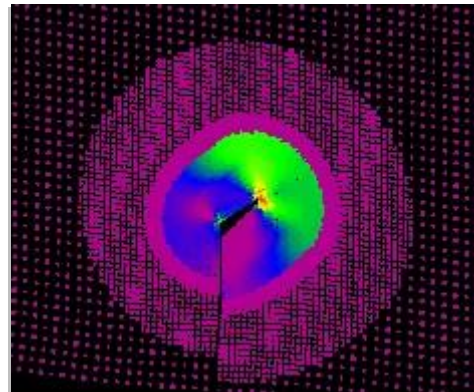
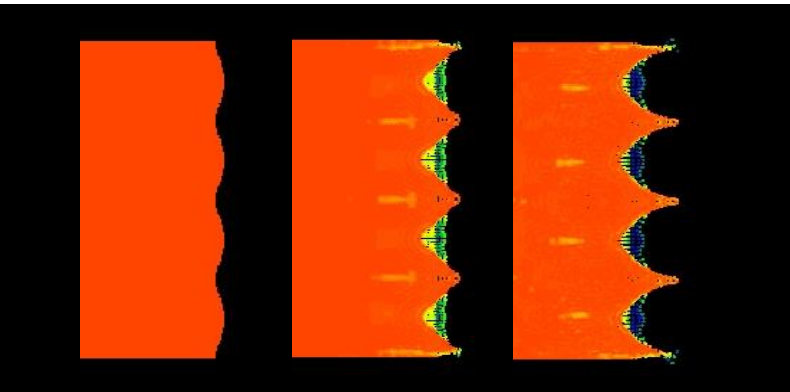


Exceptional service in the national interest



Variable horizon in a peridynamic body

Stewart Silling

Sandia National Laboratories

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University of Texas, Austin

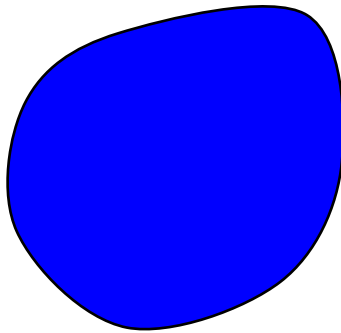
USNCCM12, Raleigh, NC, July 23, 2013

Outline

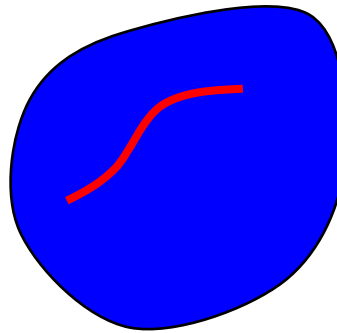
- Peridynamics background
 - States, horizon
- Rescaling a material model (at a point)
- Variable length scale (over a region)
- Partial stress
- Local-nonlocal coupling examples

Purpose of peridynamics

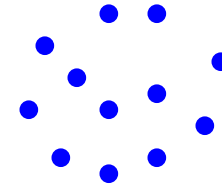
- To unify the mechanics of continuous and discontinuous media within a single, consistent set of equations.



Continuous body



Continuous body
with a defect



Discrete particles

- Why do this? Develop a mathematical framework that help in modeling...
 - Discrete-to-continuum coupling
 - Cracking, including complex fracture patterns
 - Communication across length scales.

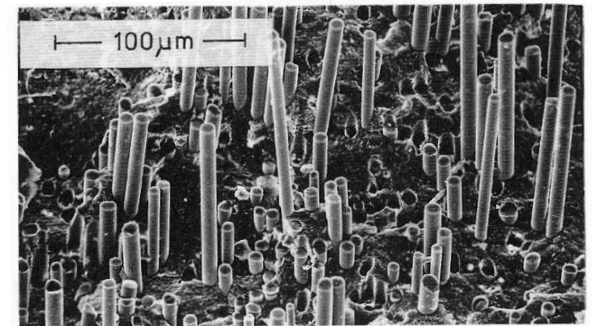


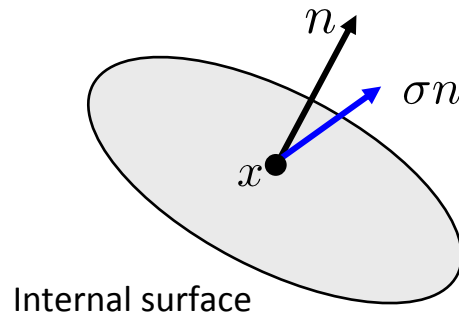
Figure 11.20 Pull-out: (a) schematic diagram; (b) fracture surface of 'Silceram' glass-ceramic reinforced with SiC fibres. (Courtesy H. S. Kim, P. S. Rogers and R. D. Rawlings.)

Peridynamics basics:

The nature of internal forces

Standard theory

Stress tensor field
(assumes contact forces and
smooth deformation)

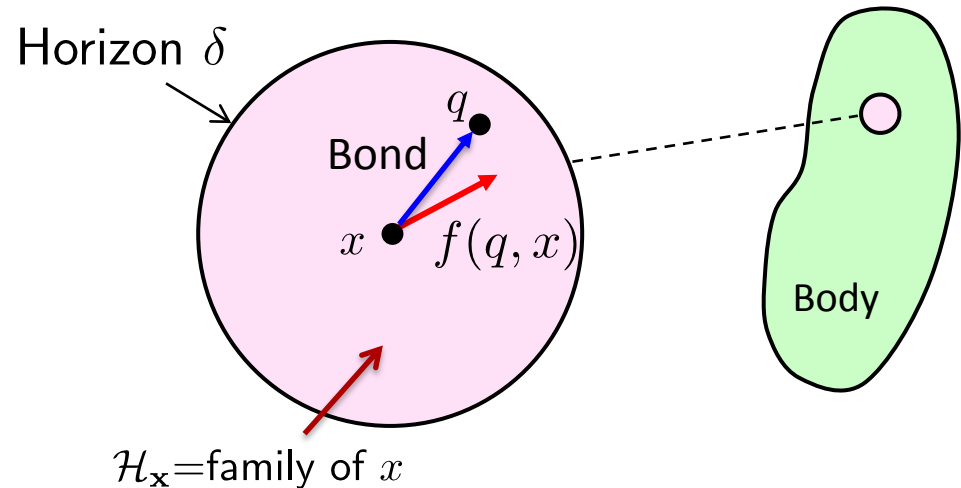


$$\rho \ddot{u}(x, t) = \nabla \cdot \sigma(x, t) + b(x, t)$$

Differentiation of contact forces

Peridynamics

Bond forces within small neighborhoods
(allow discontinuity)



$$\rho \ddot{u}(x, t) = \int_{\mathcal{H}_x} f(q, x) dq + b(x, t)$$

Summation over bond forces

Peridynamics basics:

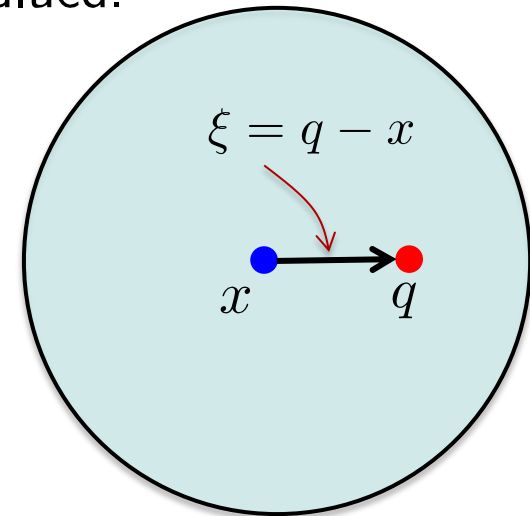
States

- A *state* is a mapping on bonds in a family.
- Notation:

$$\underline{A}[x]\langle\xi\rangle$$

where \underline{A} is a state at a point x in a body, and ξ is a bond in the family of x .

- Most of the states we deal with are vector-valued.



Peridynamics basics:

Deformation state and force state

- The *deformation state* maps each bond to its deformed image.

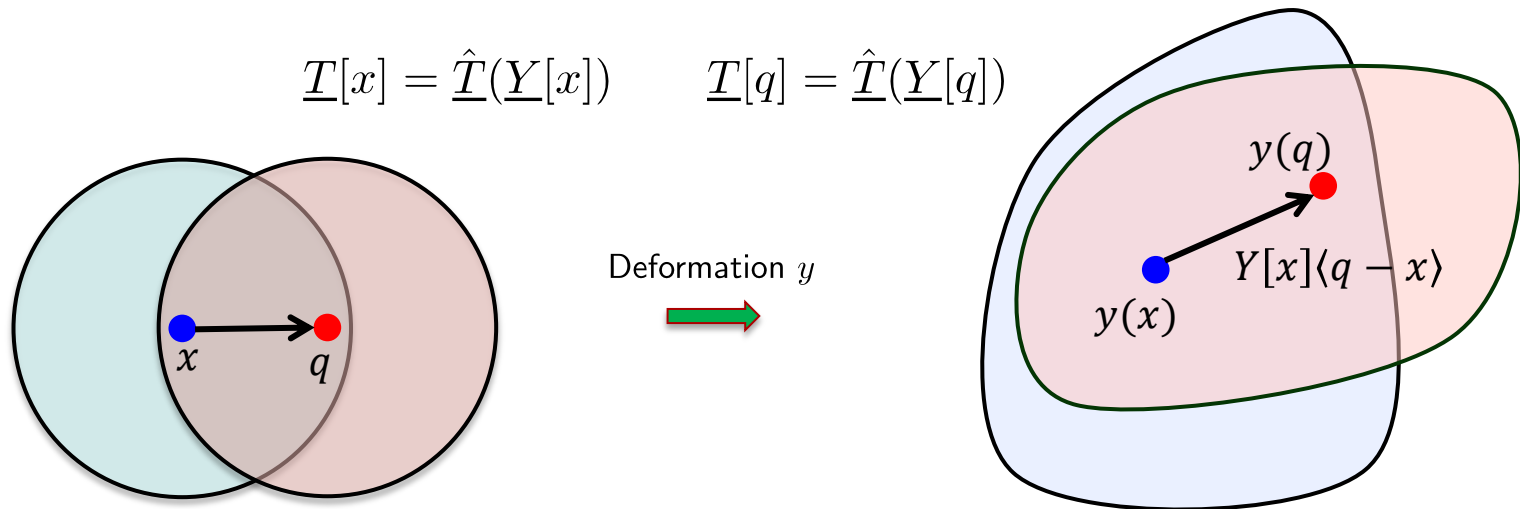
$$\underline{Y}[x]\langle q - x \rangle = y(q) - y(x)$$

- The *force state* maps each bond to bond force densities.

$$f(q, x) = \underline{T}[x]\langle q - x \rangle - \underline{T}[q]\langle x - q \rangle$$

- The *material model* at x maps deformation states to force states.

$$\underline{T}[x] = \hat{\underline{T}}(\underline{Y}[x]) \quad \underline{T}[q] = \hat{\underline{T}}(\underline{Y}[q])$$



Peridynamics basics:

Elastic materials

- A material is *elastic* if there is a scalar-valued *strain energy density function* $\hat{W}(\underline{Y})$ such that

$$\hat{T} = \hat{W}_{\underline{Y}}$$

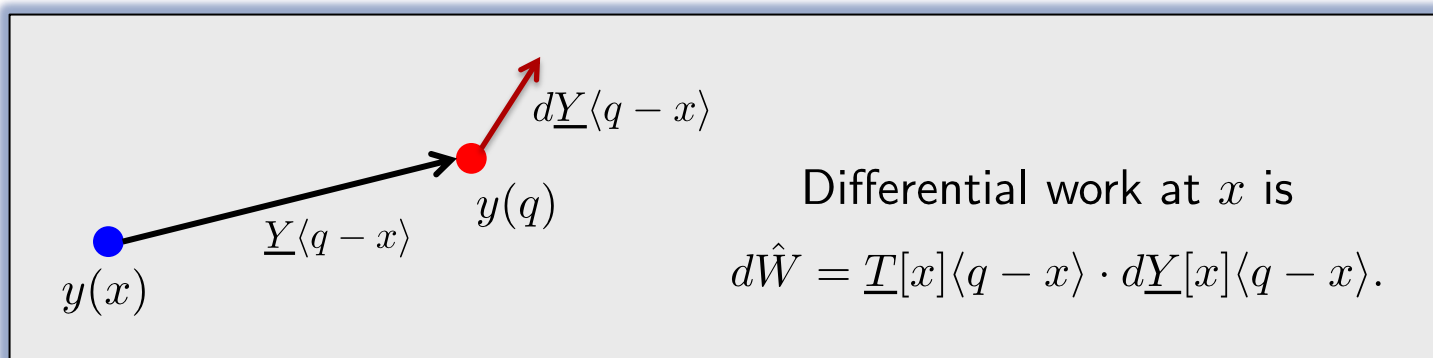
- $\hat{W}_{\underline{Y}}$ denotes the Frechet derivative with respect to the deformation state

$$\hat{W}(\underline{Y} + d\underline{Y}) = \hat{W}(\underline{Y}) + \hat{W}_{\underline{Y}} \bullet d\underline{Y}$$

for any small \underline{Y} .

- Here,

$$\underline{A} \bullet \underline{B} = \int_{\mathcal{H}_x} \underline{A}(\xi) \cdot \underline{B}(\xi) d\xi$$



Scaling of a material model

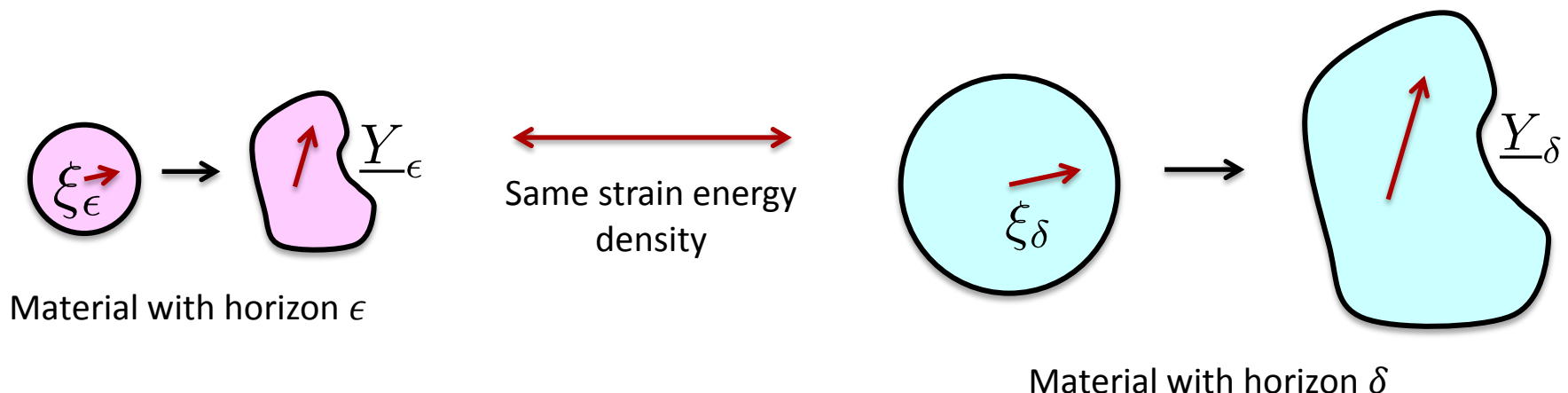
- Suppose we have a material model \hat{W}_ϵ for some given horizon ϵ .
- Try to find a material model with a new horizon δ such that $\hat{W}_\delta = \hat{W}_\epsilon$ whenever the deformation is homogeneous.

- Define:

$$r = \epsilon/\delta \quad \xi_\epsilon = r\xi_\delta \quad \underline{Y}_\epsilon\langle\xi_\epsilon\rangle = r\underline{Y}_\delta\langle\xi_\delta\rangle$$

- Our scaled material model is given by

$$\hat{W}_\delta(\underline{Y}_\delta) := \hat{W}_\epsilon(\underline{Y}_\epsilon).$$



Scaling of the force state

- Now find the scaled force state \underline{T}_δ . We know

$$dW = \underline{T}_\epsilon \bullet d\underline{Y}_\epsilon = \underline{T}_\delta \bullet d\underline{Y}_\delta.$$

- Then

$$\begin{aligned}\underline{T}_\epsilon \bullet d\underline{Y}_\epsilon &= \int_\epsilon \underline{T}_\epsilon \langle \xi_\epsilon \rangle \cdot d\underline{Y}_\epsilon \langle \xi_\epsilon \rangle d\xi_\epsilon \\ &= \int_\delta \underline{T}_\epsilon \langle \xi_\epsilon \rangle \cdot (r d\underline{Y}_\delta \langle \xi_\delta \rangle) (r^D d\xi_\delta) \\ &= r^{D+1} \underline{T}_\epsilon \bullet d\underline{Y}_\delta\end{aligned}$$

where D is the number of dimensions.

- Hence the scaled force state is

$$\underline{T}_\delta(\underline{Y}_\delta) = r^{D+1} \underline{T}_\epsilon(\underline{Y}_\epsilon), \quad r = \epsilon/\delta.$$

Rescaling works fine if the horizon is independent of position

- Example: Uniform strain in a homogeneous bar.

$$u = Hx$$

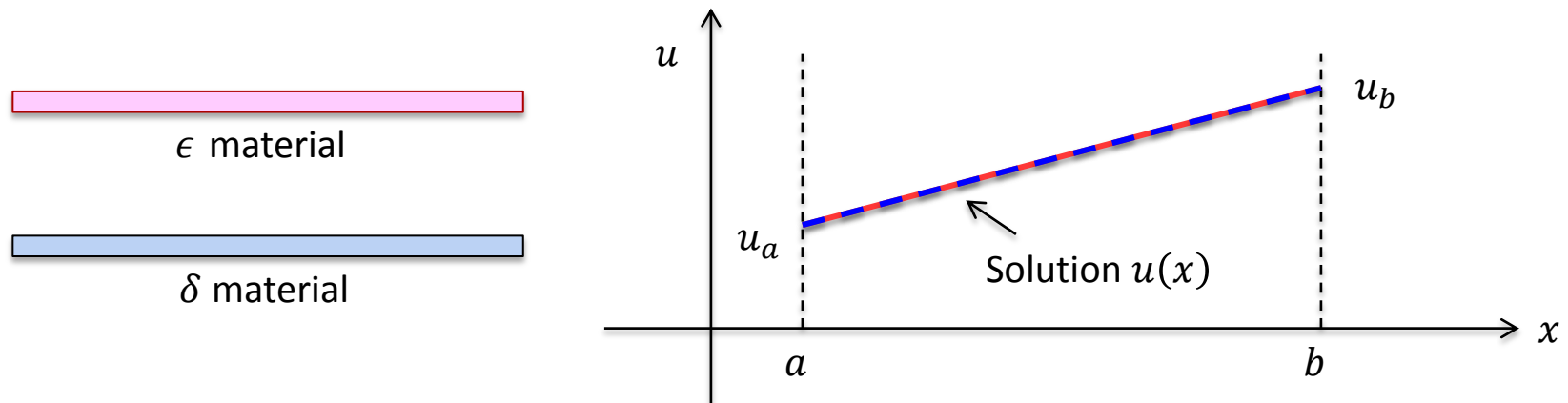
where H is a constant.

- If we set

$$\underline{T}_\delta \langle \xi_\delta \rangle = (\epsilon/\delta)^2 \underline{T}_\epsilon \langle \xi_\epsilon \rangle$$

as derived above, we are assured the strain energy is independent of horizon.

- It follows that stress is also independent of horizon.



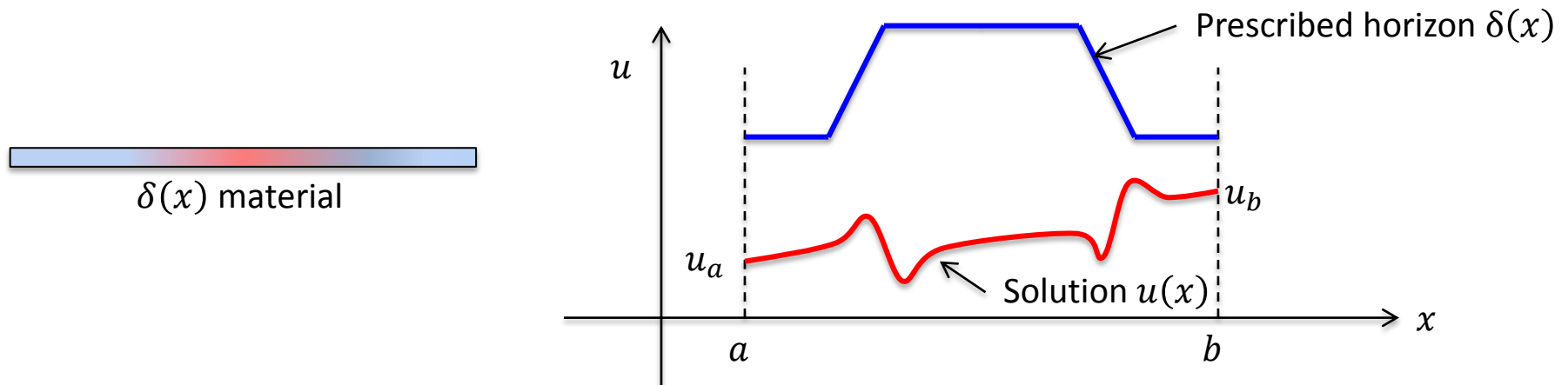
Variable horizon: the problem

- The scaling discussed above holds at any point x .
- We might anticipate that we can let horizon $\delta(x)$ vary with position and get the same result.
- Specifically, set

$$\underline{T}[x] = \delta^{-2}(x) \underline{Z} \left\langle \frac{\xi}{\delta(x)} \right\rangle$$

where \underline{Z} is a reference force state with horizon $\epsilon = 1$.

- Surprise! The uniform strain deformation is not in equilibrium even though it has uniform strain energy density.



Search for a different scaling

- Instead of

$$\underline{T}[x] = \delta^{-2}(x) \underline{Z} \left\langle \frac{\xi}{\delta(x)} \right\rangle$$

search for pairs of functions $\delta(x)$ and $F(x)$ such that \underline{T} given by

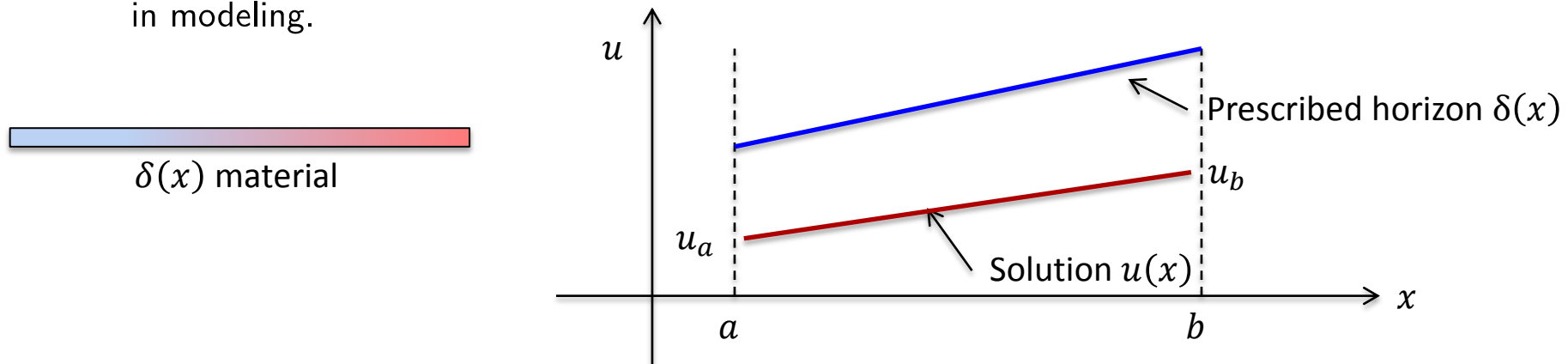
$$\underline{T}[x] = F(x) \underline{Z} \left\langle \frac{\xi}{\delta(x)} \right\rangle$$

is equilibrated with no body force.

- This search has so far turned up only the following result:

$$\delta(x) = \delta_0 + ax, \quad F(x) = \delta^{-2}(x).$$

- This is interesting but not very practical since it doesn't allow enough flexibility in modeling.



Interactions with variable horizon

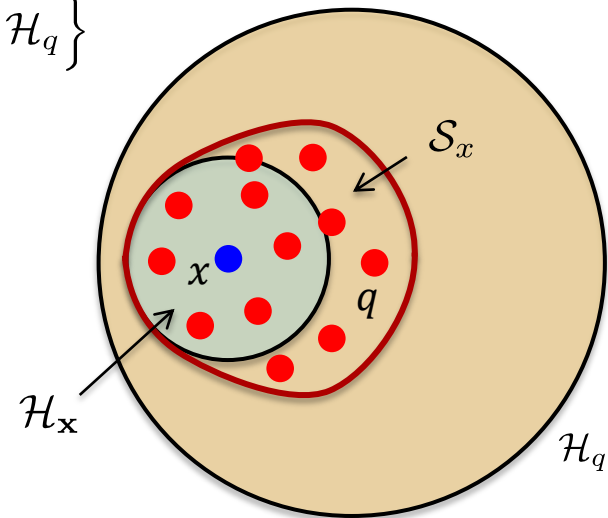
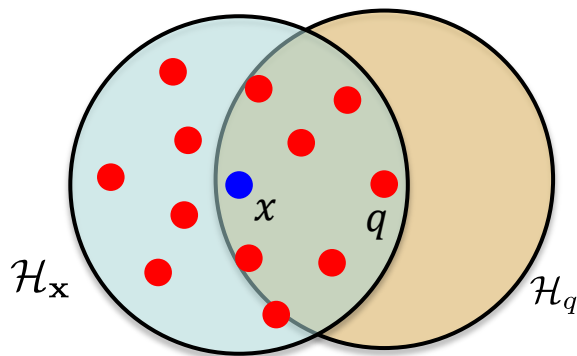
- If the horizon is constant throughout the body, we write

$$\int_{\mathcal{H}_x} \left[\underline{T}[x] \langle q - x \rangle - \underline{T}[q] \langle x - q \rangle \right] dq + b(x) = 0$$

where \mathcal{H}_x is the family of x .

- If the horizon varies, we need to change the volume of integration to \mathcal{S}_x , the “superfamily” of x consisting of all the points that x interacts with either through its own material model or the material model at the other points.

$$\mathcal{S}_x = \mathcal{H}_x \cup \left\{ q \in \mathcal{B} \mid x \in \mathcal{H}_q \right\}$$

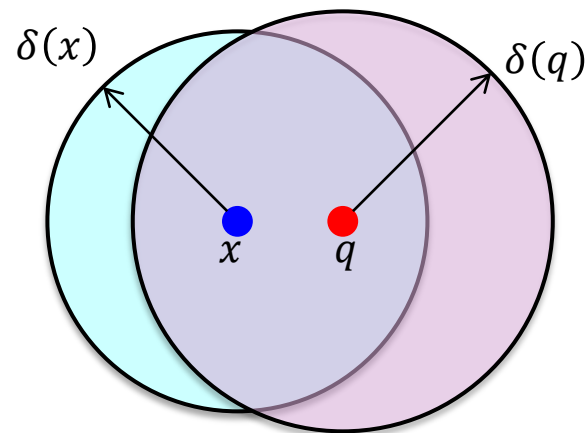


Origin of artifacts

- The peridynamic force density operator $L(x)$ involves the force state not only at x but also the force states at all points within the horizon.

$$0 = L(x) + b, \quad L(x) = \int_{-\infty}^{\infty} \{T_{\delta(x)}[x]\langle q - x \rangle - T_{\delta(q)}[q]\langle x - q \rangle\} dq$$

so simply scaling the material model at x is not sufficient.



Variable horizon

Consider altering the equilibrium equation

- Use the 1D bar problem to define a “continuum patch test”.
- In a deformation of the form

$$u(x) = u_0 + Hx$$

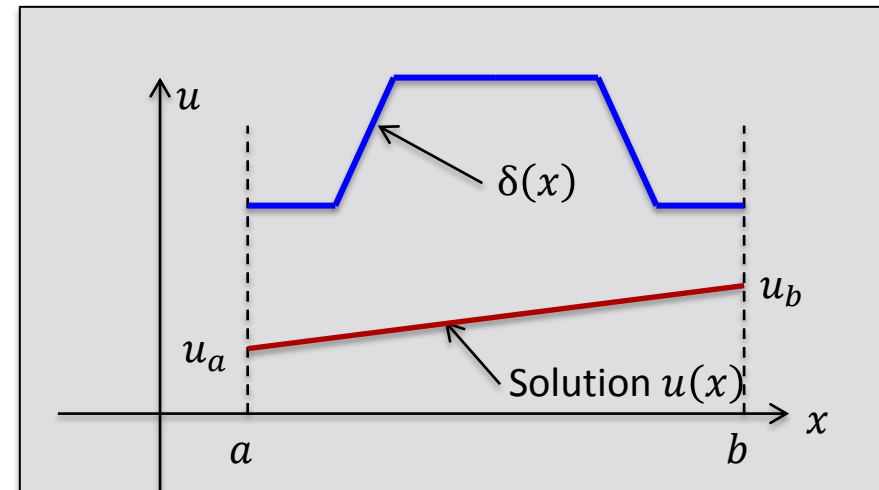
where u_0 and H are constants, and the material model is of the form

$$\underline{T}[x]\langle\xi\rangle = \delta^{-2}(x)\underline{Z}(H)\left\langle\frac{\xi}{\delta(x)}\right\rangle$$

where \underline{Z} is a state that depends on H only, require

$$L(x) = 0$$

for all x and for any prescribed $\delta(x)$.



Peridynamic stress tensor

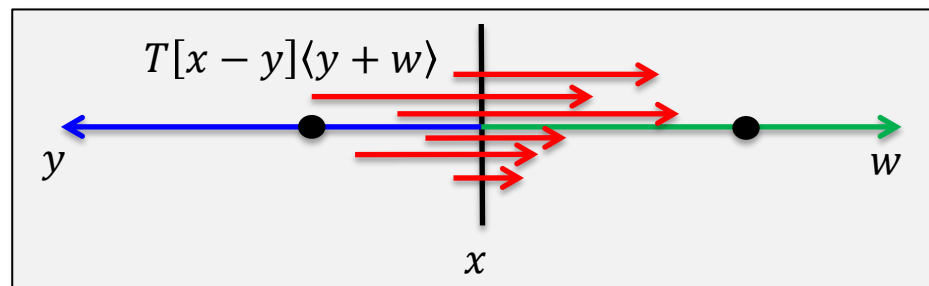
- The peridynamic stress tensor field* is defined by

$$\nu(x) = \int_0^\infty \int_0^\infty \left\{ \underline{T}[x-y] \langle y+w \rangle - \underline{T}[x+y] \langle y-w \rangle \right\} dy dz.$$

- ν allows the peridynamic force operator to be written in a form similar to the local theory:

$$L(x) = \int_{-\infty}^\infty \left\{ \underline{T}[x] \langle q-x \rangle - \underline{T}[q] \langle x-q \rangle \right\} dq = \frac{d\nu}{dx}(x).$$

- ν is the force per unit area carried by nonlocal interactions.



*R. B. Lehoucq & SS, "Force flux and the peridynamic stress tensor," JMPS (2008)

Partial stress field

- If δ and \underline{T} are constant, the peridynamic stress field simplifies to

$$\nu(x) = \int_{-\infty}^{\infty} \xi \underline{T} \langle \xi \rangle d\xi.$$

- Now use this expression to *define* a new field ν_0 for any force state field:

$$\nu_0(x) := \int_{-\infty}^{\infty} \xi \underline{T}[x] \langle \xi \rangle d\xi.$$

- In the patch test,

$$\underline{T}[x] \langle \xi \rangle = \delta^{-2}(x) \underline{Z} \left\langle \frac{\xi}{\delta(x)} \right\rangle \quad \Rightarrow \quad \nu_0(x) = \int_{-\infty}^{\infty} \xi \underline{Z} \langle \xi \rangle d\xi$$

independent of x .

- ν_0 is called the *partial stress field*.

Equilibrium equation based on the partial stress field

- The previous observation (that ν_0 is independent of x in the patch test) leads to the following proposed expression for internal force density:

$$L_0(x) = \frac{d\nu_0}{dx}(x)$$

where the equilibrium equation is

$$L_0(x) + b(x) = 0$$

and

$$\nu_0(x) = \int_{-\infty}^{\infty} \xi \underline{T}[x] \langle \xi \rangle d\xi$$

for any deformation, any $\delta(x)$, and any material model (possibly heterogeneous).

- Trivially, this model passes the patch test.

Concept for coupling method

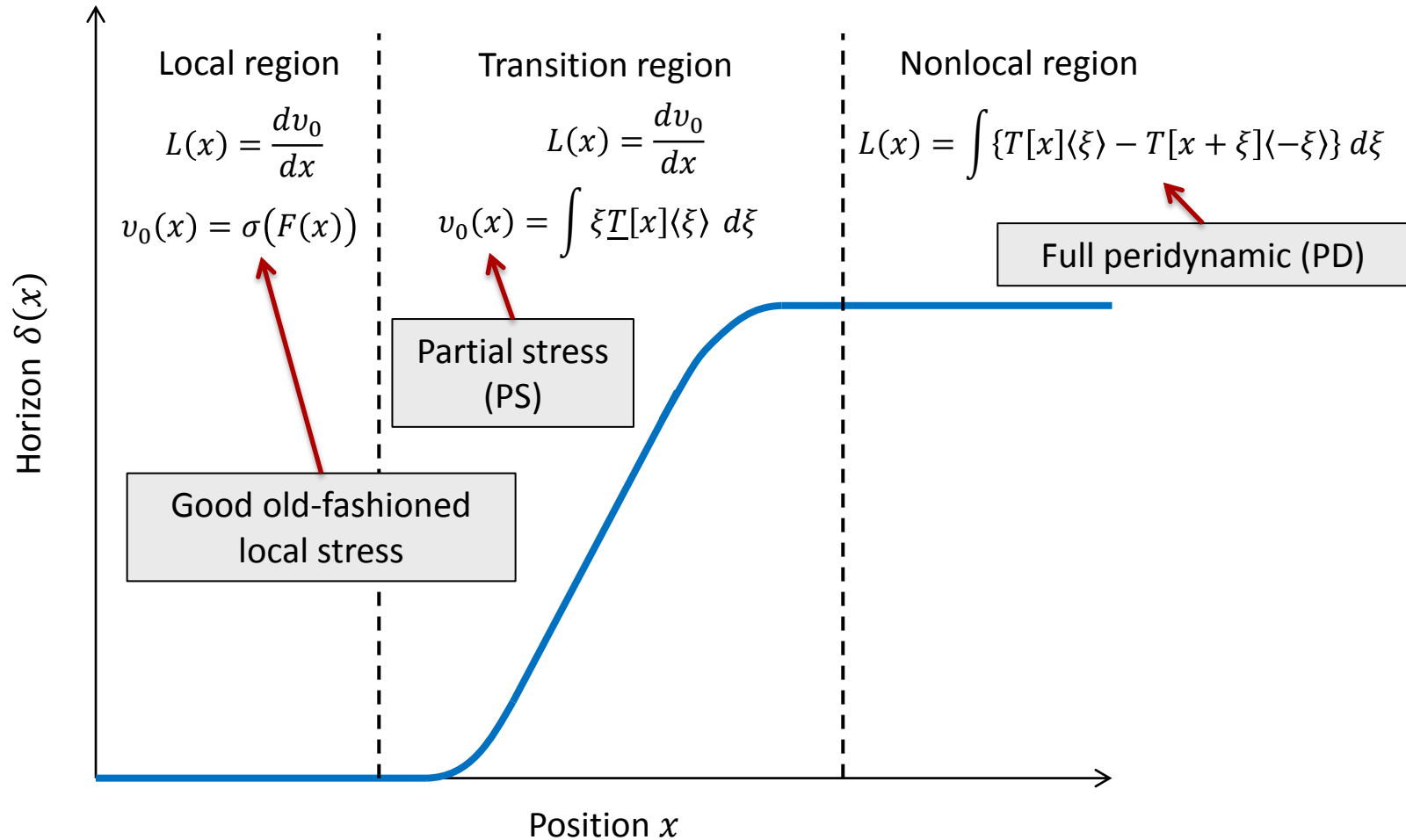
- Idea: within a coupling region in which δ is changing, compute the internal force density from

$$L(x) = \frac{dv_0}{dx}(x), \quad v_0(x) := \int_{-\infty}^{\infty} \xi T[x] \langle \xi \rangle d\xi$$

instead of the full PD nonlocal integral.

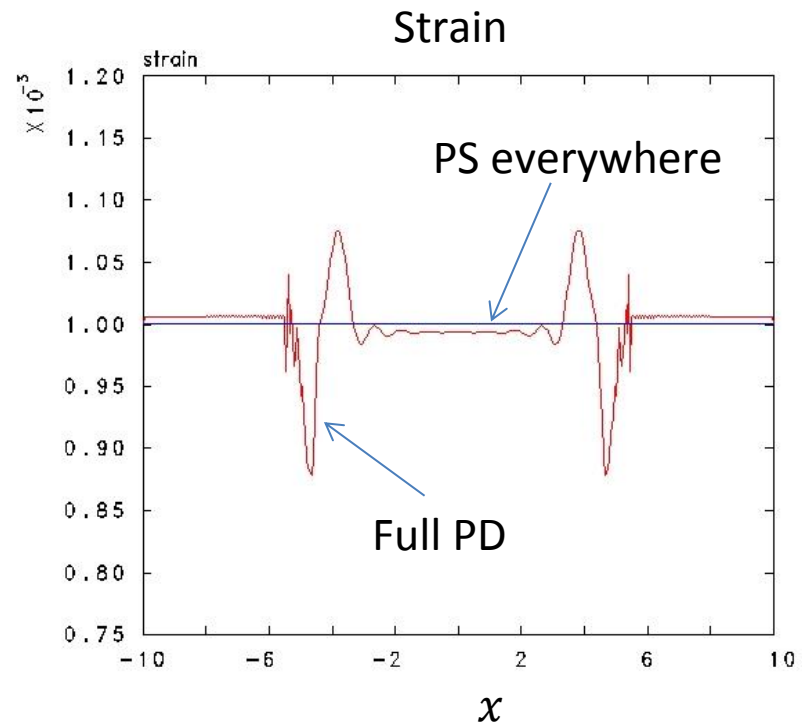
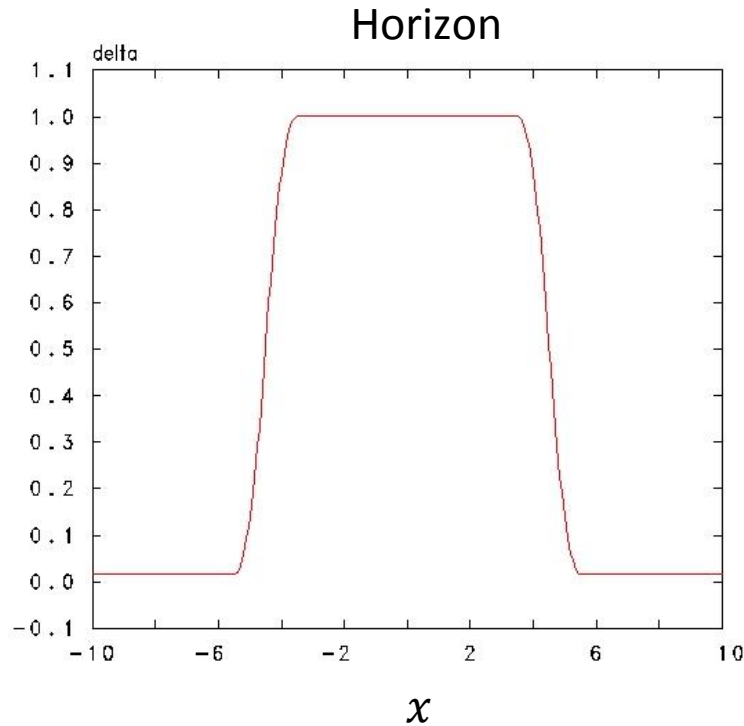
- Here, Tx is determined from whatever the deformation happens to be near x .
 - Z is no longer involved.
 - The material model has not changed from full PD, but the way of computing L has.

Local-nonlocal coupling idea



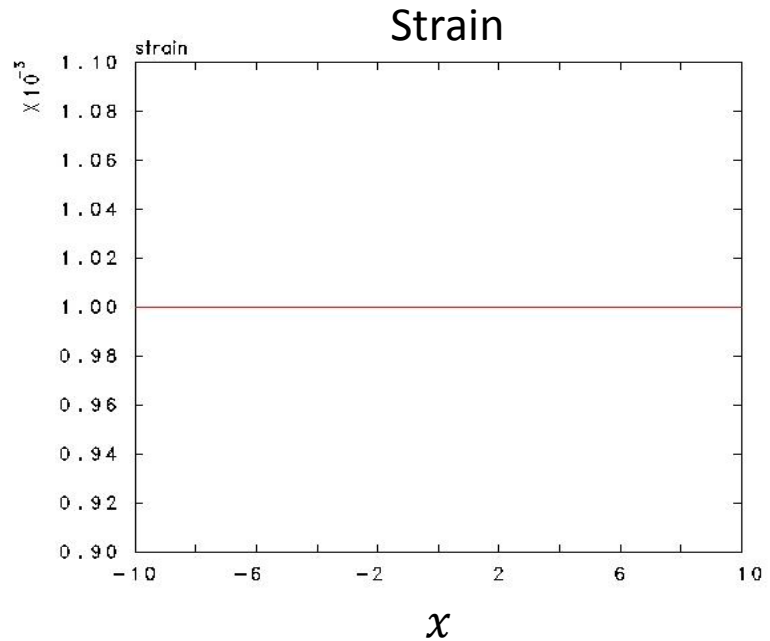
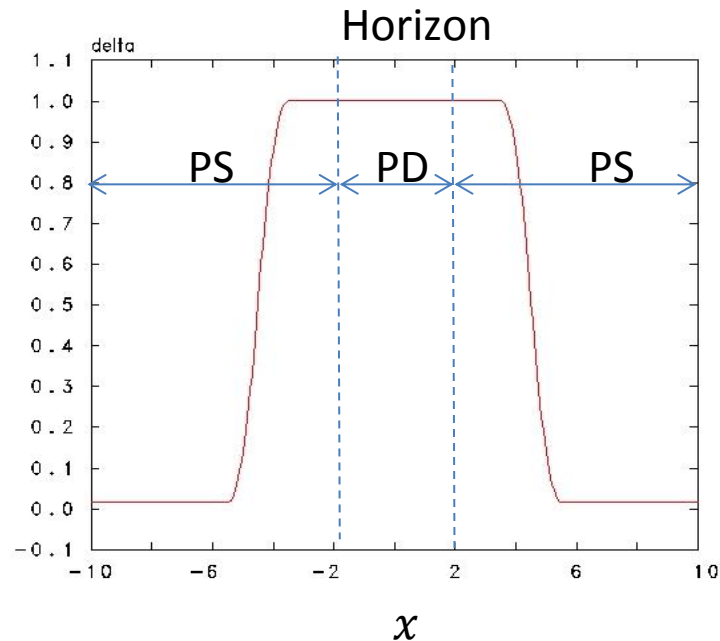
Continuum patch test results

- Full PD shows artifacts, as expected.
- PS shows no artifacts, as promised.



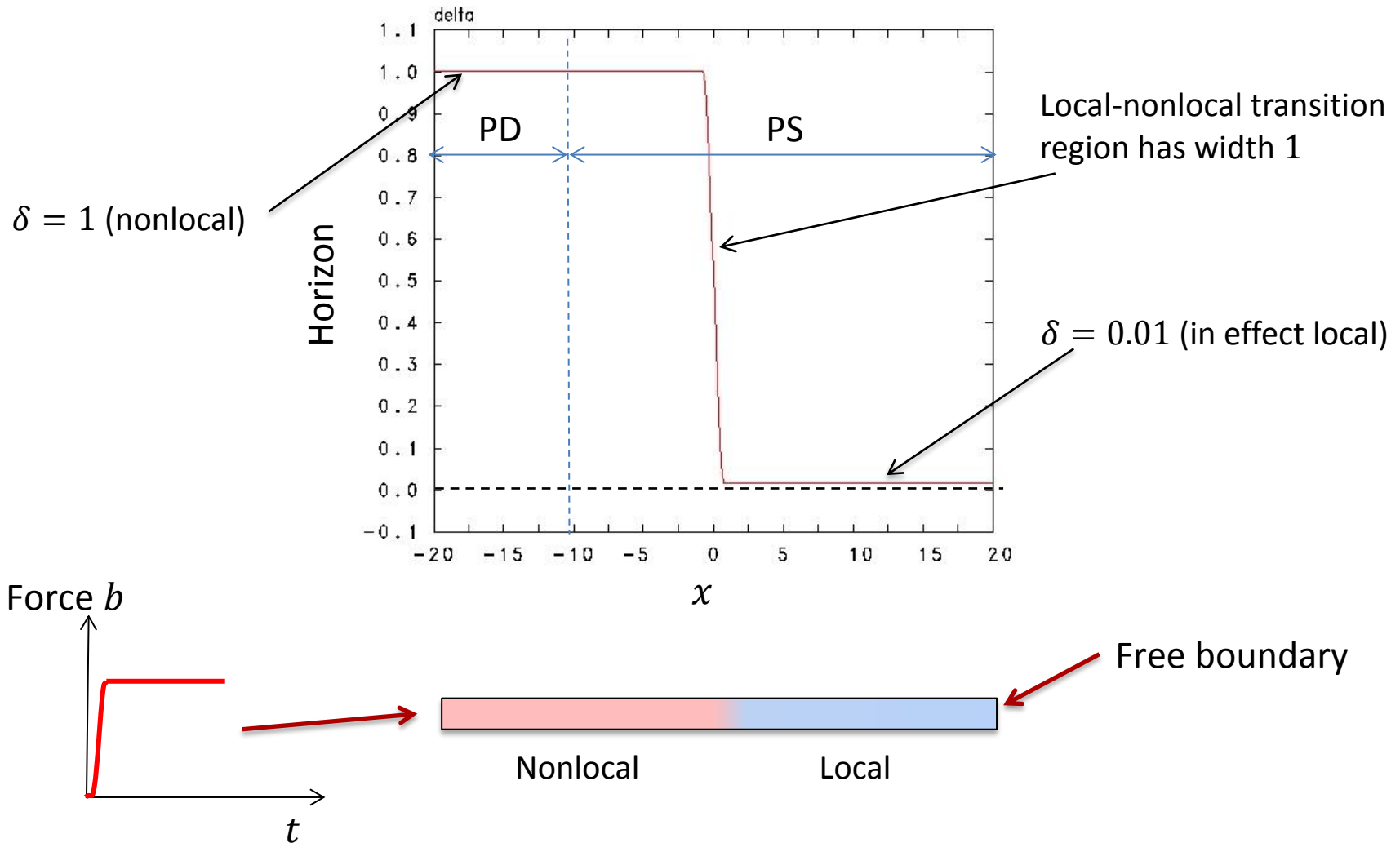
Continuum patch test with coupling

- No artifacts with PD-PS coupling (this was hoped for but not guaranteed).



Pulse propagation test problem

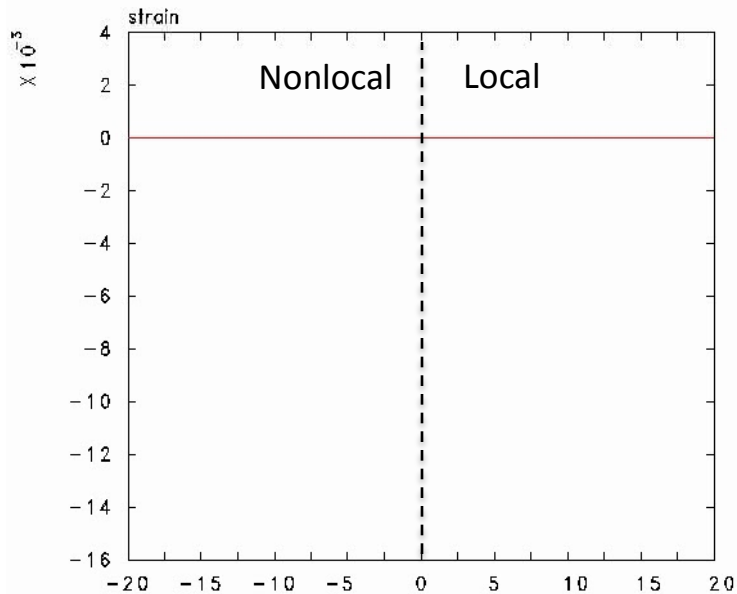
- Does our coupling method work for dynamics as well as statics with variable horizon?



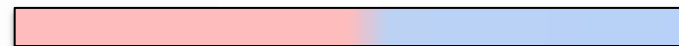
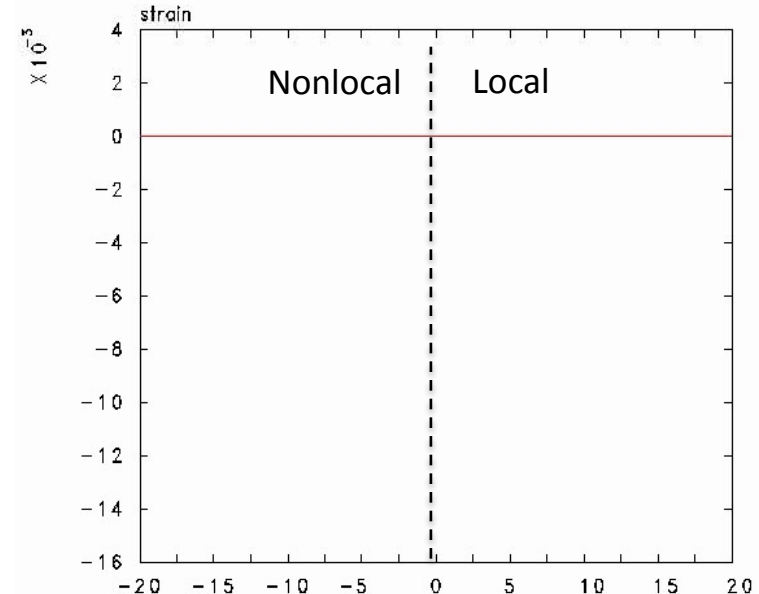
Pulse propagation test results

- Movies of strain field evolution

Full PD everywhere



Coupled PD-PS

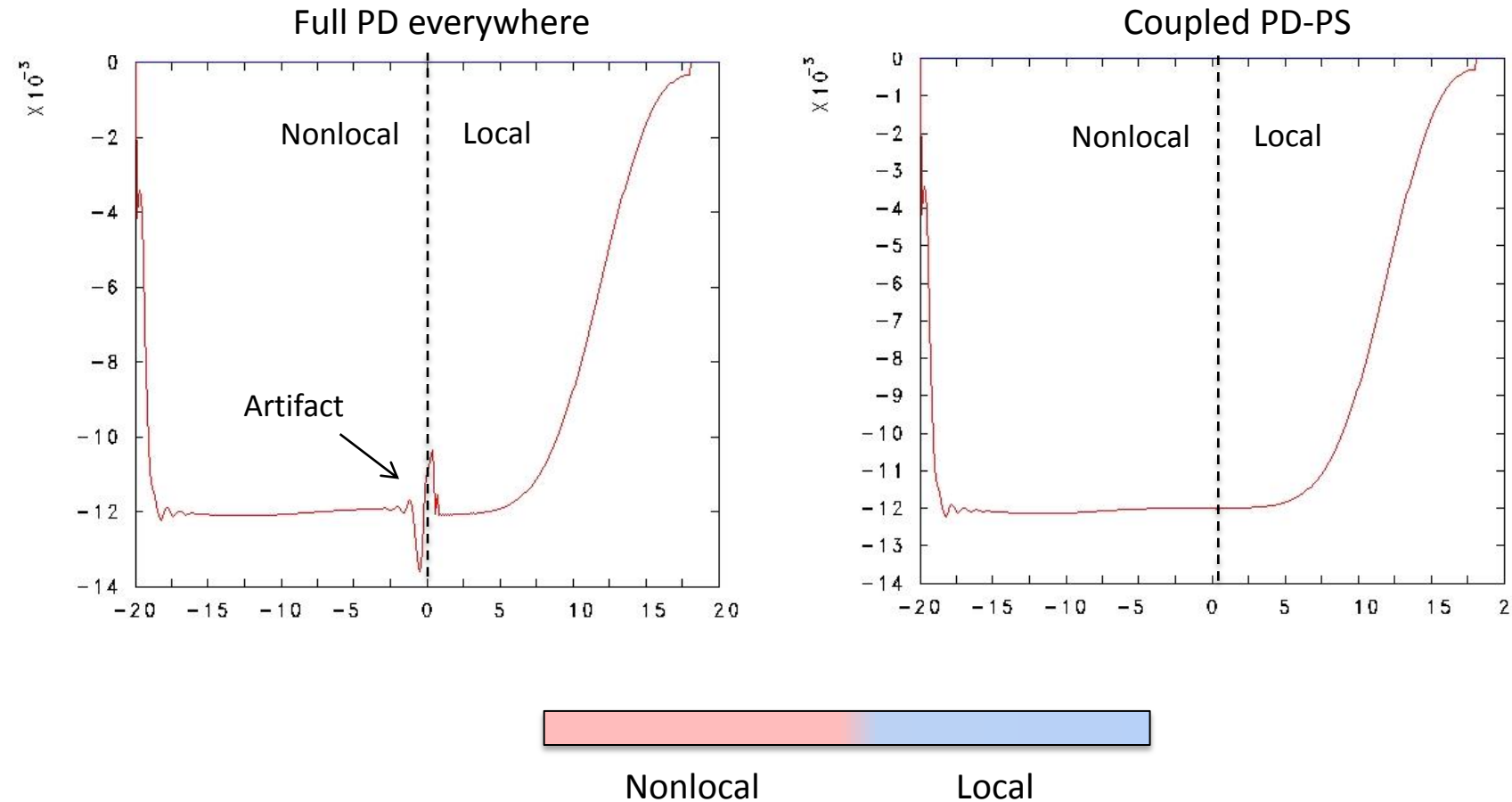


Nonlocal

Local

Pulse propagation test results

- Strain field: no artifacts appear in the coupled model the local-nonlocal transition.



Discussion

- The partial stress approach may provide a means for local-nonlocal coupling within the continuum equations.
 - Uses the underlying peridynamic material model but modifies the way internal force density is computed.
 - Expected to work in 2D & 3D, linear & nonlinear.
- PS is inconsistent from an energy minimization point of view.
 - Not suitable for a full-blown theory of mechanics and thermodynamics (as full PD is).
 - Not yet clear what implications this may have in practice.
 - We still need to use full PD for crack progression.

Extra slides

Peridynamic vs. local equations

State notation: $\underline{\text{State}}\langle \text{bond} \rangle = \text{vector}$

<i>Relation</i>	<i>Peridynamic theory</i>	<i>Standard theory</i>
Kinematics	$\underline{\mathbf{Y}}\langle \mathbf{q} - \mathbf{x} \rangle = \mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{x})$	$\mathbf{F}(\mathbf{x}) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}(\mathbf{x})$
Linear momentum balance	$\rho \ddot{\mathbf{y}}(\mathbf{x}) = \int_{\mathcal{H}} \left(\mathbf{t}(\mathbf{q}, \mathbf{x}) - \mathbf{t}(\mathbf{x}, \mathbf{q}) \right) dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x})$	$\rho \ddot{\mathbf{y}}(\mathbf{x}) = \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) + \mathbf{b}(\mathbf{x})$
Constitutive model	$\mathbf{t}(\mathbf{q}, \mathbf{x}) = \underline{\mathbf{T}}\langle \mathbf{q} - \mathbf{x} \rangle, \quad \underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})$	$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{F})$
Angular momentum balance	$\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle \mathbf{q} - \mathbf{x} \rangle \times \underline{\mathbf{T}}\langle \mathbf{q} - \mathbf{x} \rangle dV_{\mathbf{q}} = \mathbf{0}$	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
Elasticity	$\underline{\mathbf{T}} = W_{\underline{\mathbf{Y}}} \text{ (Fréchet derivative)}$	$\boldsymbol{\sigma} = W_{\mathbf{F}} \text{ (tensor gradient)}$
First law	$\dot{\varepsilon} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} + q + r$	$\dot{\varepsilon} = \boldsymbol{\sigma} \cdot \dot{\mathbf{F}} + q + r$

$$\underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} := \int_{\mathcal{H}} \underline{\mathbf{T}}\langle \boldsymbol{\xi} \rangle \cdot \dot{\underline{\mathbf{Y}}}\langle \boldsymbol{\xi} \rangle dV_{\boldsymbol{\xi}}$$