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A Locally Conservative, Discontinuous Least-Squares Finite Element Method for the Stokes Equations

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Least-squares finite element methods

- Cast any PDE into an unconstrained minimization problem
- Let X, Y be Hilbert spaces, and let $\mathcal{L} : X \to Y$ be a differential operator, consider

$$\mathcal{L}u = f$$

- Least-squares finite element solution given by minimizing *least-squares* functional

$$\mathcal{J}(u; f) = \|\mathcal{L}u - f\|_Y^2$$

i.e.,

$$\min_X \mathcal{J}(u; f)$$

Norm equivalence

Least-squares functional norm equivalent on X if

$$c_1\|u\|_X \le \mathcal{J}(u;0) \le c_2\|u\|_X \qquad \forall u \in X$$

Norm equivalence $\Rightarrow$ strongly coercive $\Rightarrow$ well-posed

Advantages of LSFEM

- General approach to solving PDEs
- Discrete system always symmetric positive definite
- Norm equivalence ⇒ automatic stability and well-posednes of LSFEM
  - Compatibility conditions between the FE spaces are not required
  - Variables can be approximated independently from each other by any conforming space
  - Stabilization and/or regularization of weak problems not required
- Automatic error-estimates: functional serves as error estimator

Disadvantages

Disadvantages

- Minimum of least-squares functional nonzero for discrete problem
- Some equations may not be exactly satisfied
- Standard methods not conservative
- Mass loss - solutions are not accurate

Mass loss - remedies

Strategies to improve mass conservation:

- Refine mesh
- Increase weight of conservation equation
- Restricted least-squares
- Mimetic discretizations
- Discontinuous elements

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Model problem - Stokes equations

Primitive variable formulation

$$\begin{cases} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \end{cases}$$

Velocity boundary condition

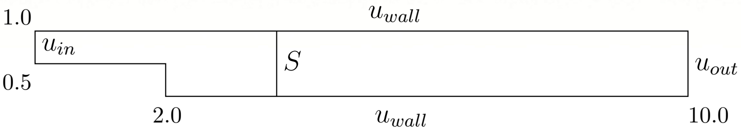
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega$$

Zero mean pressure

$$\int_\Omega p \, d\Omega = 0 \, .$$

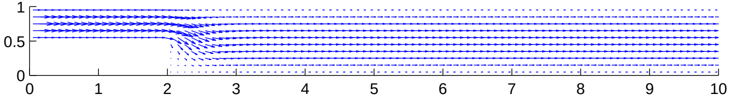
- $\mathbf{u}$: velocity
- p : pressure

Backward facing step: geometry and reference solution

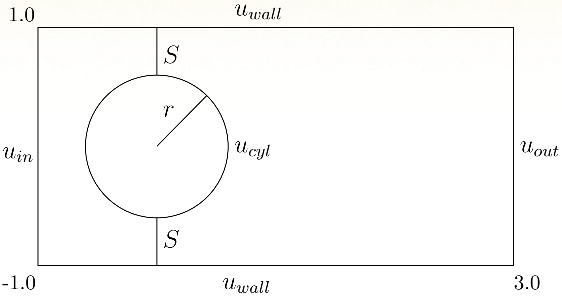


$$\mathbf{u}_{in} = \begin{bmatrix} 8(y-0.5)(1-y) \\ 0 \end{bmatrix} \quad , \quad \mathbf{u}_{out} = \begin{bmatrix} y(1-y) \\ 0 \end{bmatrix} ,$$

$$\mathbf{u}_{wall} = \mathbf{0}$$



Flow past cylinder: geometry



$$\mathbf{u}_{in} = \mathbf{u}_{out} = \mathbf{u}_{wall} = \begin{bmatrix} (1-y)(1+y) \\ 0 \end{bmatrix},$$

$$\mathbf{u}_{cyl} = \mathbf{0}$$

Flow past cylinder: reference solutions

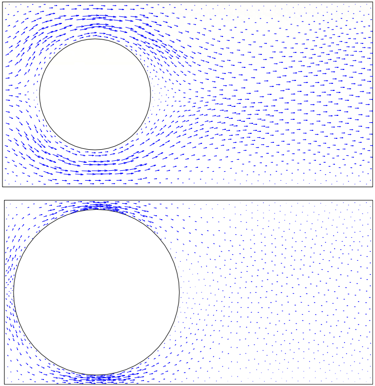


Figure: Velocity field, $r = 0.6$ (top) and $r = 0.9$ (bottom).

First order formulation

Define vorticity

$$\omega = \nabla \times \mathbf{u}$$

Use identity

$$\nabla \times \nabla \times \mathbf{u} = -\triangle \mathbf{u} + \nabla(\nabla \cdot \mathbf{u})$$

Velocity-vorticity-pressure formulation

$$\begin{cases} \nabla \times \omega + \nabla p &= \mathbf{f} \quad \text{on } \Omega \\ \omega - \nabla \times \mathbf{u} &= 0 \quad \text{on } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{on } \Omega \end{cases}$$

A well-posed least-squares principle for Stokes

1. **A priori bound:**

$$\|\mathbf{u}\|_1+\|\omega\|_0+\|p\|_0 \le C(\|\nabla\times\omega+\nabla p\|_{-1}+\|\omega-\nabla\times\mathbf{u}\|_0+\|\nabla\cdot\mathbf{u}\|_0)$$

for any $\mathbf{u}\in\mathbf{H}_0^1(\Omega)=[H_0^1(\Omega)]^2$, $\omega\in L^2(\Omega)$, and $p\in L_0^2(\Omega)$

2. **Norm-equivalent functional:**

$$\mathcal{J}_{-1}(\mathbf{u},\omega,p;\mathbf{f})=\|\nabla\times\omega+\nabla p-\mathbf{f}\|_{-1}^2+\|\nabla\times\mathbf{u}-\omega\|_0^2+\|\nabla\cdot\mathbf{u}\|_0^2 \quad (1)$$

norm equivalent on $X=\mathbf{H}_0^1(\Omega)\times L^2(\Omega)\times L_0^2(\Omega)$

3. **Least-squares principle:** *find* $(\mathbf{u},\omega,p)\in X$ *such that*

$$\mathcal{J}_{-1}(\mathbf{u},\omega,p;\mathbf{f})\le\mathcal{J}_{-1}(\mathbf{v},\xi,q;\mathbf{f}) \quad \forall(\mathbf{v},\xi,q)\in X$$

Finite element spaces

Approximate

$$X = \mathbf{H}^1_0(\Omega) \times L^2(\Omega) \times L^2_0(\Omega)$$

with the equal order space

$$X^h_r = \mathring{\mathbf{P}}_r(\Omega) \times \mathcal{P}_r(\Omega) \times \breve{\mathcal{P}}_r(\Omega), \quad r = 2$$

- $\mathring{\mathbf{P}}_r$ - vector finite element space, each component piecewise polynomial of degree r, boundary nodes fixed
- $\mathcal{P}_r$ - scalar finite element space, piecewise polynomial of degree r
- $\breve{\mathcal{P}}_r$ - $\mathcal{P}_r$ with single node constrained on boundary

Velocity must be quadratic or higher

- All methods are implemented using the Intrepid package: http://trilinos.sandia.gov/packages/intrepid/

Mass loss

We examine mass loss in the C^0 weighted LSFEM

$$\mathcal{J}_\mu^h(\mathbf{u}^h, \omega^h, p^h; \mathbf{f}^h) = h^2\|\nabla\times\omega^h + \nabla p^h - \mathbf{f}^h\|_0^2 + \|\nabla\times\mathbf{u}^h - \omega^h\|_0^2 + \mu\|\nabla\cdot\mathbf{u}^h\|_0^2 \quad (2)$$

implemented with equal order space

$$X_r^h = \mathring{\mathbf{P}}_r(\Omega)\times\mathcal{P}_r(\Omega)\times\breve{\mathcal{P}}_r(\Omega) \quad r = 2 \quad (3)$$

- Expect better mass conservation with large μ

Mass loss is estimated by computing mass flow along vertical lines placed at every 0.1 units along the x-axis. A total of 100 lines are used for the backward-facing step and 40 lines are used for the flow past a cylinder.

Mass loss - backward facing step

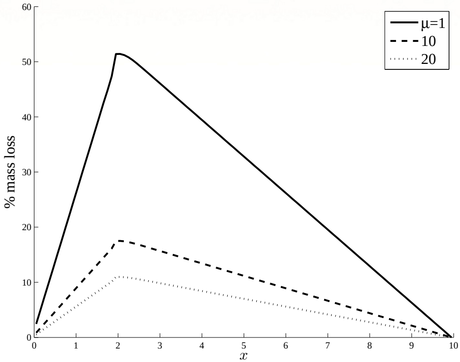


Figure: Percent mass loss of standard LSFEM for backward facing step with different weights on continuity equation.

Mass loss - cylinder

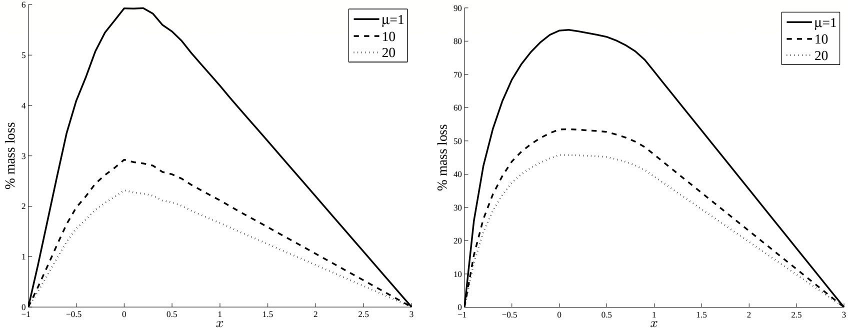


Figure: Percent mass loss of standard LSFEM for flow past a cylinder with $r = 0.6$ (left) and $r = 0.9$ (right)

LSFEM with improved mass conservation: Take 1 - discontinuous velocity

Let velocity be discontinuous across element boundaries

$$\widetilde{X}_r^h = [\mathring{\mathbf{P}}_r] \times \mathcal{P}_r \times \breve{\mathcal{P}}_r$$

Discontinuous velocity LSFEM:

$$\widetilde{J}_{-1}^h(\mathbf{u}^h, \omega^h, p^h; \mathbf{f}^h) =$$

$$h^2 \|\nabla \times \omega^h + \nabla p^h - \mathbf{f}^h\|_0^2$$

$$+ \sum_{K \in \mathcal{K}} \left(\|\nabla \times \mathbf{u}^h - \omega^h\|_{0,K}^2 + \|\nabla \cdot \mathbf{u}^h\|_{0,K}^2 \right)$$

$$+ \sum_{e_i \in \mathcal{E}(\Omega)} h^{-1} \left(\alpha_1 \|[\mathbf{u}^h \cdot \mathbf{n}_i]\|_{0,e_i}^2 + \alpha_2 \|[\mathbf{u}^h \times \mathbf{n}_i]\|_{0,e_i}^2 \right)$$

Mass loss - backward step

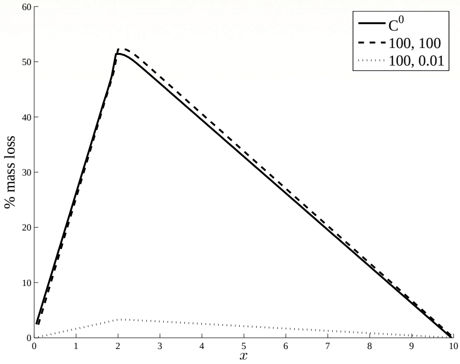


Figure: Percent mass loss of discontinuous velocity formulation for the backward-facing step. Legend values are α_1, α_2.

Problem solved :)

Mass loss - cylinder

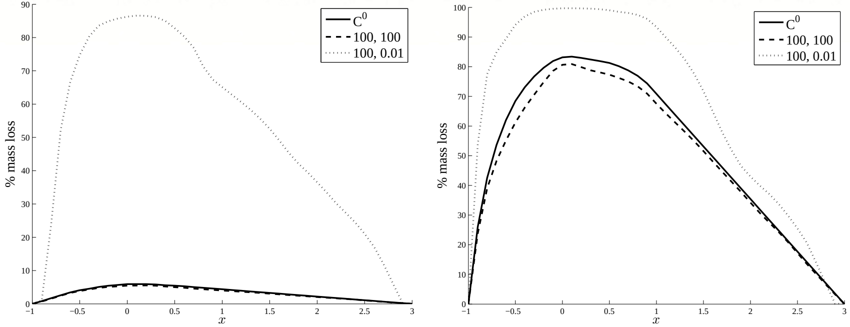


Figure: Percent mass loss of discontinuous velocity formulation for cylinder flow with $r = 0.6$ (left) and $r = 0.9$ (right). Legend values are α_1, α_2.

Not really :(

LSFEM with improved mass conservation: Take 2 - Stream function

Force $\mathbf{u}^h$ to be *pointwise divergence-free* on each element by setting

$$\mathbf{u}^h|_K = \nabla \times \psi^h|_K \quad \forall K \in \mathcal{K}\,,$$

where $\psi^h \in [\mathcal{P}^{r+1}]$ is discontinuous *stream function*

- Need $\psi^h \in [\mathcal{P}^{r+1}]$ so $\mathbf{u}^h = \nabla \times \psi^h \in [\mathbf{P}^r]$, where $r \geq 2$

Since $\nabla \cdot (\nabla \times \phi) = 0$ for any ϕ, $\nabla \cdot \mathbf{u}^h = 0$ automatically satisfied on each element

Discontinuous SVP formulation

Since $\nabla \cdot \mathbf{u}^h = 0$ on each element, terms can be dropped

$$\widetilde{J}^h_{-1}(\psi^h, \omega^h, p^h; \mathbf{f}^h) =$$

$$h^2 \|\nabla \times \omega^h + \nabla p^h - \mathbf{f}^h\|_0^2 + \sum_{K \in \mathcal{K}} \|\nabla \times \nabla \times \psi^h - \omega^h\|_{0,K}^2$$

$$+ \sum_{e_i \in \mathcal{E}(\Omega)} h^{-1} \left(\|[(\nabla \times \psi^h) \cdot \mathbf{n}_i]\|_{0,e_i}^2 + \|[(\nabla \times \psi^h) \times \mathbf{n}_i]\|_{0,e_i}^2 \right)$$

$$+ \sum_{e_i \in \mathcal{E}(\Gamma)} h^{-1} \|(\nabla \times \psi^h) \times \mathbf{n}_i\|_{0,e_i}^2 + \sum_{e_i \in \mathcal{E}(\Omega)} h^{-3} \|[\psi^h]\|_{0,e_i}^2$$

Mass loss - backward step

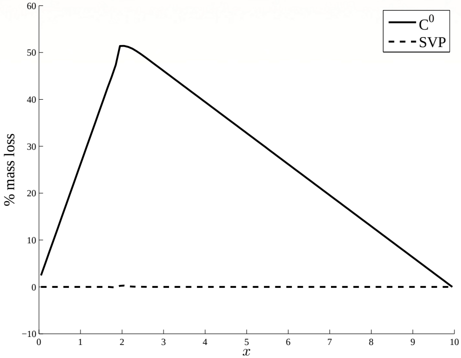


Figure: Comparison of mass loss in C^0 and SVP formulations for the backward-facing step.

Mass loss - cylinder

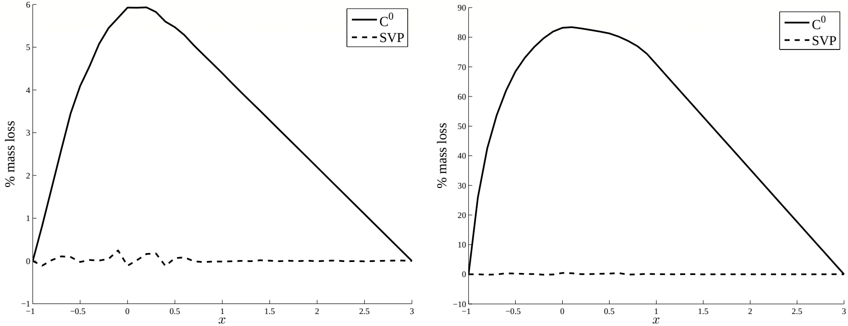


Figure: Comparison of mass loss in C^0 and SVP formulations for cylinder with $r = 0.6$ (left) and $r = 0.9$ (right).

Pressure - backward step

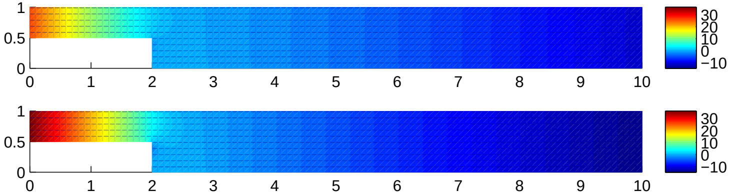


Figure: Pressure for C^0 formulation (top) and SVP formulation (bottom)

Vorticity - backward step

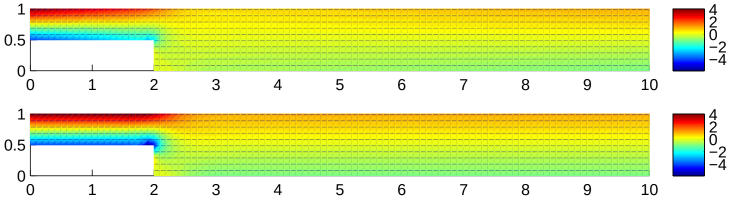


Figure: Vorticity for C^0 formulation (top) and SVP formulation (bottom)

Pressure - cylinder

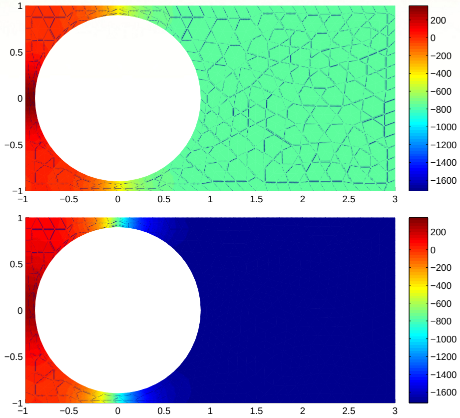


Figure: Pressure for C^0 formulation (top) and SVP formulation (bottom)

Vorticity - cylinder

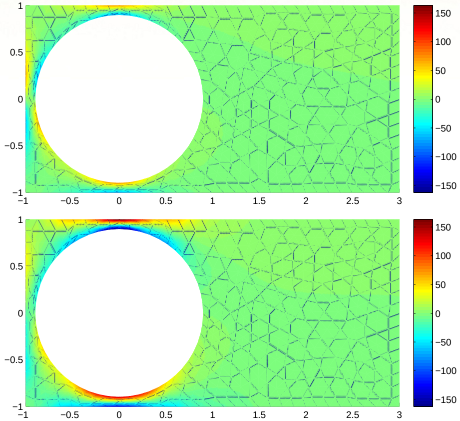


Figure: Vorticity for C^0 formulation (top) and SVP formulation (bottom)

Summary

Summary

- Standard LSFEM for Stokes can experience significant mass loss
- Surprisingly, making velocity discontinuous does not fix the problem
- Using discontinuous stream-function works much better!

Ongoing work

- Divergence-free basis vs. stream-function formulation
- Formal error analysis of the method
- Extension to 3-D and Navier-Stokes

P. Bochev, J. Lai, and L. Olson. *Int. J. Numer. Meth. Fluids*, 2011.