SAND2011-5163C

An Approach to Modeling Waves in Fractal Media

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Presentation at
United States National Congress on Computational Mechanics
Minneapolis, MN
July 27, 2011

[1]Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000..

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  - Demmie, Ostoja-Starzewski, "Waves in Fractal Media", *Journal of Plasticity*, volume 104, number 1-2, pp. 187–204, 2011

Background

- The term fractal was coined by Benoît Mandelbrot to denote an object that is broken or fractured in space or time.
- Fractals provide appropriate models for many media over some finite range of length scales with lower and upper cutoffs.
- Research was performed in condensed matter physics since the late 1980's for materials with fractal geometries.
  - However, a field theory, an analogue of continuum physics and mechanics, was sorely lacking.
  - Some progress towards a field theory was made by mathematicians, who began to look at classical problems like Laplace's or the heat equation on fractal sets. But, this approach is very technical from the mathematical analysis standpoint and only begins to offer an avenue to tackle simple mechanics problems.

Background

- **A very different step in the direction of a field theory and problems was taken by Tarasov [1-3].**
  - **He developed continuum-type equations of conservation of mass, linear and angular momentum, and energy for fractals, and studied several fluid mechanics and wave problems.**
  - **Tarasov's approach relies on dimensional regularization of fractal objects through fractional integrals in Euclidean space, a technique with its roots in quantum mechanics.**
  - **Another advantage of this approach is that it admits upper and lower cutoffs of fractal scaling, so that one effectively deals with a physical pre-fractal rather than a purely mathematical fractal lacking any cutoffs.**

1. Tarasov, V.E.: Continuous medium model for fractal media. Phys. Lett. A **336, 167–174 (2005)**
2. Tarasov, V.E.: Fractional hydrodynamic equations for fractal media. Ann. Phys. **318(2), 286–307 (2005)**
3. Tarasov, V.E.: Wave equation for fractal solid string. Mod. Phys. Lett. B **19(15), 721–728 (2005)**

Background

- **Whereas the original formulation of Tarasov was based on the Riesz measure, and thus more suited to isotropic fractal media, a model that is based on a product measure was introduced by Li and Ostoja-Starzewski [1,2].**
  - **This measure has different fractal dimensions in different directions.**
  - **It grasps the anisotropy of fractal geometry better than the original formulation for a range of length scales between the lower and upper cutoffs.**

1. Li, J., Ostoja-Starzewski, M.: Fractal materials, beams and fracture mechanics. Z. Angew. Math. Phys. **60, 1–12 (2009)**
2. Li, J., Ostoja-Starzewski, M.: Fractal solids, product measures and fractional wave equations. Proc. R. Soc. A **465, 2521–2536 (2009). doi:10.1098/rspa.2009.0101. Errata (2010) doi:10.1098/rspa.2010.0491**

Developing Continuum Mechanics for Fractal Media

- We extended continuum thermomechanics to a fractal medium which is characterized by a mass (or spatial) fractal dimension, a surface fractal dimension, and a resolution length scale.
- The continuum theory is based on dimensional regularization, in which global mass, momentum, and energy laws employ fractional integrals.
- The global forms of governing equations are cast in forms involving conventional (integer-order) integrals, while the local forms are expressed by partial differential equations with derivatives of integer order.

Mass Power Law and Fractional Integrals

- **By a fractal medium, we mean a medium with a pre-fractal geometric structure.**
- **In order to deal with general anisotropic, fractal media, we use a more general power law relation with respect to each coordinate and the mass is specified via a product measure as**

$$m(W) = \int_W \rho(x_1, x_2, x_3)\, dl_{\alpha_1}(x_1)\, dl_{\alpha_2}(x_2)\, dl_{\alpha_3}(x_3).$$

- **The length measure in each coordinate is provided using the transformation coefficients**

$$dl_{\alpha_k}(x_k) = c_1^{(k)}(\alpha_k, x_k)\, dx_k, \quad k = 1,2,3 \quad \text{(no sum)}.$$

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Transformation Coefficients

- We adopt the modified Riemann-Liouville fractional integral of Jumarie [1,2] for the transformation coefficients

$$c_1^{(k)} = \alpha_k \left(\frac{l_k - x_k}{l_{k0}} \right)^{\alpha_k - 1}, \quad k = 1,2,3, \quad \text{(no sum)},$$

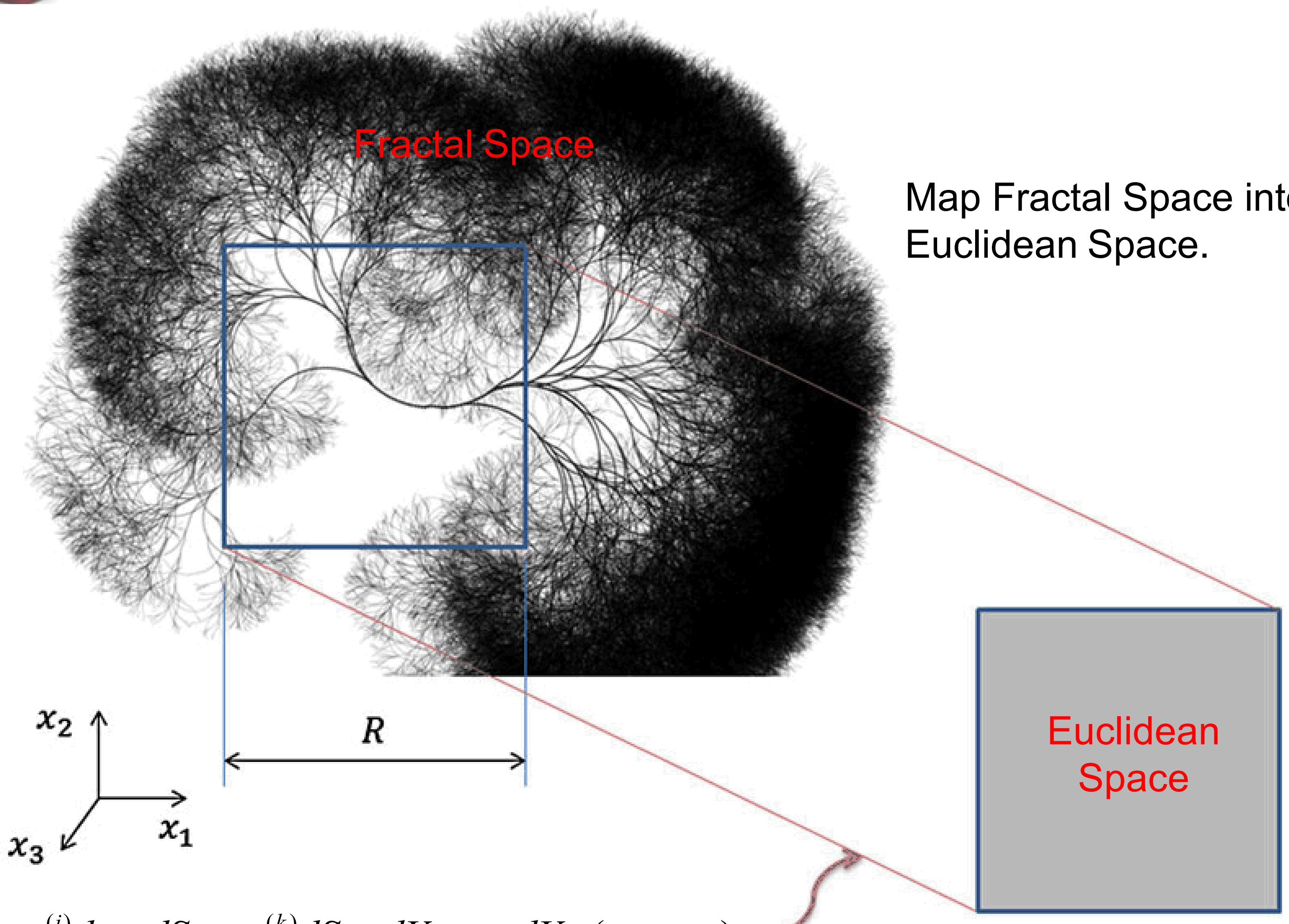
where l_k is the total length (integral interval) along x_k and l_{k0} is the characteristic length in the given direction, like the mean pore size. In the product measure formulation, the resolution length scale is

$$R = \sqrt{l_k l_k}$$

1. Jumarie, G.: On the representation of fractional Brownian motion as an integral with respect to *(dt)a* . Appl. Math. Lett. **18, 739–748 (2005)**

2. Jumarie, G.: Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. Appl. Math. Lett. **22(3), 378–385 (2009)**

Homogenization Process for Fractal Media



Map Fractal Space into Euclidean Space.

$$dl_{\alpha_i} = c_1^{(i)} dx_i,\ dS_d = c_2^{(k)} dS_2,\ dV_D = c_3 dV_3 \ \text{(no sum)}.$$

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Fractional Integral Theorems and Fractal Derivatives

- **By the conventional Gauss theorem, and noting that $c_2^{(k)}$ does not depend on the coordinate x_k, we obtain**

$$\int_{\partial W} \boldsymbol{f} \cdot \boldsymbol{n}\, dS_d = \int_{\partial W} f_k n_k dS_d = \int_W [f_k c_2^{(k)}],_k \, dV_3 = \int_W \frac{f_{k},_{k}}{c_1^{(k)}} dV_D.$$

- **Based on this expression, we define the fractal derivative, ∇_k^D *as***

$$\nabla_k^D = \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} \quad \text{(no sum)}.$$

- **Based on this definition, Gauss Theorem for Fractal media becomes**

$$\int_{\partial W} \boldsymbol{f} \cdot \boldsymbol{n}\, dS_d = \int_W \nabla_k^D f_k dV_D = \int_W \left(\nabla^D \cdot \boldsymbol{f} \right) dV_D.$$

Fractional Integral Theorems and Fractal Derivatives

- **To define the fractal material time derivative, we consider the fractional generalization of Reynolds transport theorem.**
  - **Consider any quantity, *P*, accompanied by a moving fractal material system, W_t, with velocity vector field *v* (= v_k).**
  - **It is straightforward to show that**

$$\frac{d}{dt}\int_{W_t} P dV_D = \int_{W_t}\left[\frac{\partial P}{\partial t} + \left(v_k P\right),_k\right] dV_D \;\; = \int_{W_t}\left[\frac{\partial P}{\partial t} + c_1^{(k)} \nabla_k^D \left(v_k P\right)\right] dV_D .$$

- **The result given by the first equality is identical to the conventional representation, hence, the fractal material time derivative and the conventional material time derivative are the same**

$$\left(\frac{d}{dt}\right)_D P = \frac{\partial P}{\partial t} + v_k P,_k = \frac{\partial P}{\partial t} + c_1^{(k)} v_k \nabla_k^D P = \frac{dP}{dt}.$$

Continuum Mechanics for Fractal Media

- We specify the relationship between surface force, F^S (= F_k^S), and the Cauchy stress tensor, σ_{kl}, using fractional integrals as

$$F_k^S = \int_{S_d} \sigma_{lk} n_l dS_d = \int_{S_2} \sigma_{lk} n_l c_2^{(l)} dS_2 .$$

- To specify the strain, we observe that

$$\frac{\partial}{\partial l_{\alpha_k}} = \frac{\partial x_k}{\partial l_{\alpha_k}} \frac{\partial}{\partial x_k} = \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} = \nabla_k^D .$$

- Thus, for small deformations, we define the strain, ε_{ij}, in terms of the displacement u_k as

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_j^D u_i + \nabla_i^D u_j\right) = \frac{1}{2}\left[\frac{1}{c_1^{(j)}} u_{i,j} + \frac{1}{c_1^{(i)}} u_{j,i}\right] \text{ (no sum).}$$

- The fractal equations for continuity, momentum, and energy follow from the balance laws for mass, momentum, and energy.

Fractal Continuity Equation

- Consider the equation for conservation of mass for *W*

$$\frac{d}{dt}\int_W \rho dV_D = 0.$$

- Using the fractional Reynolds transport theorem, we obtain

$$\frac{d}{dt}\int_W \rho dV_D = \int_W \left[\frac{\partial\rho}{\partial t} + \left(v_k\rho\right),_k\right] dV_D = 0.$$

- Since *W* is arbitrary, the fractal continuity equation is

$$\frac{\partial\rho}{\partial t} + \left(v_k\rho\right),_k = \frac{d\rho}{dt} + \rho v_k,_k = 0.$$

- Or in terms of the fractal derivative

$$\frac{d\rho}{dt} + \rho c_1^{(k)} \nabla_k^D v_k = 0.$$

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Fractal Momentum Equation

- Consider the balance law of linear momentum for *W*, with F_B is the body force, and F_S is the surface force,

$$\frac{d}{dt}\int_W \rho \boldsymbol{v} dV_D = \boldsymbol{F}^B + \boldsymbol{F}^S,$$

- Using Reynold's transport and Gauss theorems and the continuity equation, we obtain

$$\int_W \rho \frac{dv_k}{dt} dV_D = \int_W b_k dV_D + \int_{\partial W} \sigma_{lk} n_l dS_d = \int_W b_k dV_D + \int_W \nabla_l^D \sigma_{lk} dV_D.$$

- Since W is arbitrary, the fractal linear momentum equation is

$$\rho \frac{dv_k}{dt} = b_k + \nabla_l^D \sigma_{lk}.$$

Fractal Energy Equation

- **The most general form of balance (conservation) of energy must be used to obtain the fractal energy equation is**
  - *The time rate of change of the kinetic energy plus the internal energy of a region, W, in a continuum equals the sum of the rate of work performed on W by external agencies plus the flux of all other energies supplied to or removed from W by external agencies across the boundary of W.*
- **In terms of the kinetic energy K, internal energy K, and heat flux,**

$$K = \frac{1}{2}\int_W \rho v_i v_i dV_D, \; E = \int_W \rho e dV_D, \text{ and } -\int_{\partial W} q_i n_i dS_d,$$

and using the fractal momentum equation, we obtain

$$\int_W \rho \frac{de}{dt} dV_D = \int_W \sigma_{lk}\left(\nabla_l^D v_k\right) dV_D - \int_W \nabla_i^D q_i dV_D.$$

- **Since W is arbitrary, the fractal energy equation is**

$$\rho \frac{de}{dt} = \sigma_{lk}\left(\nabla_l^D v_k\right) - \nabla_i^D q_i.$$

Fractal Angular Momentum Equation

- The conservation of angular momentum in a fractal medium is stated as

$$\frac{d}{dt}\int_W \rho e_{ijk} x_j v_k dV_D = \int_W e_{ijk} x_j b_k dV_D + \int_{\partial W} e_{ijk} x_j \sigma_{lk} n_l dS_d,$$

- Using the fractal momentum equation and Gauss Theorem, we obtain the fractal angular momentum equation,

$$e_{ijk} \frac{\sigma_{jk}}{c_1^{(j)}} = 0.$$

- *Since* $c_1^{(j)} \neq c_1^{(k)}$, $j \neq k$, in general, the Cauchy stress is generally asymmetric in fractal media.

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Fractal Elastic Solid Under Finite Strains

- **We obtain the equations of motion using Hamilton's Principle for the Lagrangian *L* = *K* - *E* of a fractal solid *W* isolated from external interactions**

$$\delta I = \delta \int_{t_1}^{t_2} [K - E] dt = 0,$$

- **The deformation gradient in fractal space is**

$$F_{kI} = \frac{1}{c_1^{(I)}} x_k,_I = \nabla_I^D x_k.$$

- **Assuming the specific internal energy *e* is a function of the deformation gradient only and after lots of math, we obtain the governing equation**

$$\nabla_I^D \left[\rho \frac{\partial e}{\partial F_{kI}} \right] - \rho \frac{dv_k}{dt} = 0 \quad \text{or} \quad \frac{1}{c_1^{(I)}} \frac{\partial}{\partial X_I} \left[\rho \frac{\partial e}{\partial F_{kI}} \right] - \rho \frac{dv_k}{dt} = 0,$$

Nonlinear Waves in a Fractal Elastic Solid

- According to our prescription, the deformation gradient in one dimension is

$$F = \frac{1}{c_1}\frac{\partial x(X,t)}{\partial X}.$$

- Differentiating with respect to time, we obtain the equation

$$\frac{\partial F}{\partial t} - \frac{1}{c_1}\frac{\partial v}{\partial X} = 0 \quad \text{which implies} \quad \frac{\partial F}{\partial t} - \frac{\partial v}{\partial l_\alpha} = 0,$$

- Assuming the stress S is a function only of F, we obtain

$$\frac{\partial v}{\partial t} - \frac{E}{\rho c_1}\frac{\partial F}{\partial X} = 0 \quad \text{which implies} \quad \frac{\partial v}{\partial t} - \frac{E}{\rho}\frac{\partial F}{\partial l_\alpha} = 0. \qquad E = \frac{\partial S(F)}{\partial F}.$$

- These equations can be solved by the method of characteristics in fractal spacetime using characteristics

$$\frac{dl_\alpha}{dt} = \pm C \quad \text{with propagation speed} \quad C = \sqrt{\frac{E}{\rho}}.$$

Shock Front in a Fractal Linear Viscoelastic Solid

- Consider shock fronts in viscoelastic fractal solids. In one dimension, the motion in a fractal rod *W* is

$$(1/c_1)\sigma,_x = \rho u,_{tt}$$

- The dynamics compatibility condition implies the discontinuity in stress is

$$[[\sigma]] = -\rho C[[u,_t]] \quad \text{in} \quad (l_\alpha, t) - plane,$$

- The linear viscoelastic stress-strain relation for a process that started at time $t = t_0^+$ is

$$\sigma(t) = E(0)\varepsilon(t) + \int_{t_0^+}^{t} E,_t(t-s)\varepsilon(s)ds = E(0)\frac{1}{c_1}u,_x(t) + \int_{t_0^+}^{t} E,_t(t-s)\frac{1}{c_1}u,_x(s)ds,$$

- It can be shown that the discontinuity in stress and the propagation speed are given by

$$[[\sigma]] = \sigma_0 \exp\left\{\frac{1}{2}\frac{E,_t(0)}{E(0)}t\right\} \quad \text{and} \quad C = \sqrt{\frac{E(0)}{\rho}}.$$

Summary and Conclusions

- We extended continuum thermomechanics to fractal media.
- The continuum theory is based on dimensional regularization, in which we employ fractional integrals to state global balance laws.
- We derived fractal continuity, linear momentum, and energy equations, which through dimensional regularization can be cast into equations in E^3.
- We showed that the Cauchy stress tensor is, in general, not symmetric in fractal media.

Summary and Conclusions

- Using Hamilton's principle, we obtained the equations of motion of a fractal elastic solid undergoing finite strains.
- We obtained equations governing the nonlinear waves in such a solid and showed that the equations can be solved by the method of characteristics in fractal space-time.
- We studied shock fronts in linear viscoelastic solids under small strains and showed that the discontinuity in stress across a shock front in a fractal medium is identical to the classical result.

Future Directions

- We plan to extend these ideas to develop a peridynamic theory for fractal media.