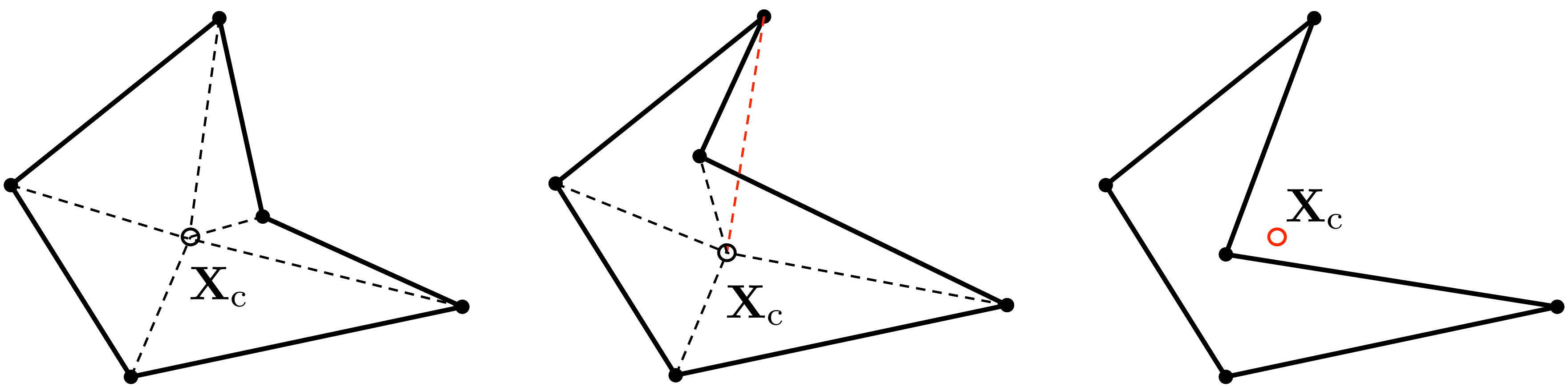
Polyhedral Finite-Element Formulation

- Applicable to nonlinear solid mechanics
- General polyhedra: non-convex with non-planar faces
- Compatible with standard trilinear hexahedron
- Use harmonic shape functions
- “Correct” shape-function derivatives to pass the patch-test
- Mean-dilation formulation for nearly-incompressible materials

Star Convexity

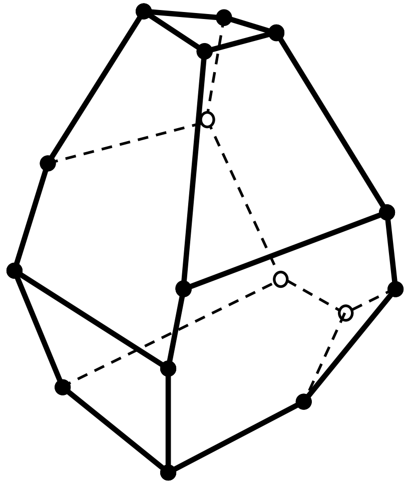
For ease of construction, present formulation assumes star-convexity with respect to vertex-averaged centroid.

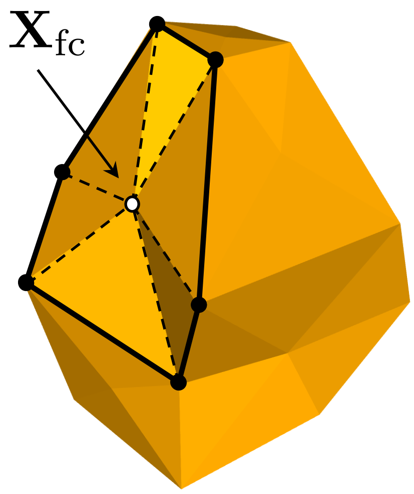


star convex | not star convex | not star convex

How to Fully Specify Face Geometry?

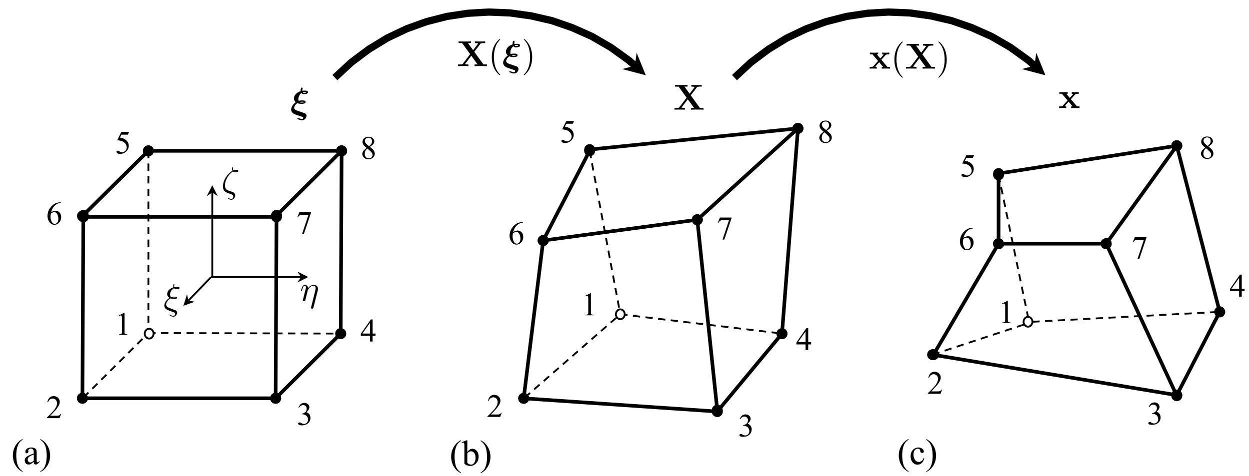
- Use vertex-averaged centroid.
- Could also use a bilinear mapping for quadrilateral faces.



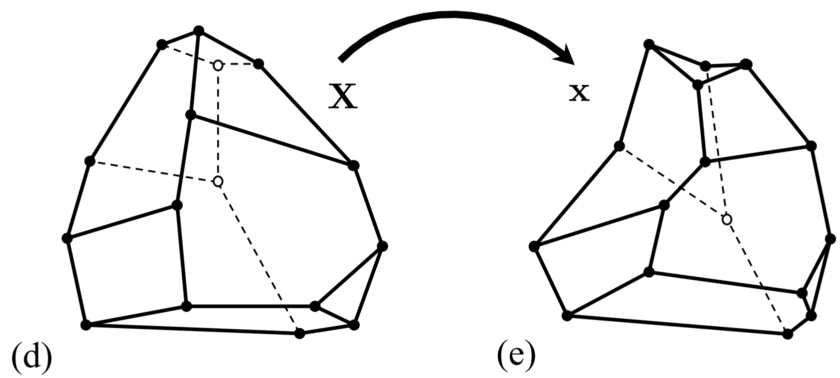


Define Shape Functions Directly on Initial Configuration

standard trilinear hexahedral mapping using a parent c.s.



present formulation defines shape functions directly on initial configuration.

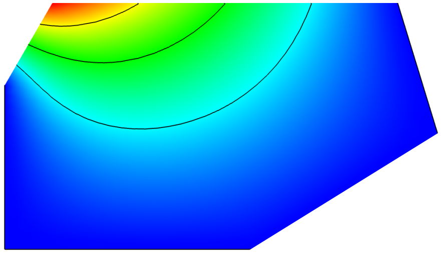


Harmonic Shape Functions

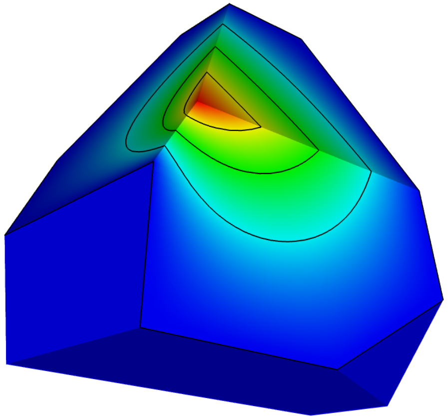
A harmonic function is a solution of Laplace's equation.

$$\nabla^2 \psi = 0$$

Can solve efficiently using BEM, or can just use FEM.



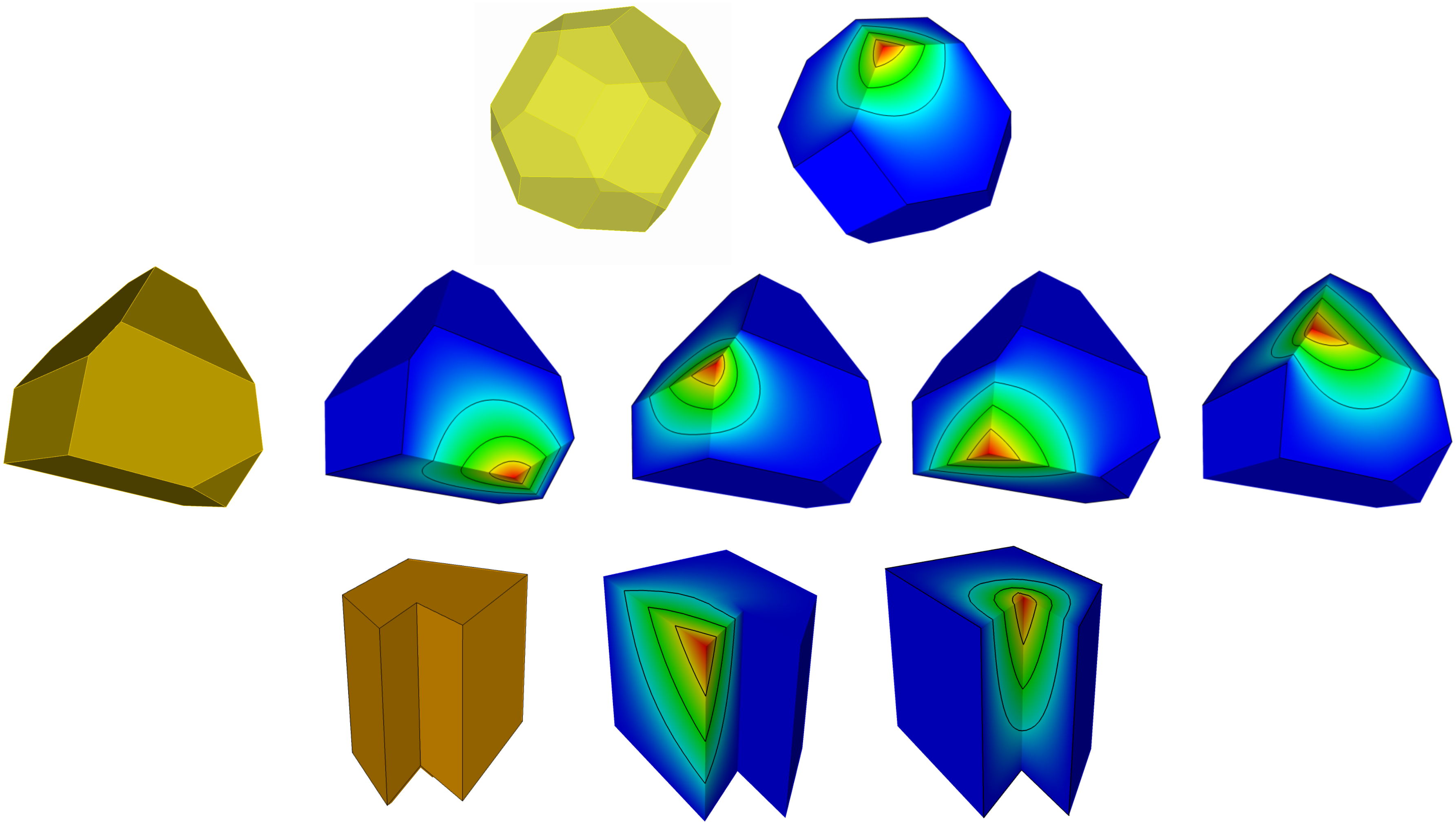
example in 2D



example in 3D

Note: Only need to store shape function values and derivatives at the quadrature points.

Harmonic Shape Function Examples



Only need to store shape functions and derivatives at integration points.
Discard everything else.

Harmonic Shape Function Properties

$$\sum_{a=1}^{N_\mathrm{v}} \psi^a(\mathbf{X}) = 1\,, \quad \mathbf{X} \in \Omega_\mathrm{e}$$

partition of unity

$$\sum_{a=1}^{N_\mathrm{v}} \psi^a(\mathbf{X})\,\mathbf{X}^a = \mathbf{X}\,, \quad \mathbf{X} \in \Omega_\mathrm{e}$$

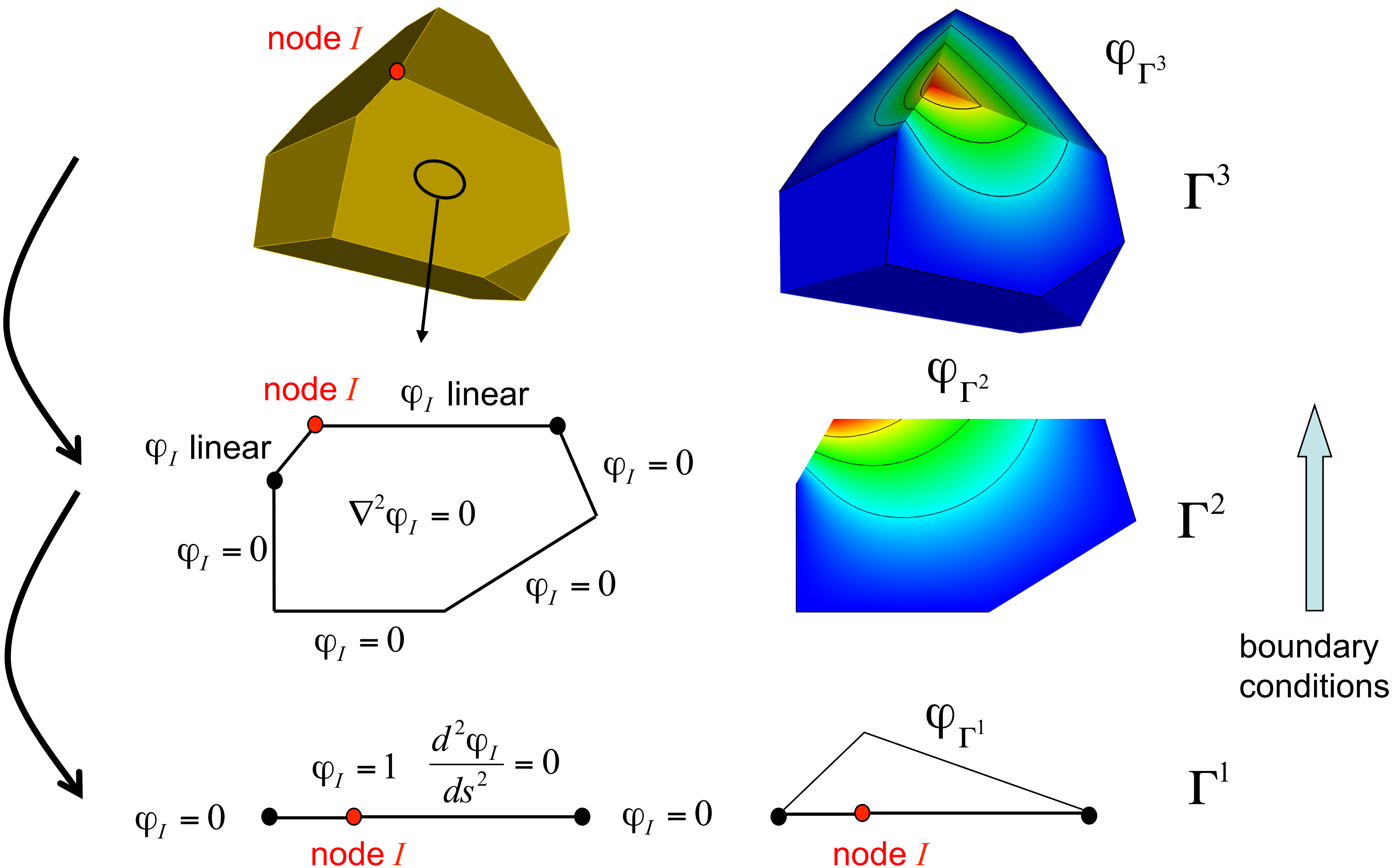
reproduce linear fields

$$\psi^a(\mathbf{X}^b) = \delta_{ab}$$

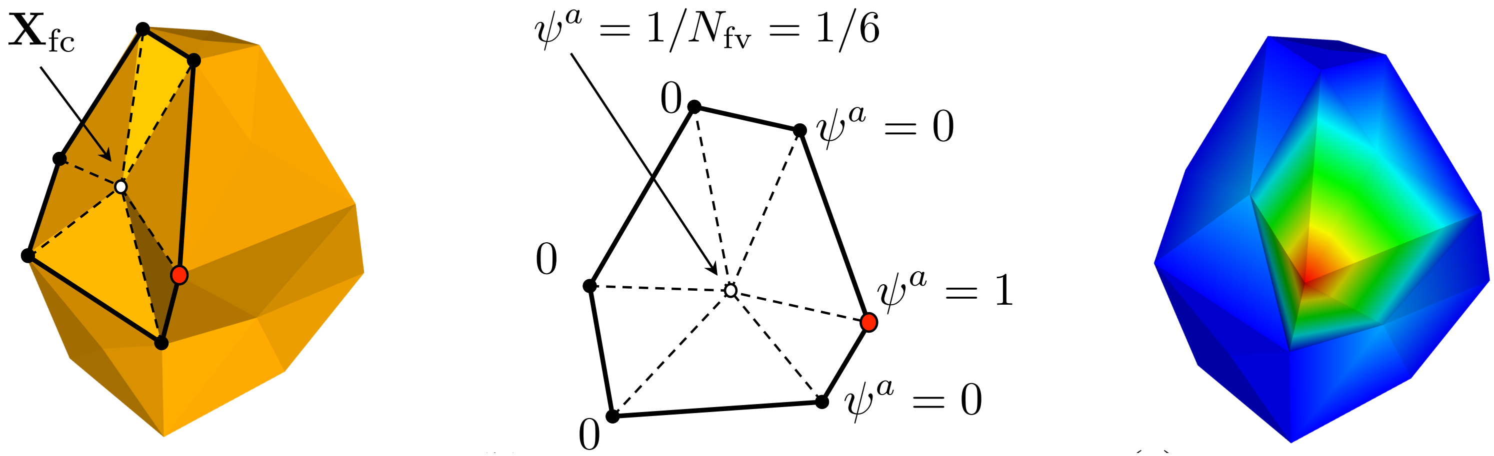
Kronecker-delta property at nodes

Hierarchical Construction of Harmonic Shape Functions

(Joshi, 2007)



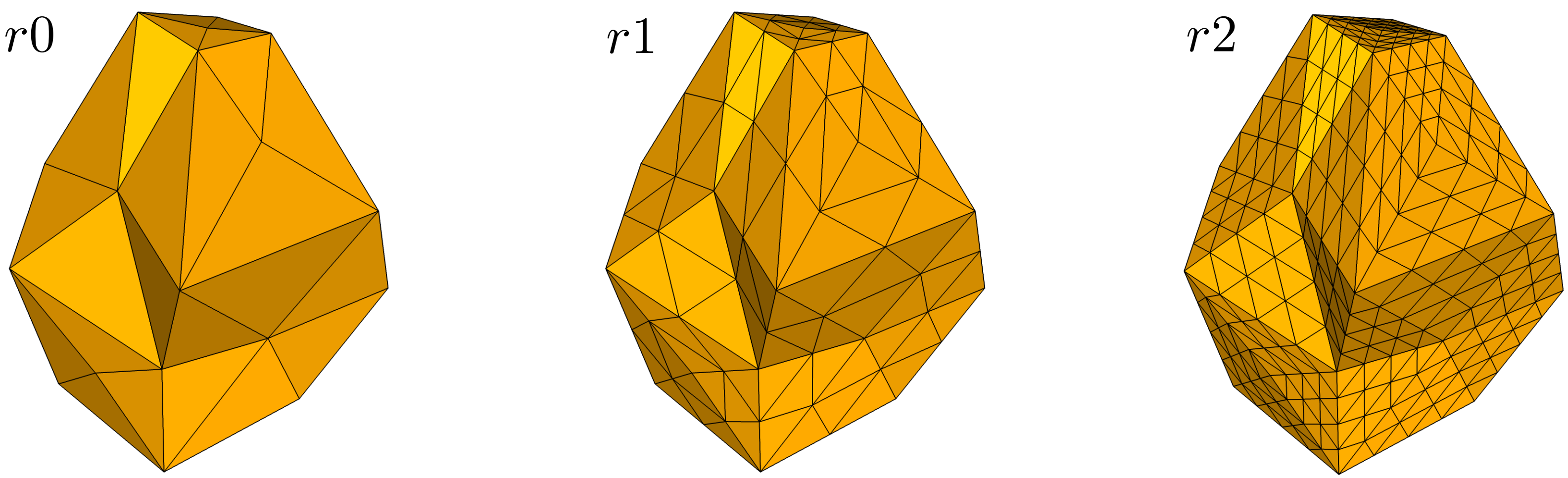
Harmonic Shape Functions for Non-planar Faces



Can also use other barycentric face mappings.

How to Solve for Harmonic Shape Functions using FEA

Use a temporary tetrahedral submesh.

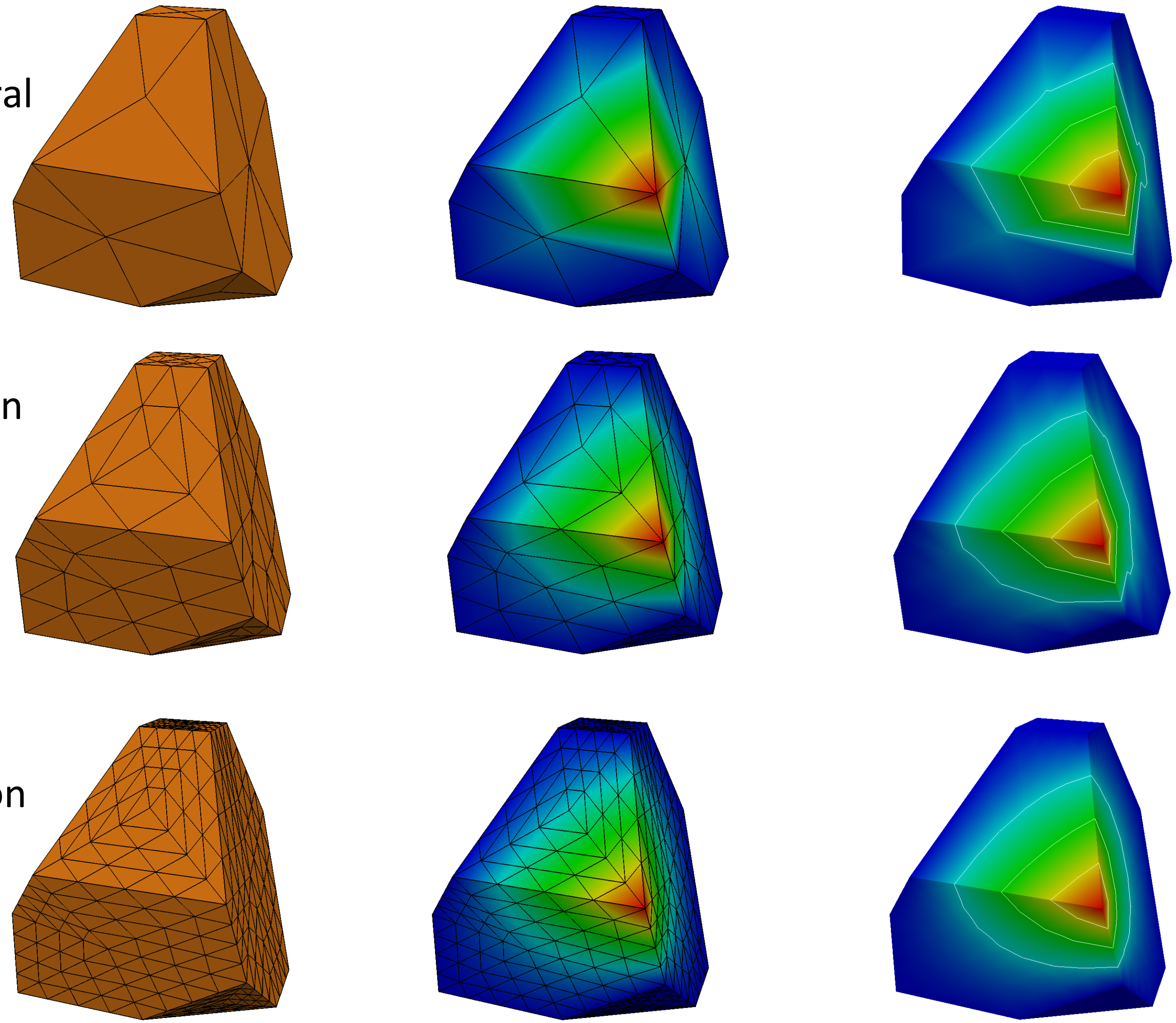


Accuracy of Harmonic Shape Functions?

Base tetrahedral subdivision — R0

1 : 8 subdivision — R1

1 : 8 subdivision — R2

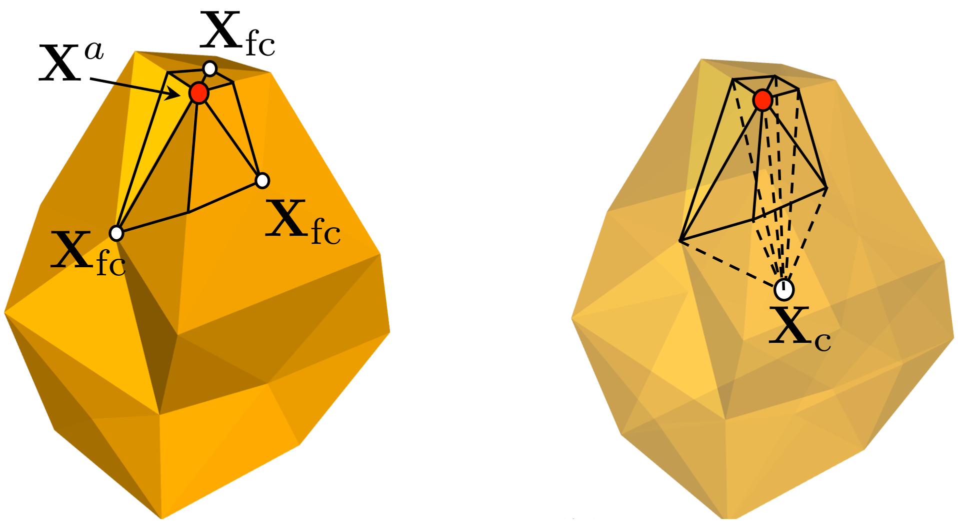


Numerical Precision in Reproducing Properties

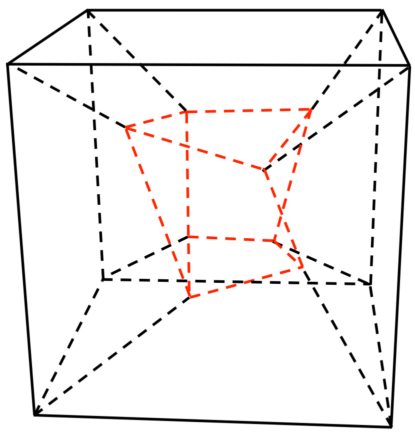
	Partition of Unity	Reproduction of Linear Fields
subdivision	$\max_{k \in \{1,\ldots,N_{\text{i.p.}}\}} \left[\sum_{a=1}^{N_\text{v}} \psi^a(\mathbf{X}^k) - 1 \right]$	$\max_{\substack{k \in \{1,\ldots,N_{\text{i.p.}}\} \\ j \in \{1,2,3\}}} \left[\sum_{a=1}^{N_\text{v}} \psi^a(\mathbf{X}^k) X_j^a - X_j^k \right]$
$r0$	3.33×10^{-16}	5.55×10^{-16}
$r1$	6.66×10^{-16}	5.55×10^{-16}
$r2$	1.55×10^{-15}	5.55×10^{-16}

Element Integration

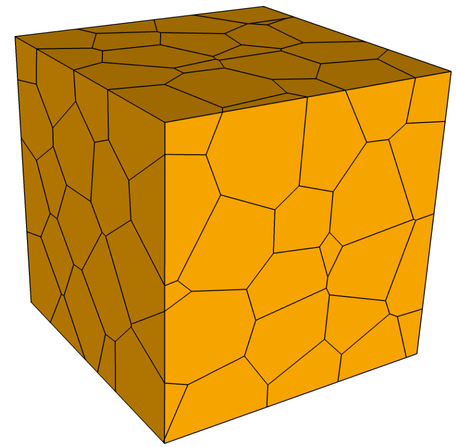
- Due to computational expense of plasticity models, want to minimize the number of quadrature points.
- Follow approach of Rashid and Selimotec, 2006.
- Each node is associated with a “tributary” volume.
- Number of quadrature points is equal to the number of vertices.
- Quadrature weight = volume of tributary volume.
- First-order accurate, but quadrature weights are positive (avoids Runge’s phenomenon)



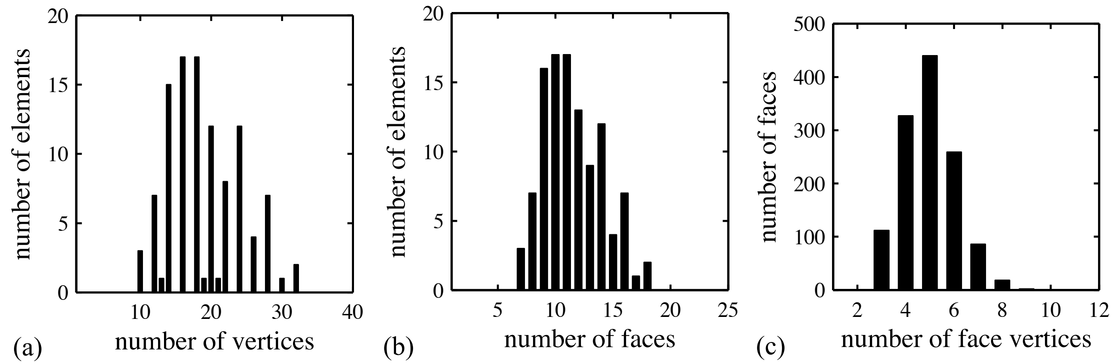
Patch Test



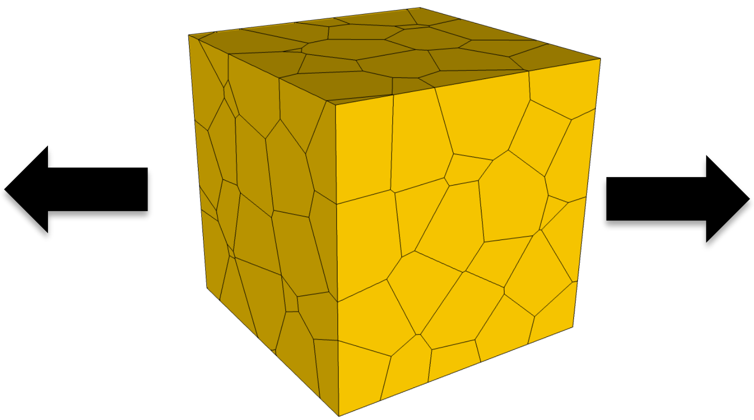
distorted hex patch



random close-packed Voronoi patch

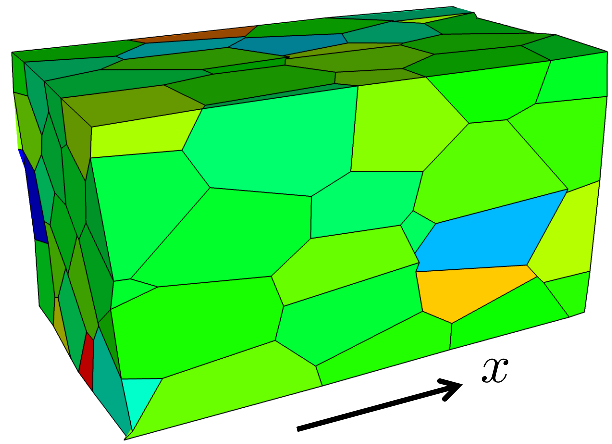


Patch Test



patch of elements

Failed patch test!



stress error > 10%

Patch Test and Integration Consistency

Divergence theorem

$$\int_{\Omega_e} \psi^a_{,i}\, d\Omega = \int_{\Gamma_e} \psi^a\, n_i\, d\Gamma\ , \quad a = 1, \ldots, N_v\ , \quad i = 1, 2, 3$$

Discrete divergence theorem

$$\sum_{k=1}^{N_{\mathrm{i.p.}}} w_k \psi^{ak}_{,i} = \sum_{l=1}^{N^{\Gamma}_{\mathrm{i.p.}}} w^{\Gamma}_l\, \psi^{al} n^l_i\ , \quad a = 1, \ldots, N_v \quad i = 1, 2, 3$$

Maximum error in integration constraint

subdivision	before derivative correction	after derivative correction
$r0$	0.0609	2.77×10^{-17}
$r1$	0.0138	2.77×10^{-17}
$r2$	0.0106	2.77×10^{-17}

(error over all shape functions and coordinate directions)

Derivative Correction to Pass the Patch Test

- "Tweak" the shape function derivatives to satisfy the integration consistency condition.
- Maintain the reproducing properties of the derivatives.
- Minimize the difference between the new derivatives and the old.
- Local solve at the element level; performed once.
- Performed for each direction and shape function independently.

$$\min_{\xi^k \in \Re} \sum_{k=1}^{N_{\mathrm{i.p.}}} \left(\xi^k - \psi^{ak}_{,i}\right)^2 \quad \text{subject to the constraints} \quad \sum_{k=1}^{N_{\mathrm{i.p.}}} w_k \xi^k - \sum_{l=1}^{N^{\Gamma}_{\mathrm{i.p.}}} w^{\Gamma}_l \, \psi^{al} n^l_i = 0$$

Derivative Correction to Pass the Patch Test

Maximum error in integration constraint

subdivision	before derivative correction	after derivative correction
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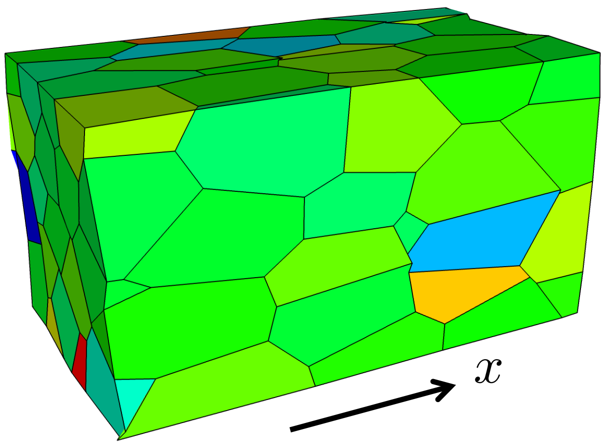
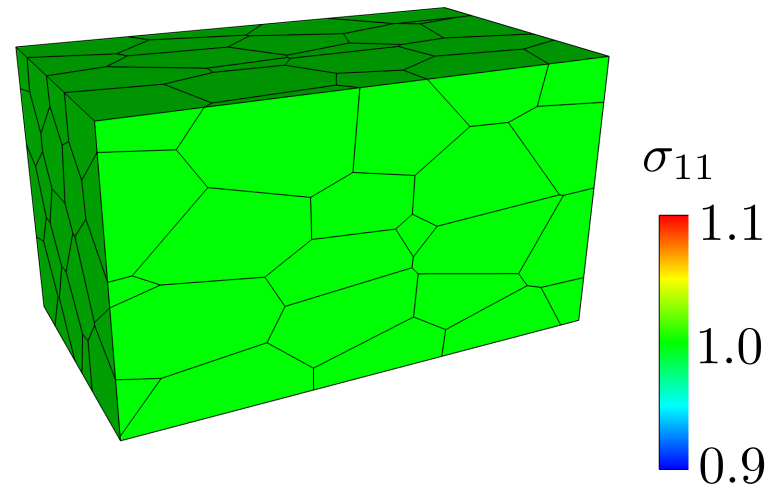
(error over all shape functions and coordinate directions)

Patch Test: Before and After



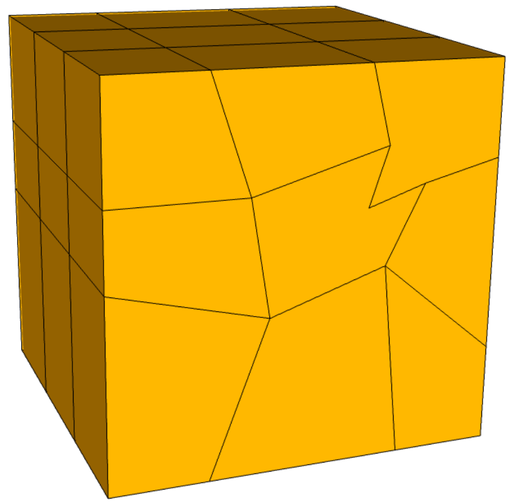
failed patch test

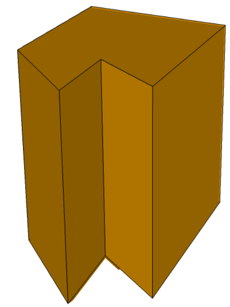


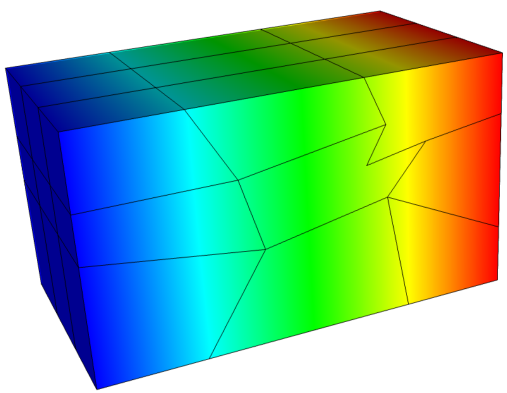
successful patch test

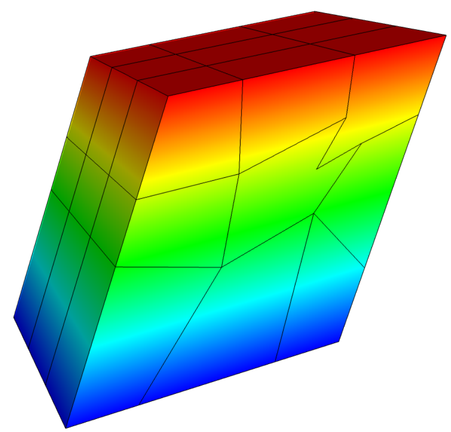
case	without derivative correction	with derivative correction
hex patch, trilinear formulation	1.11×10^{-15}	–
hex patch, poly formulation	0.0863	5.55×10^{-16}
hex patch, trilinear and poly	0.0152	8.88×10^{-16}
random Voronoi patch	0.1844	1.41×10^{-12}

Patch Test with Non-Convex Elements







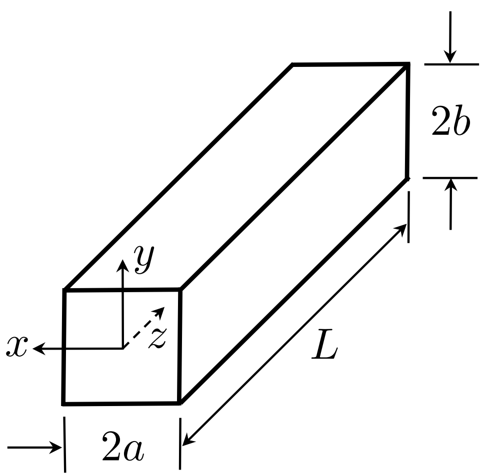


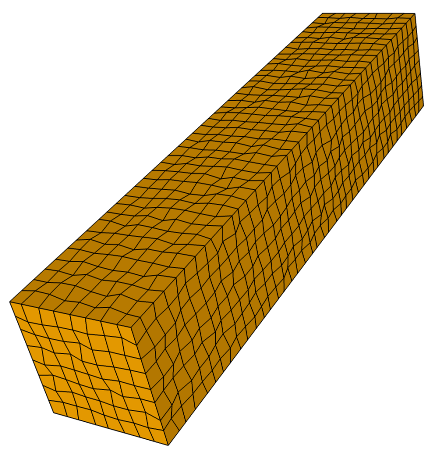
Verification Tests

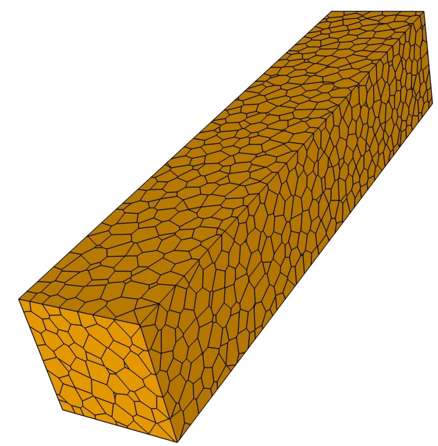
<u>loading</u>: beam bending with shear load

<u>meshes and element formulations</u>:

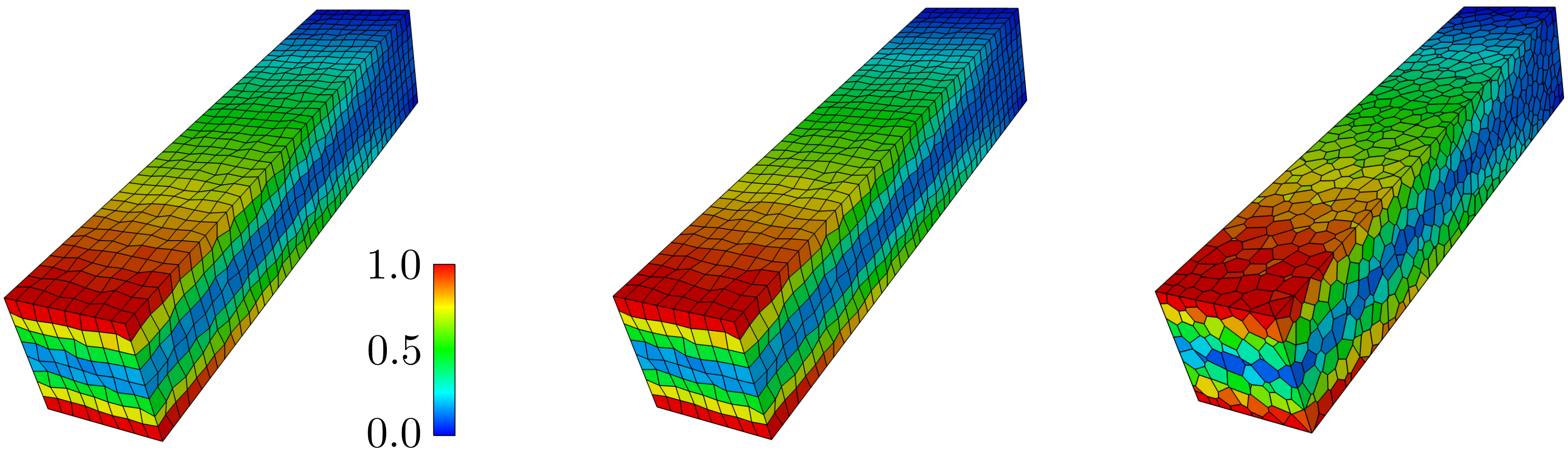
1. distorted hex mesh, trilinear hex formulation
2. distorted hex mesh, poly formulation
3. Voronoi mesh, poly formulation





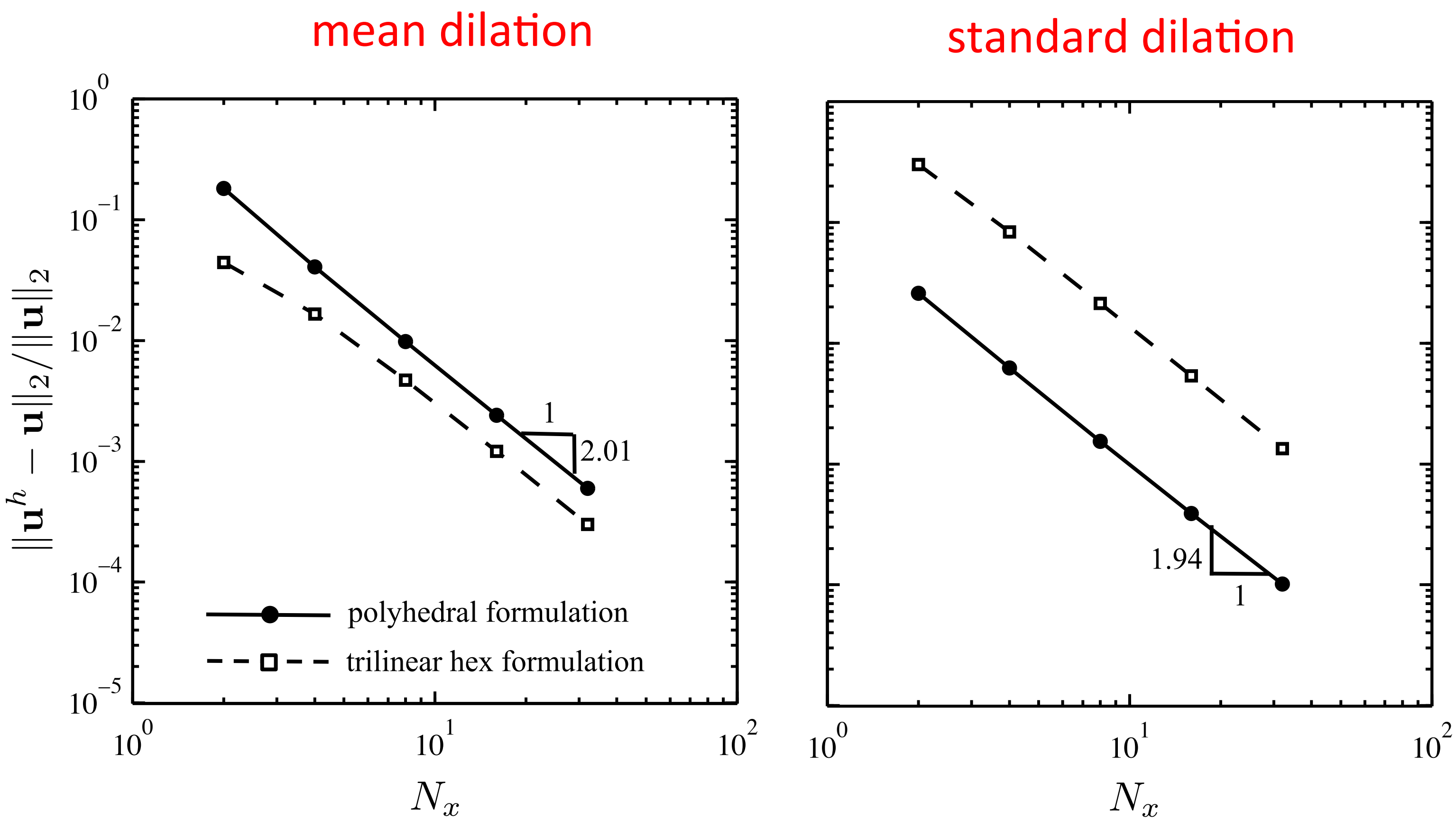


Verification Test: Beam Bending with Shear Load

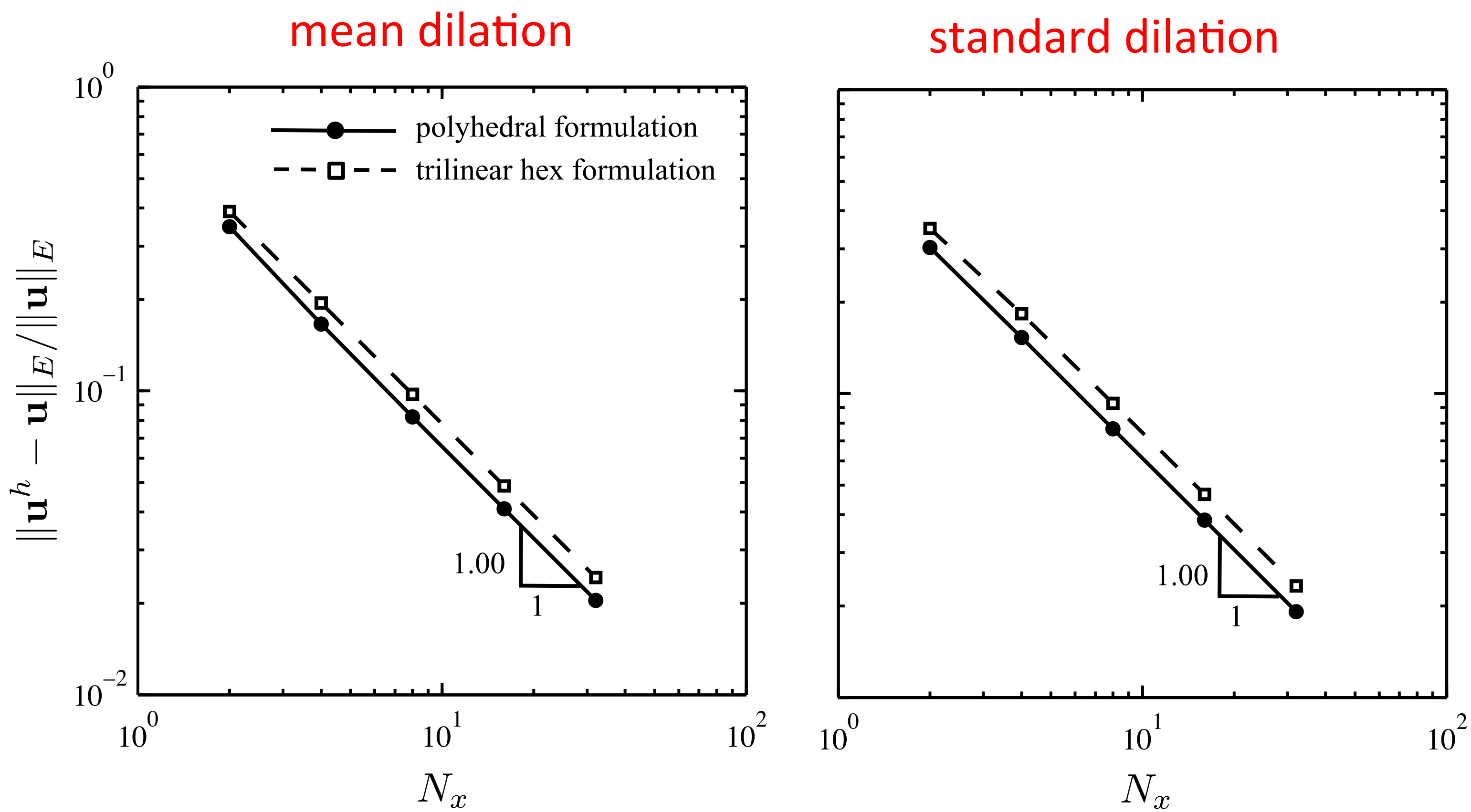


trilinear hex formulation   poly formulation   poly formulation

Beam Bending with Shear Load



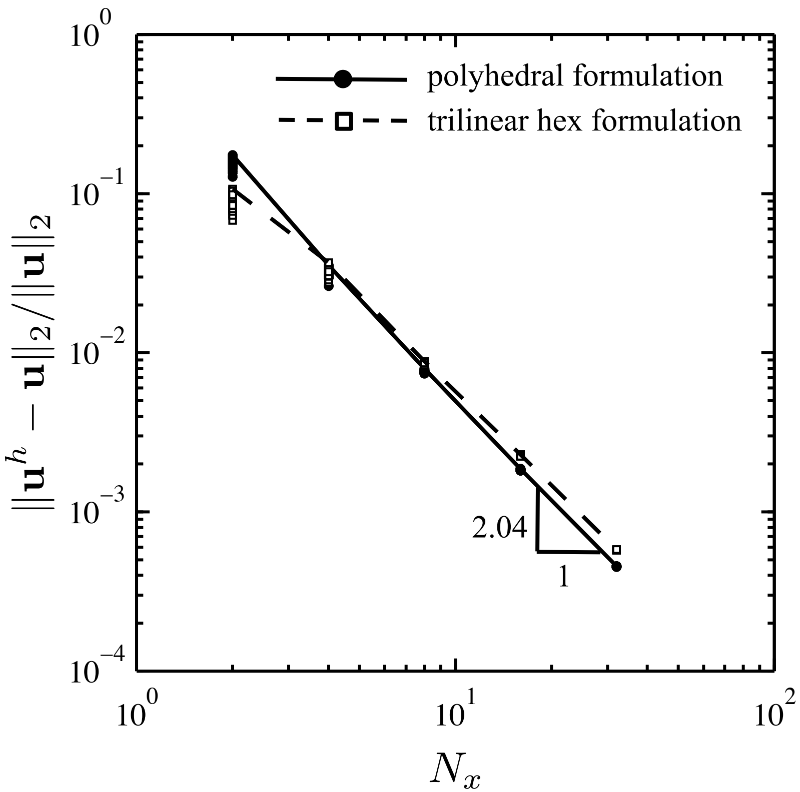
Energy Norm



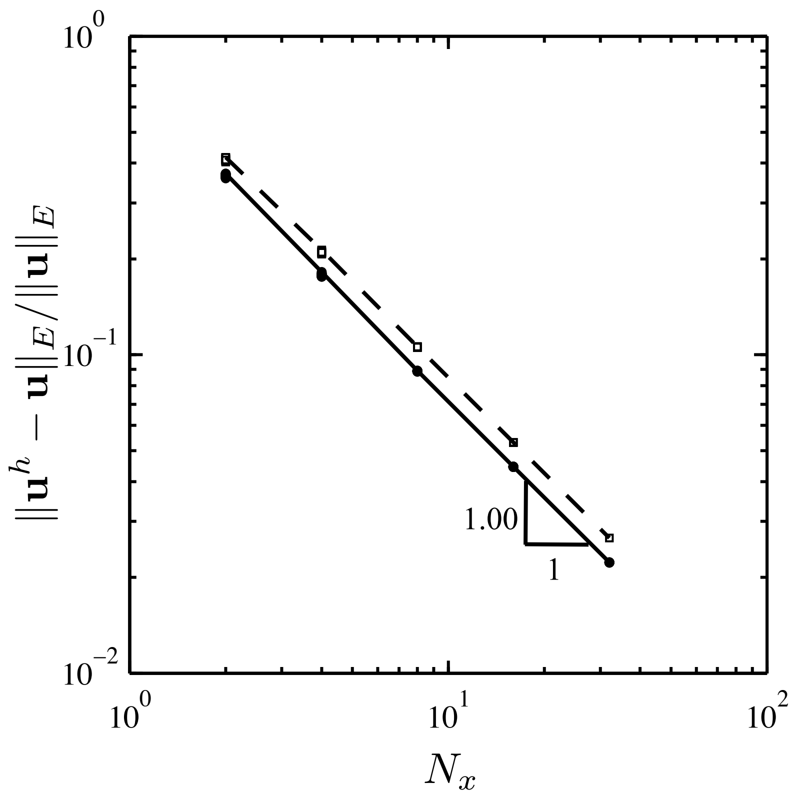
mean dilation
standard dilation
polyhedral formulation
trilinear hex formulation
1.00
1
1.00
1
10^0
10^{-1}
10^{-2}
10^0
10^1
10^2
10^0
10^1
10^2
$\|\mathbf{u}^h - \mathbf{u}\|_E / \|\mathbf{u}\|_E$
N_x
N_x

Random Hex Mesh (20 realizations)

L2 norm

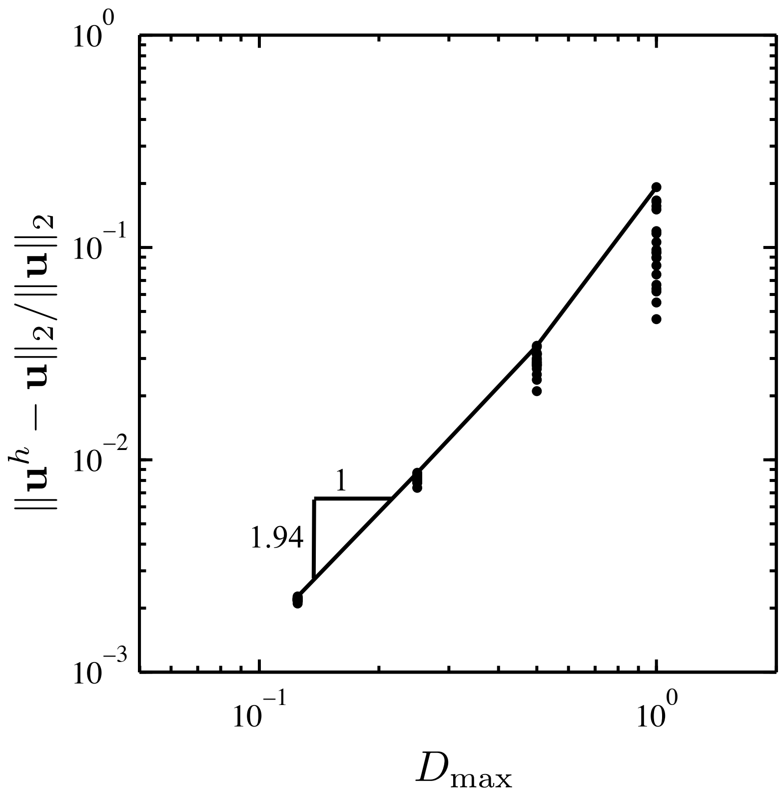


Energy Norm

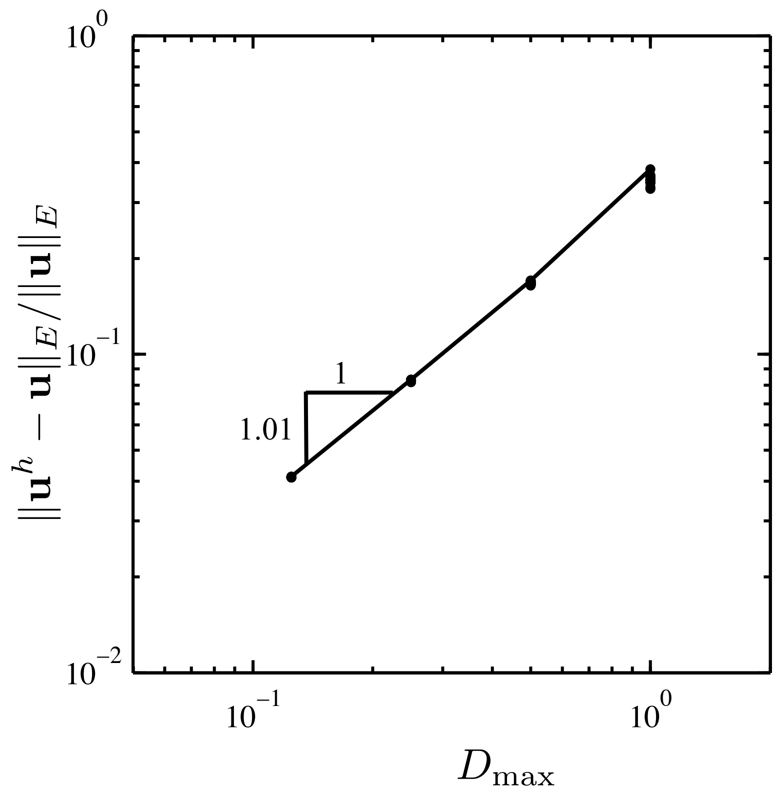


Random Voronoi Mesh (20 realizations)

L2 norm

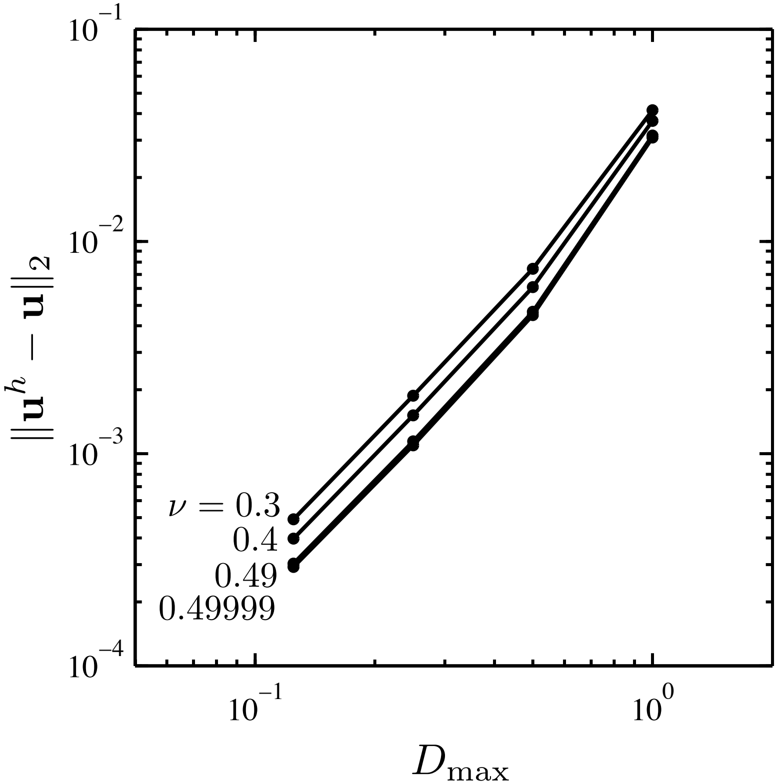


Energy Norm

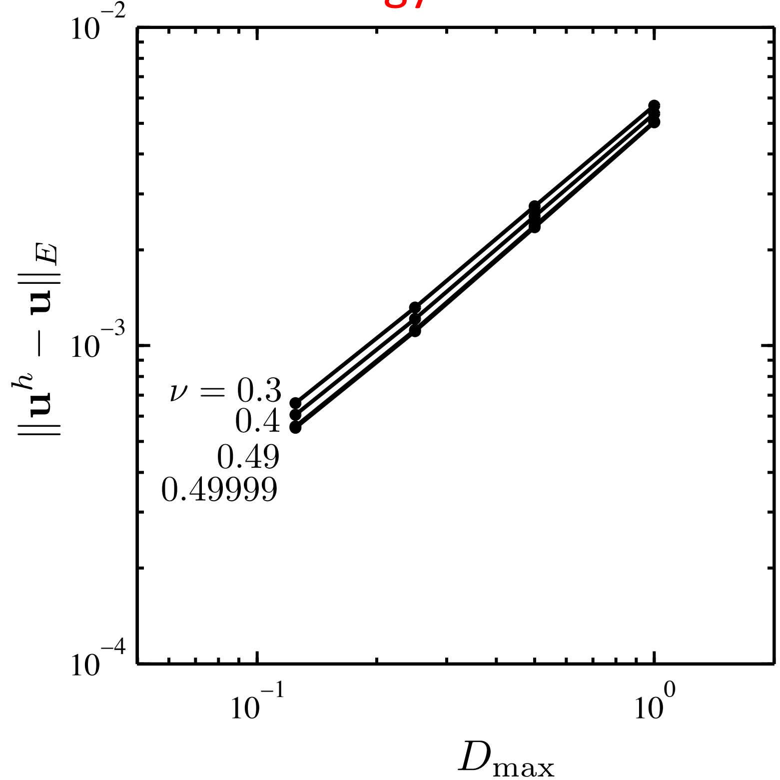


Near Incompressibility

L2 norm

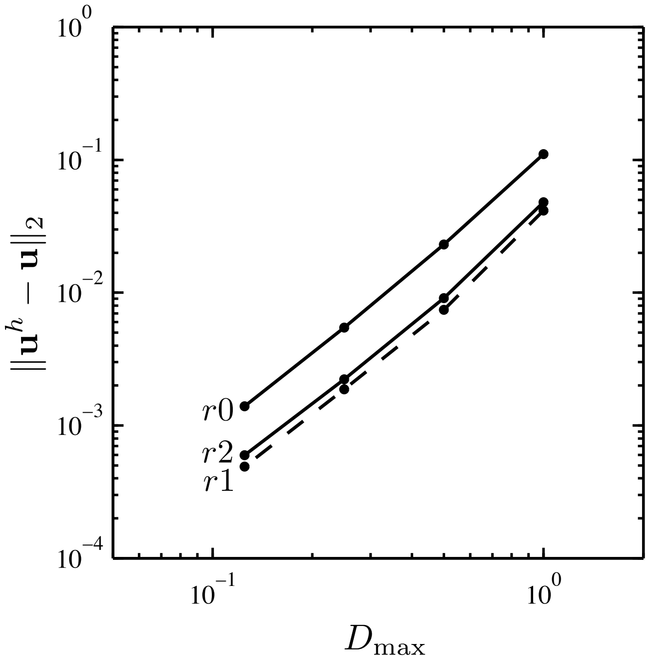


Energy Norm

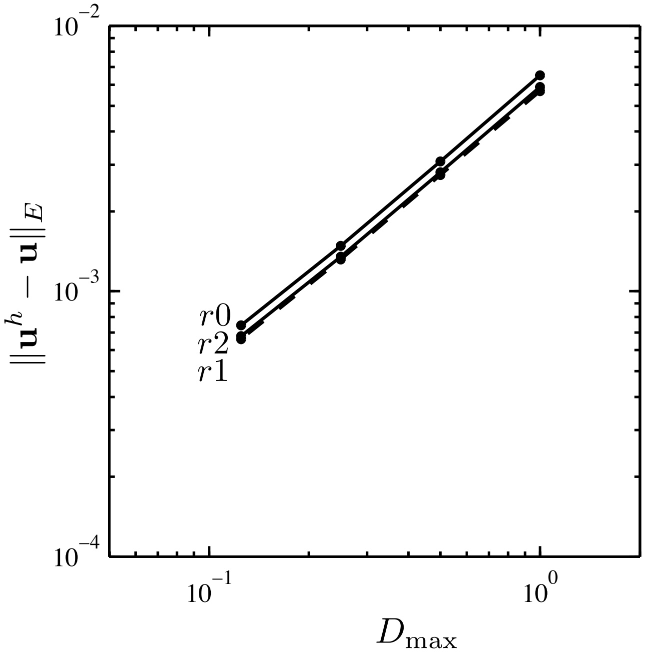


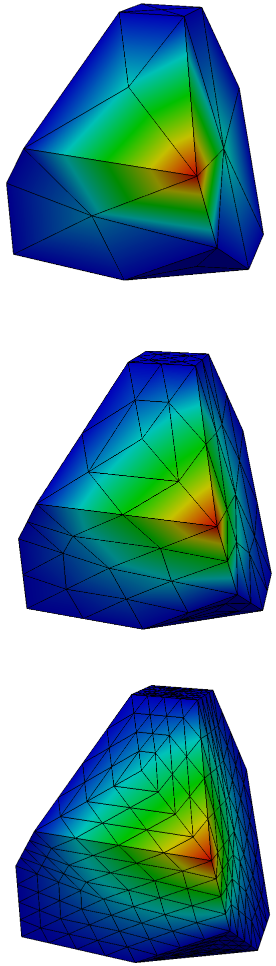
Effect of Shape Function Accuracy

L2 norm



Energy Norm





R0

R1

R2

Summary

1. Presented a polyhedral finite-element formulation based on harmonic shape functions.
2. Applicable to non-convex elements with non-planar faces.
3. Adopted quadrature scheme of Rashid (number of quadrature points = number of vertices.
4. In order to pass the patch test, needed to use "pseudo-derivatives".

Bishop, J., 2013, "A Displacement-Based Finite Element Formulation for General Polyhedra using Harmonic Shape Functions," IJNME, (accepted)