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Variable length scale in a peridynamic body

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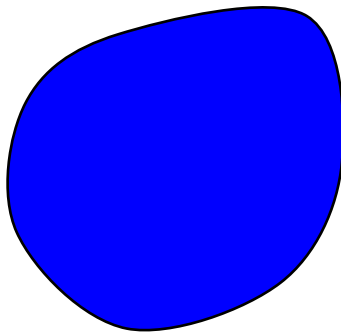University of Texas, Austin

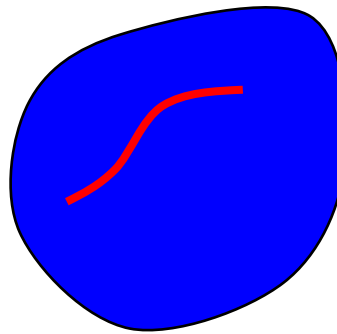SIAM Conference on Mathematical Aspects of Materials Science, Philadelphia, PA, June 12, 2013

Purpose of peridynamics

- To unify the mechanics of continuous and discontinuous media within a single, consistent set of equations.

Continuous body

Continuous body with a defect
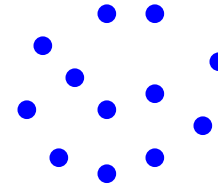
Discrete particles

- Why do this? Develop a mathematical framework that help in modeling…
  - Discrete-to-continuum coupling
  - Cracking, including complex fracture patterns
  - Communication across length scales.
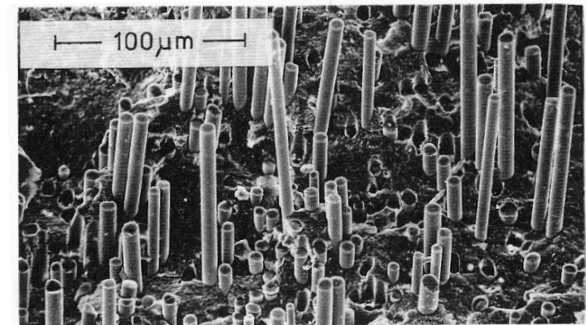
Figure 11.20 Pull-out: (a) schematic diagram; (b) fracture surface of 'Silceram' glass-ceramic reinforced with SiC fibres. (Courtesy H. S. Kim, P. S. Rogers and R. D. Rawlings.)

Peridynamics basics:
The nature of internal forces

Standard theory
Stress tensor field
(assumes contact forces and
smooth deformation)

n

σn

Internal surface
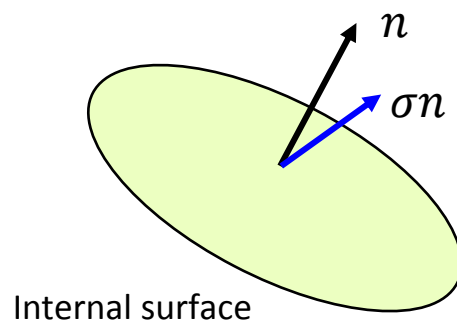
$$\rho \ddot{u}(x,t) = \nabla \cdot \sigma(x,t) + b(x,t)$$

Differentiation of contact forces

Peridynamics
Bond forces within small neighborhoods
(allow discontinuity)

Horizon δ

q

Bond

x $f(q,x)$

Family of x

Body
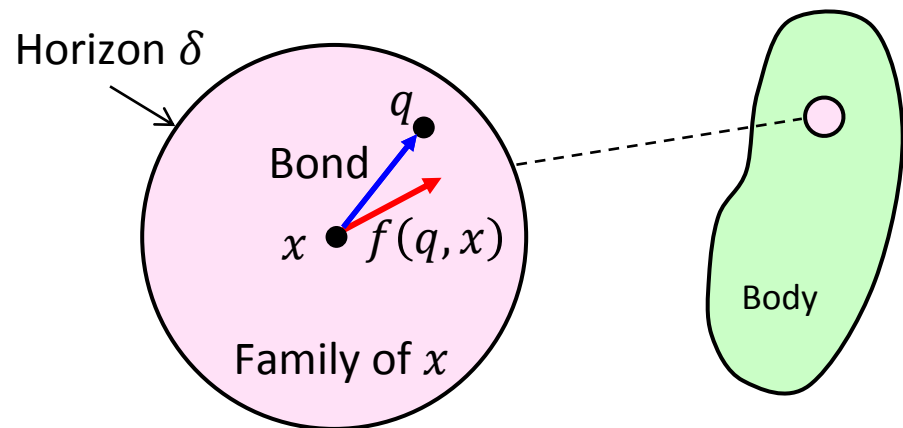
$$\rho \ddot{u}(x,t) = \int_{H_x} f(q,x)\,dV_q + b(x,t)$$

Summation over bond forces

Peridynamics basics:
Deformation state and force state

- The <u>deformation state</u> maps each bond to its deformed image.

$$Y[x]\langle q - x \rangle = y(q) - y(x)$$

- The <u>force state</u> maps bonds to bond forces according to the constitutive model.

$$f(q, x) = T[x]\langle q - x \rangle - T[q]\langle x - q \rangle$$

- The <u>constitutive model</u> maps deformation states to force states.

$$T[x] = \hat{T}(Y[x]) \qquad\qquad T[q] = \hat{T}(Y[q])$$

Peridynamic vs. local equations

State notation: $\underline{\text{State}}\langle\text{bond}\rangle = \text{vector}$
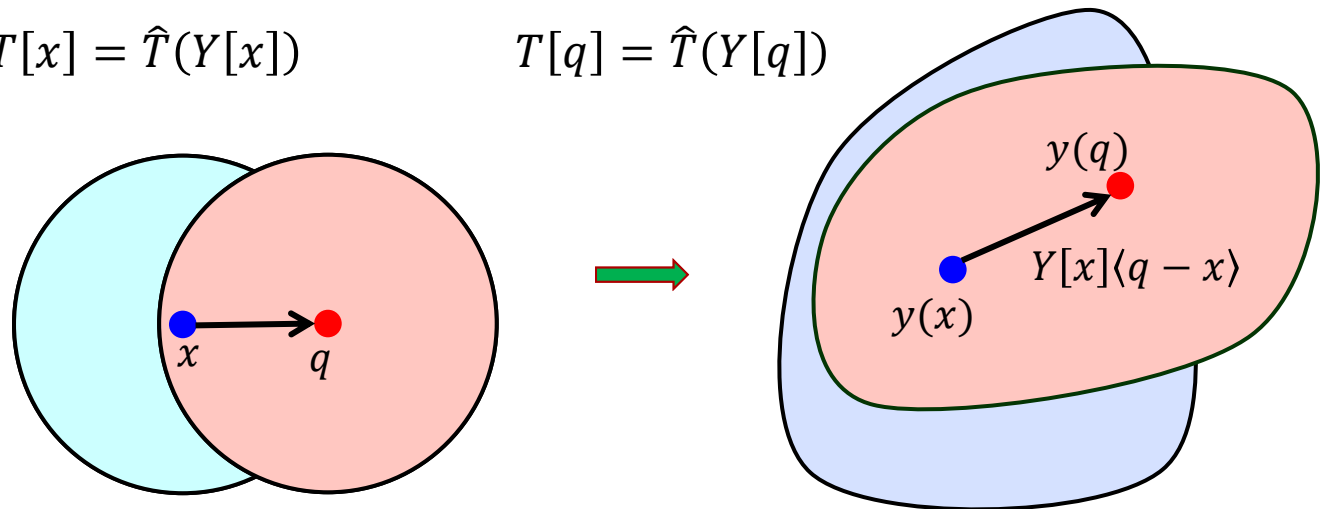
Relation	Peridynamic theory	Standard theory
Kinematics	$\underline{\mathbf{Y}}\langle\mathbf{q}-\mathbf{x}\rangle = \mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{x})$	$\mathbf{F}(\mathbf{x}) = \dfrac{\partial\mathbf{y}}{\partial\mathbf{x}}(\mathbf{x})$
Linear momentum balance	$\rho\ddot{\mathbf{y}}(\mathbf{x}) = \displaystyle\int_{\mathcal{H}} \Big(\mathbf{t}(\mathbf{q},\mathbf{x}) - \mathbf{t}(\mathbf{x},\mathbf{q})\Big)\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x})$	$\rho\ddot{\mathbf{y}}(\mathbf{x}) = \nabla\cdot\boldsymbol{\sigma}(\mathbf{x}) + \mathbf{b}(\mathbf{x})$
Constitutive model	$\mathbf{t}(\mathbf{q},\mathbf{x}) = \underline{\mathbf{T}}\langle\mathbf{q}-\mathbf{x}\rangle, \qquad \underline{\mathbf{T}} = \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}})$	$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{F})$
Angular momentum balance	$\displaystyle\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle\mathbf{q}-\mathbf{x}\rangle \times \underline{\mathbf{T}}\langle\mathbf{q}-\mathbf{x}\rangle\, dV_{\mathbf{q}} = \mathbf{0}$	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
Elasticity	$\underline{\mathbf{T}} = W_{\underline{\mathbf{Y}}}$ (Fréchet derivative)	$\boldsymbol{\sigma} = W_{\mathbf{F}}$ (tensor gradient)
First law	$\dot{\varepsilon} = \underline{\mathbf{T}} \bullet \underline{\dot{\mathbf{Y}}} + q + r$	$\dot{\varepsilon} = \boldsymbol{\sigma}\cdot\dot{\mathbf{F}} + q + r$

$$\underline{\mathbf{T}} \bullet \underline{\dot{\mathbf{Y}}} := \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\dot{\mathbf{Y}}}\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}}$$

Mixed blessing of nonlocality

- Nonlocality is necessary to achieve the goals of peridynamics , but it entails some practical difficulties.

- Example: nonuniform horizon in a bar with "homogeneous bulk properties".

- We know how to scale a material model so the Young's modulus is independent of horizon.

$$T_\delta\langle\xi\rangle = \delta^{-2}Z\langle\xi/\delta\rangle$$

where Z is a reference force state that depends only on strain.

- But when you use this to model equilibrium of a bar with variable horizon, you get a "wrong" result:

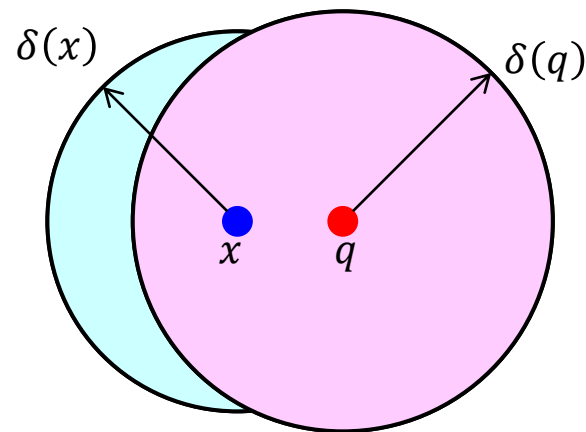Prescribed horizon $\delta(x)$

Solution $u(x)$
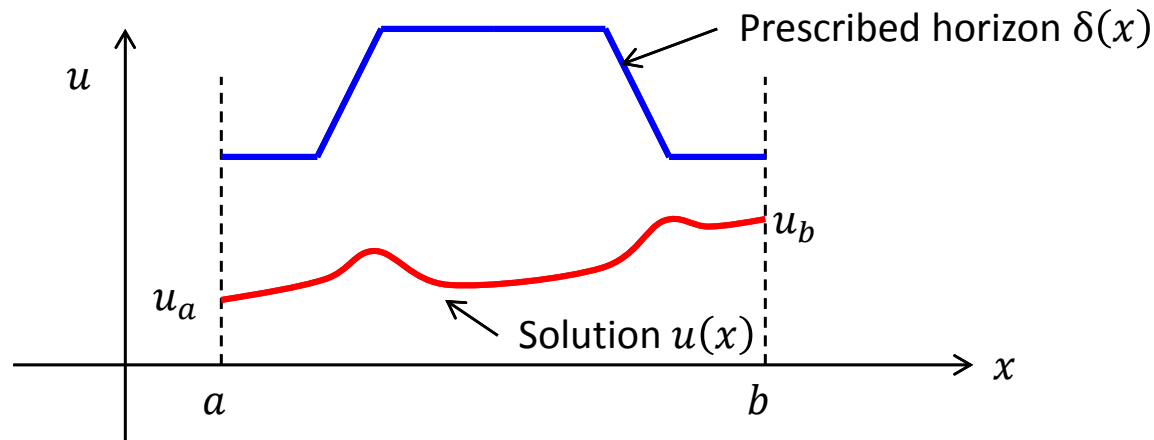
Origin of artifacts

- The peridynamic force density operator $L(x)$ involves the force state not only at x but also the force states at all points within the horizon.

$$0 = L(x) + b, \qquad L(x) = \int_{-\infty}^{\infty} \{T_{\delta(x)}[x]\langle q - x \rangle - T_{\delta(q)}[q]\langle x - q \rangle\} dq$$

so simply scaling the material model at x is not sufficient.

Variable horizon

"Patch test" requirement for a coupling method

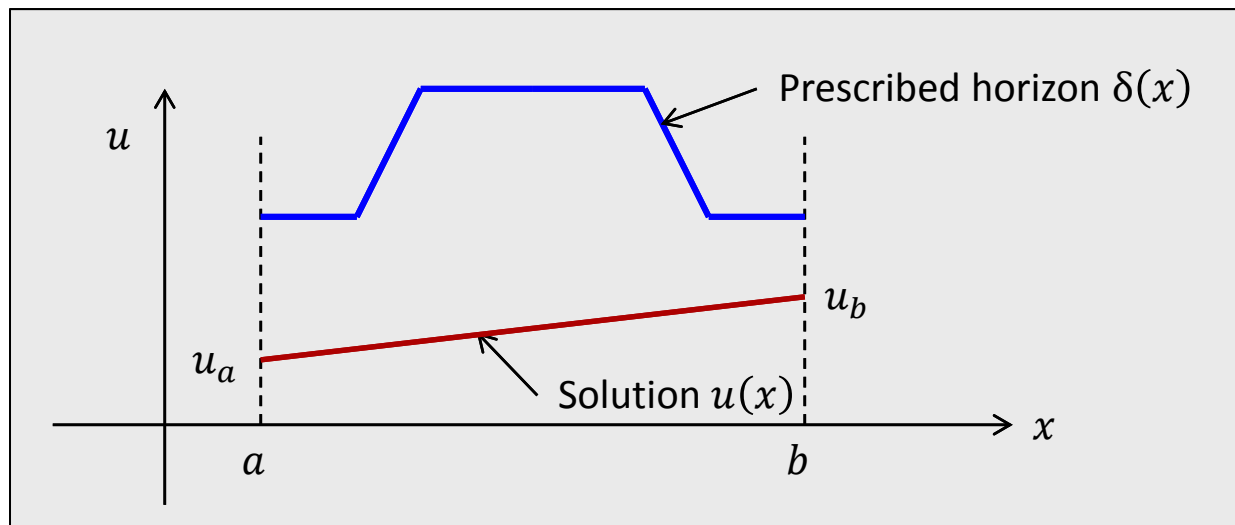- In a deformation of the form

$$u(x) = a + Hx$$

where H is a constant and the material model is of the form

$$T[x]\langle\xi\rangle = \delta^{-2}(x)Z\langle\xi/\delta(x)\rangle$$

where $\delta(x)$ is a prescribed function and Z is a state that depends only on H, we require

$$L(x) = 0 \quad \text{for all } x.$$

Prescribed horizon $\delta(x)$
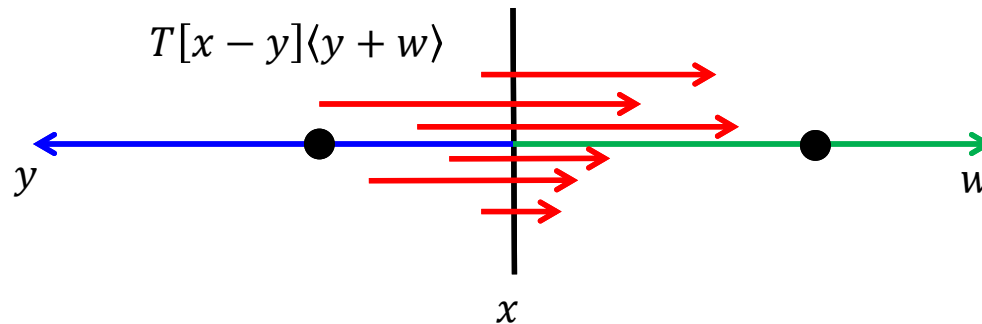
Solution $u(x)$

u_a

u_b

Peridynamic stress tensor

- Define the peridynamic stress tensor field by

$$v(x) = \int_0^\infty \int_0^\infty \{T[x-y]\langle y+w\rangle - T[x+y]\langle -y-w\rangle\} \, dy \, dz$$

- Identity:

$$\frac{dv}{dx} = \int_{-\infty}^\infty \{T[x]\langle q-x\rangle - T[q]\langle x-q\rangle\}dq$$

- $v(x)$ is the force per unit area carried by all the bonds that cross x.

$T[x-y]\langle y+w\rangle$

Partial stress field

- Under our assumption that

$$T[x]\langle\xi\rangle = \delta^{-2}(x)Z\langle\xi/\delta(x)\rangle$$

one finds directly that

$$v_0(x) := \int_{-\infty}^{\infty} \xi T[x]\langle\xi\rangle \, d\xi = \int_{-\infty}^{\infty} \xi Z\langle\xi\rangle \, d\xi$$

which is independent of x, so $dv_0/dx = 0$.

- v_0 is called the **partial stress** field.

- Clearly the internal force density field computed from

$$L_0(x) := dv_0/dx$$

passes the "patch test."
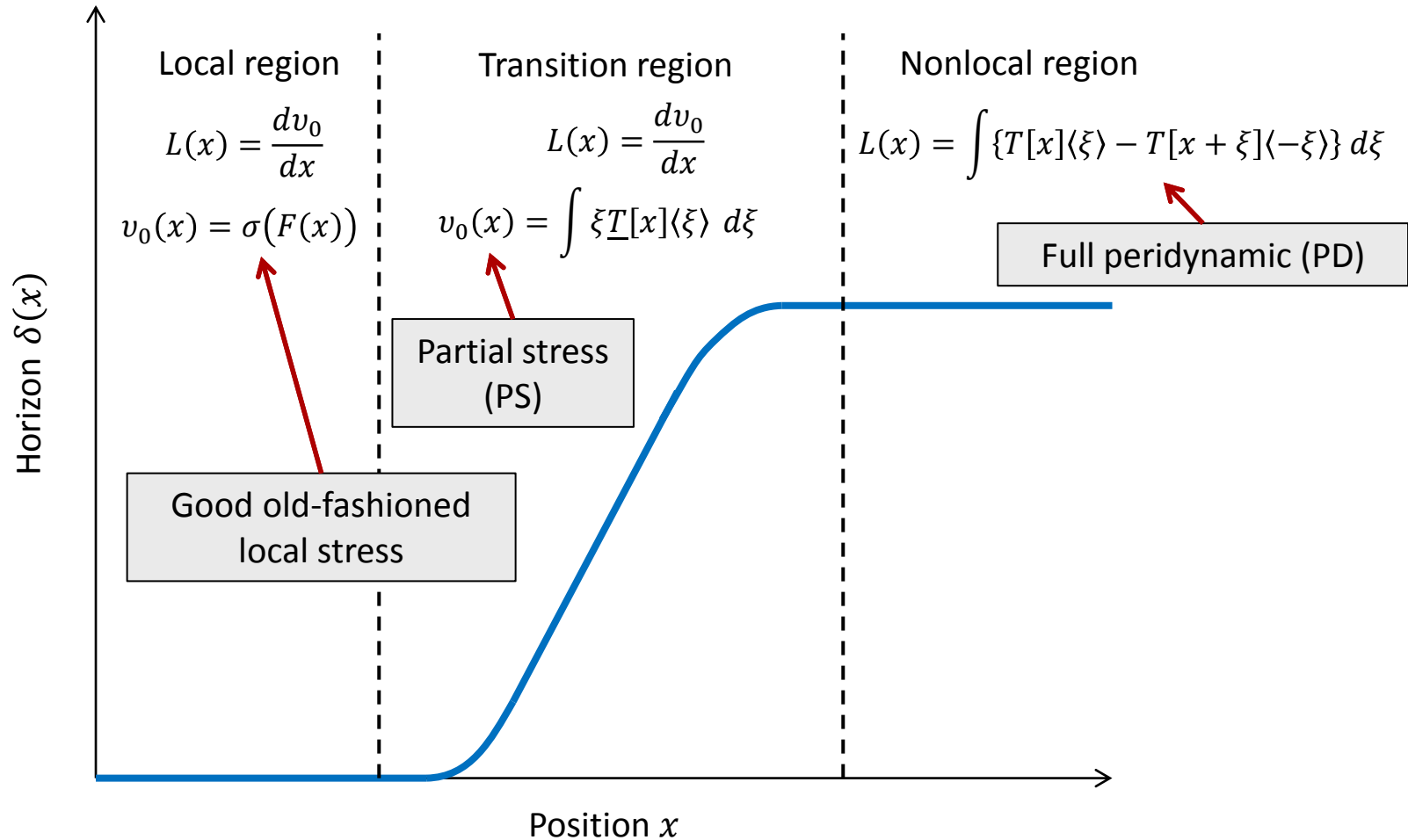
- This observation leads to the following idea…

Concept for coupling method

- Idea: within a coupling region in which δ is changing, compute the internal force density from

$$L(x) = \frac{dv_0}{dx}(x), \qquad v_0(x) := \int_{-\infty}^{\infty} \xi T[x]\langle\xi\rangle \, d\xi$$

- Here, $T[x](x)$ is determined from whatever the deformation happens to be near x.

  - Z is no longer involved.

  - The material model has not changed from full PD, but the way of computing L has.

Local-nonlocal coupling idea

Local region

$$L(x) = \frac{dv_0}{dx}$$

$$v_0(x) = \sigma\big(F(x)\big)$$

Transition region

$$L(x) = \frac{dv_0}{dx}$$

$$v_0(x) = \int \xi \underline{T}[x]\langle\xi\rangle \, d\xi$$

Nonlocal region

$$L(x) = \int \{T[x]\langle\xi\rangle - T[x+\xi]\langle-\xi\rangle\} \, d\xi$$

Full peridynamic (PD)

Partial stress (PS)

Good old-fashioned local stress

Horizon $\delta(x)$

Position x

Continuum patch test results

- Full PD shows artifacts, as expected.
- PS shows no artifacts, as promised.
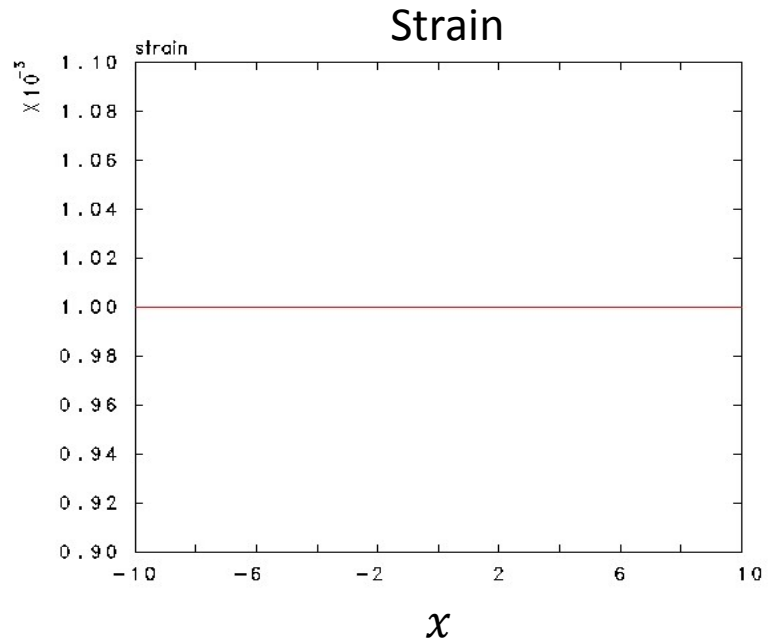
Horizon

Strain

PS everywhere

Full PD
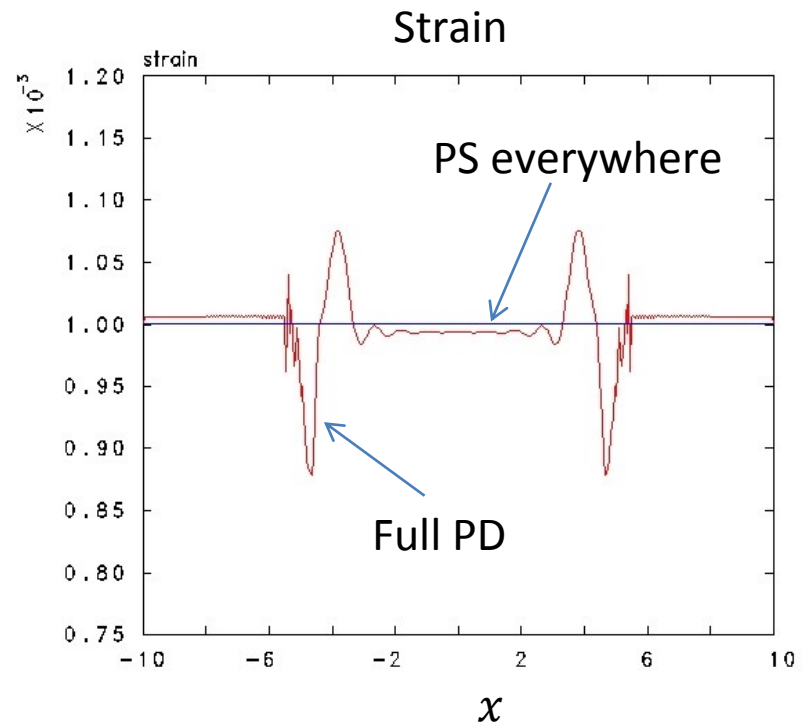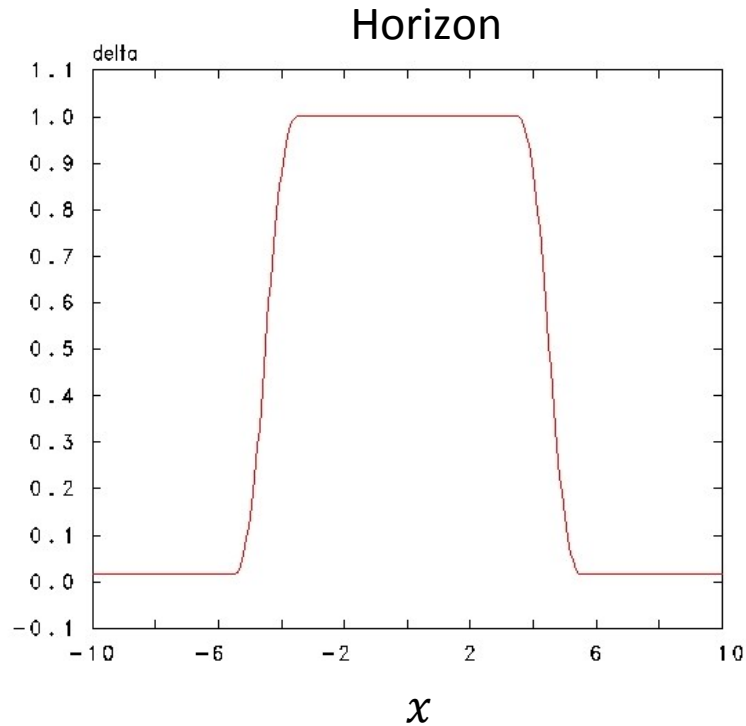
$$u = 0 \qquad\qquad u = 0.02$$
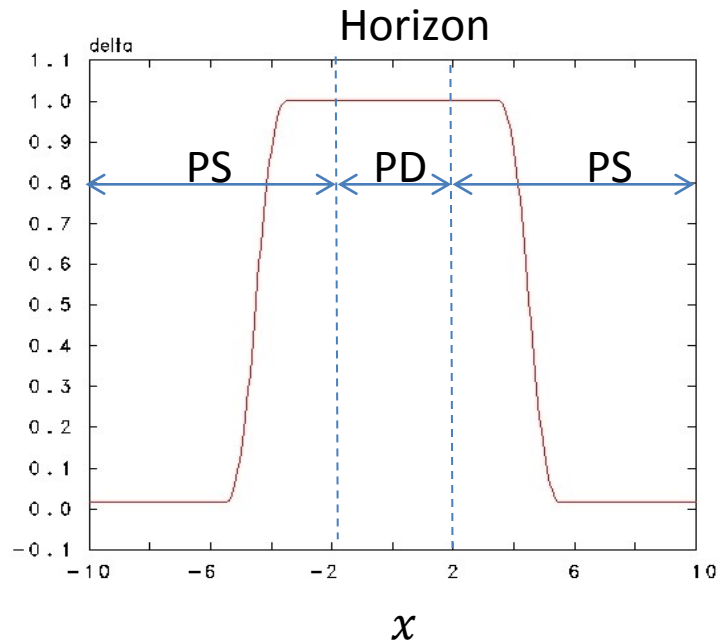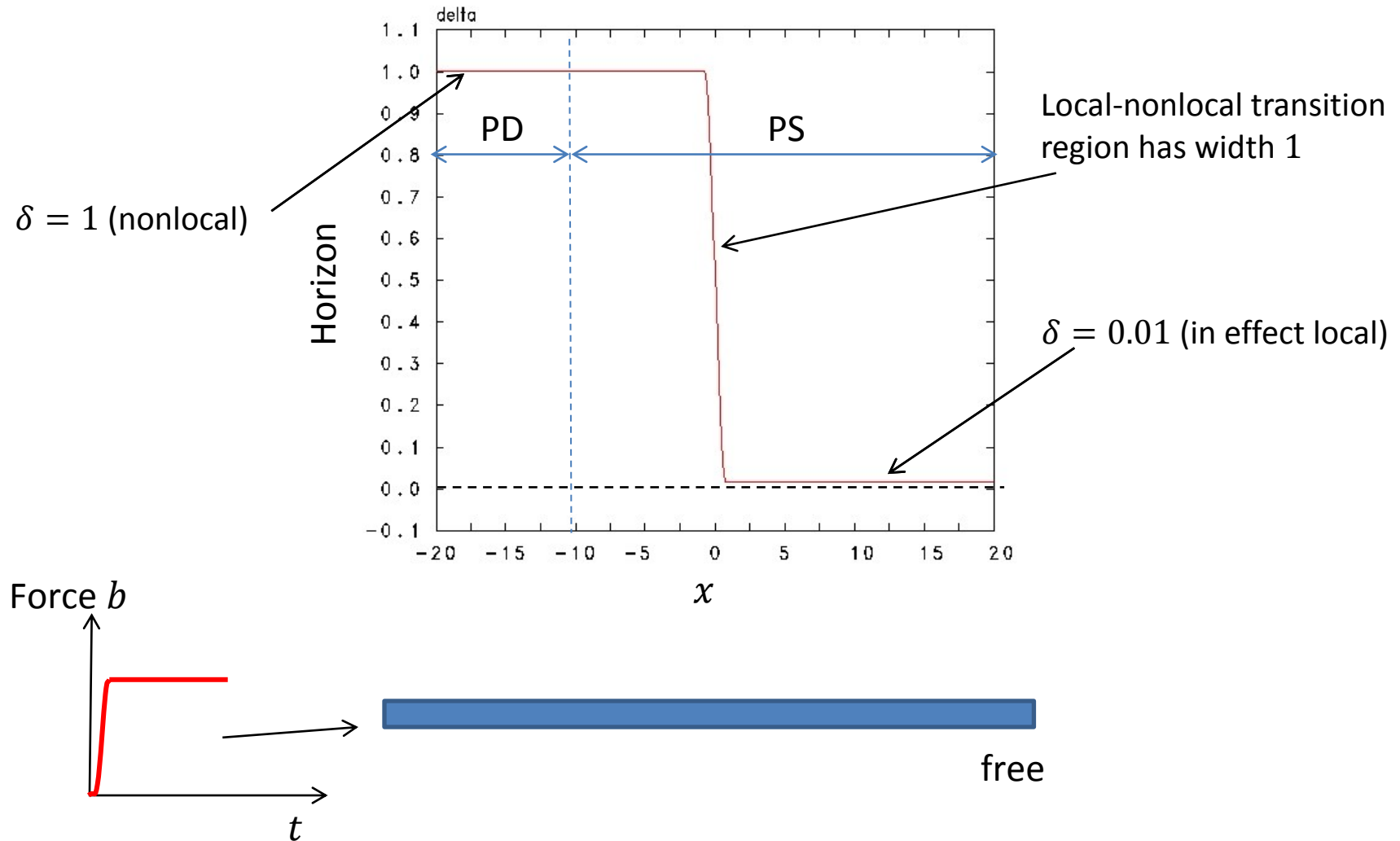
Continuum patch test with coupling

- No artifacts with PD-PS coupling (this was hoped for but not guaranteed).

Horizon

Strain

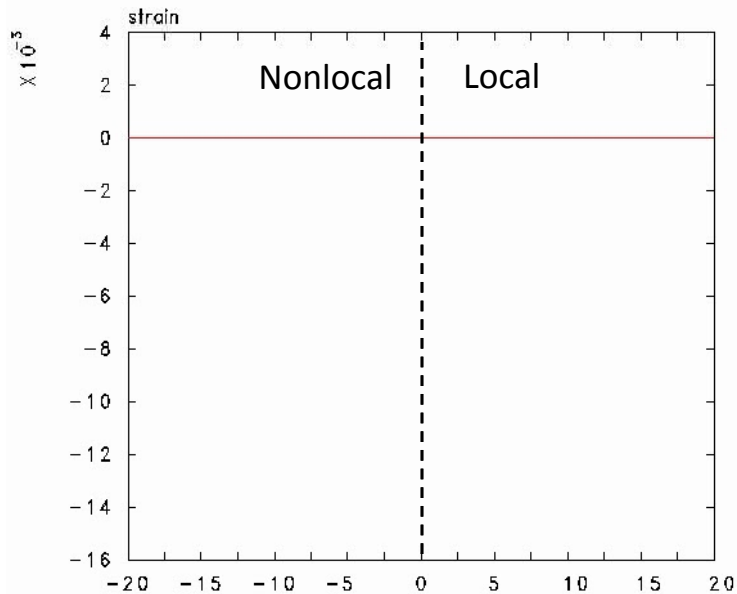

$u = 0$ $u = 0.02$

Pulse propagation test problem

- Does our coupling method work for dynamics as well as statics with variable horizon?

$\delta = 1$ (nonlocal)

PD

PS

Local-nonlocal transition region has width 1

$\delta = 0.01$ (in effect local)
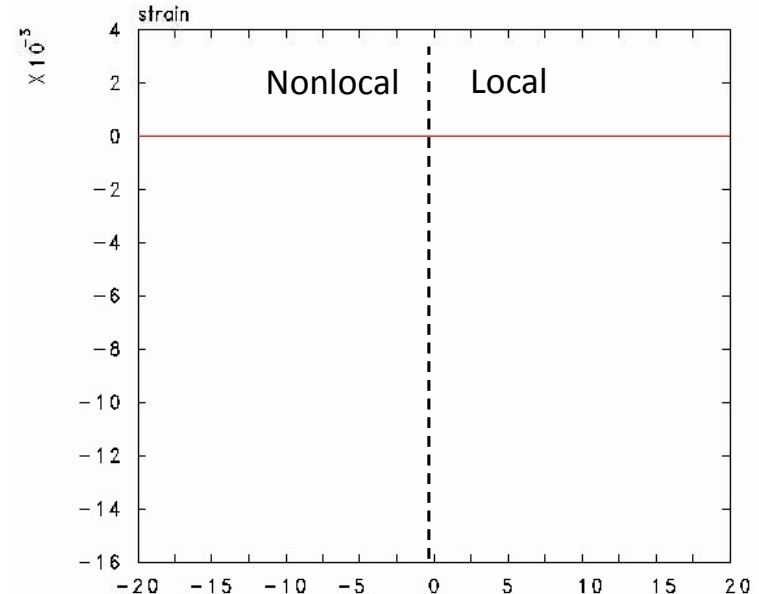
Horizon

x

Force b

t

free

Pulse propagation test results
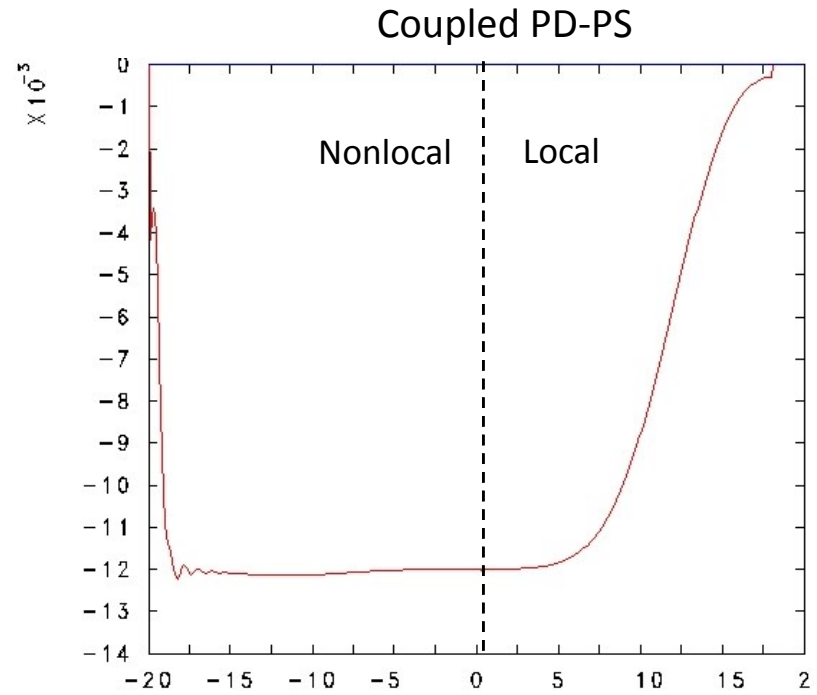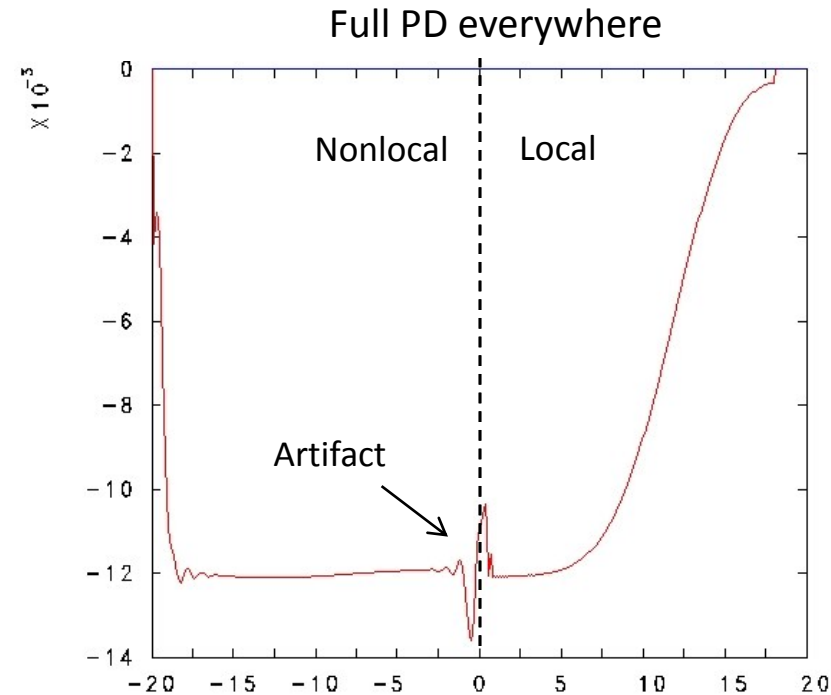
- Movies of strain field evolution

Full PD everywhere

Coupled PD-PS

Pulse propagation test results

- Strain field: no artifacts appear in the coupled model the local-nonlocal transition.

Full PD everywhere

Coupled PD-PS

Discussion

- The partial stress approach may provide a means for local-nonlocal coupling within the continuum equations.
    - Uses the underlying peridynamic material model but modifies the way internal force density is computed.
    - Expected to work in 2D and 3D too (but not yet tested).
    - No assumption of linear material response (but nonlinear not yet tested).
- PS is inconsistent from an energy minimization point of view.
    - Not suitable for a full-blown theory of mechanics (as PD is).
    - Not yet clear what implications this may have in practice.
    - We still need to use full PD for crack progression.
- Some similarities between PS and the Virtual Internal Bond method (Gao & Klein, JMPS, 1998).