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A Mixing Model for the Richtmyer-Meshkov Instability

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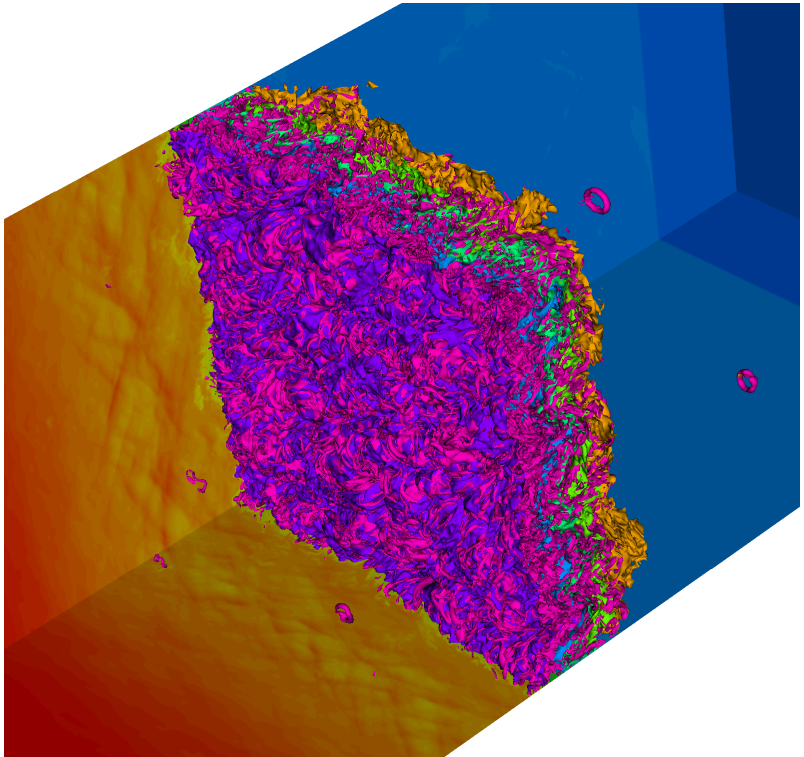
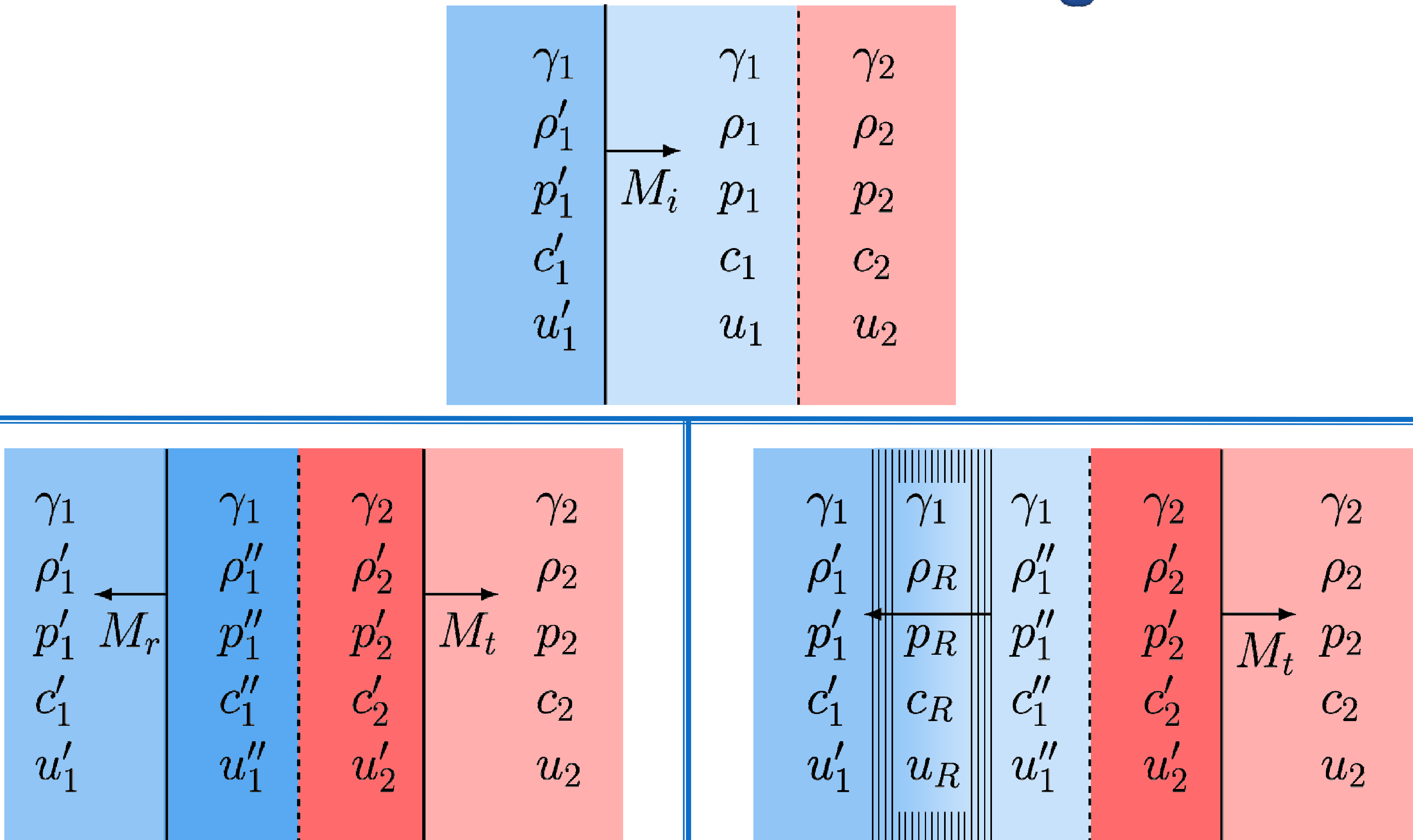


Consider a shock crossing an interface between two ideal gases



Matching pressures and velocities across the interface yields a transcendental equation for the pressure jump across the transmitted shock

$$\left[\frac{(\Lambda_2 - 1)\rho_1}{(\Lambda_1 - 1)\rho_2}\right]^{1/2} \frac{\Pi_t - 1}{(\Pi_t \Lambda_2 + 1)^{1/2}} = \frac{\Pi_i - 1}{(\Pi_i \Lambda_1 + 1)^{1/2}} - \left(\frac{\rho_1}{\rho_1'}\right)^{1/2} \frac{\Pi_t - \Pi_i}{(\Pi_t \Lambda_1 + \Pi_i)^{1/2}}$$

$$\Pi_i \equiv \frac{p_1'}{p_1}, \quad \Lambda_1 \equiv \frac{\gamma_1 + 1}{\gamma_1 - 1}$$

$$\Pi_t \equiv \frac{p_2'}{p_2}, \quad \Lambda_2 \equiv \frac{\gamma_2 + 1}{\gamma_2 - 1}$$

All other variables are readily obtained after solving this equation.

Does the RM growth rate obey a power law at late time?

$$h = ct^{\theta}$$

$$h \propto (u_s t)^{\theta}$$

$$h - h_o \propto (t - t_o)^{\theta}$$

$$0.2 \le \theta \le 0.67$$

- What are the dimensions of *c*?
- How is this growth rate derived?
- Is it a good idea to raise dimensional variables to fractional powers?
- What's missing?

The mixing with (*h*) and time (*t*) must be properly nondimensionalized

- We can eliminate the virtual origin by modeling $\dot{h}$, rather than h.
- Normalizing $\dot{h}$ by its initial value, $\dot{h}_o$, ensures that all growth curves start at unity.
- Linear stability theory and experimental evidence indicate that the growth rate depends on the dominant perturbation wavelength λ_o.
- A relevant timescale thus appears to be $\lambda_o / \dot{h}_o$.

How do we get $\dot{h}_o$?

- Assume interfacial perturbations are known.
- Define $h(t) \equiv \int_{-\infty}^{\infty} \psi(\langle\xi\rangle)\, dx$, where ψ is "product".
- *h*(*t*) is the thickness of mixed fluid that would result if the entrained gases were homogenized in the transverse plane.
- From continuity, growth rate = mass flux through equimolar plane:

$$\frac{dh}{dt} = 2\int_{-\infty}^{x_s} \frac{\partial\langle\xi\rangle}{\partial t}\, dx - 2\int_{x_s}^{\infty} \frac{\partial\langle\xi\rangle}{\partial t}\, dx = \frac{4\,\langle\rho u\rangle|_{x_s}}{\rho_1'' - \rho_2'}$$

The mass-flux definition of the growth rate has significant advantages:

- No issues of asymmetry between bubbles and spikes.
- Not sensitive to outliers (like threshold definitions).
- Valid for shocks in either direction.
- Data need only be gathered on a single plane (PLIF friendly).

The post-shock density field is obtained directly from the known perturbations

$$\dot{h}_o \approx \dot{h}^+ \equiv \frac{4\left\langle \rho^+ u^+ \right\rangle_{x^+}}{\rho_1'' - \rho_2'}$$

$$\rho^-(x, y, z) = \rho_1 + (\rho_2 - \rho_1)\, H(x - \eta(y, z))$$

$$\rho^+(x, y, z) = \rho_1'' + (\rho_2' - \rho_1'')\, H(x - \eta^+(y, z))$$

$$\eta^+ = \left(1 - \frac{u_s}{M_i c_1}\right)\eta + x^+$$

The post-shock velocity field can be obtained from Biot-Savart integration of the vorticity field

$$\boldsymbol{u}^+(\boldsymbol{x}) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\boldsymbol{\omega}^+(\boldsymbol{x}^*)\times(\boldsymbol{x}-\boldsymbol{x}^*)}{||\boldsymbol{x}-\boldsymbol{x}^*||^3}\,dx^*\,dy^*\,dz^*$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\boldsymbol{u}) = \frac{1}{\rho^2}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p + \boldsymbol{\nabla}\times\left(\frac{1}{\rho}\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\tau}}\right)$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} \approx \frac{1}{\rho^2}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p$$

The pressure gradient can be obtained by assuming an essentially planar shock

$$\frac{\partial p}{\partial x} \approx -\frac{\partial \rho u}{\partial t}$$

$$\frac{\partial p}{\partial y} \approx 0$$

$$\frac{\partial p}{\partial z} \approx 0$$

$$\frac{\partial \omega_x}{\partial t} \approx 0$$

$$\frac{\partial \omega_y}{\partial t} \approx -\frac{1}{\rho^2}\frac{\partial \rho}{\partial z}\frac{\partial \rho u}{\partial t}$$

$$\frac{\partial \omega_z}{\partial t} \approx \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial \rho u}{\partial t}$$

The impulsive approximation allows us to separate the spatial and temporal dependence of density and velocity

$$\rho \approx \rho^- + (\rho^+ - \rho^-)\mathcal{H}(t)$$

$$u \approx u_s \mathcal{H}(t)$$

$$\rho^2 = \rho^-\rho^- + (\rho^+\rho^+ - \rho^-\rho^-)\mathcal{H}(t)$$

$$\frac{\partial \rho}{\partial z} = \frac{\partial \rho^-}{\partial z} + \left(\frac{\partial \rho^+}{\partial z} - \frac{\partial \rho^-}{\partial z}\right)\mathcal{H}(t)$$

$$\frac{\partial \rho}{\partial y} = \frac{\partial \rho^-}{\partial y} + \left(\frac{\partial \rho^+}{\partial y} - \frac{\partial \rho^-}{\partial y}\right)\mathcal{H}(t)$$

$$\frac{\partial \rho u}{\partial t} = \rho^+ u_s \delta(t) \ .$$

The vorticity laid down by the shock is obtained by integrating over the impulse

$$\omega_x^+ \equiv \int_{0^-}^{0^+} \frac{\partial \omega_x}{\partial t}\, dt \approx 0$$

$$\omega_y^+ \equiv \int_{0^-}^{0^+} \frac{\partial \omega_y}{\partial t}\, dt \approx -\frac{u_s}{\rho^+} \frac{\partial \rho^+}{\partial z}$$

$$\omega_z^+ \equiv \int_{0^-}^{0^+} \frac{\partial \omega_z}{\partial t}\, dt \approx \frac{u_s}{\rho^+} \frac{\partial \rho^+}{\partial y}$$

The post-shock vorticity can be written directly in terms of the perturbations

$$\omega_y^+ \approx \frac{u_s\left(\rho_2' - \rho_1''\right)\frac{\partial \eta^+}{\partial z}\delta\left(x - \eta^+\right)}{\rho_1'' + \left(\rho_2' - \rho_1''\right)H\left(x - \eta^+\right)}$$

$$\omega_z^+ \approx -\frac{u_s\left(\rho_2' - \rho_1''\right)\frac{\partial \eta^+}{\partial y}\delta\left(x - \eta^+\right)}{\rho_1'' + \left(\rho_2' - \rho_1''\right)H\left(x - \eta^+\right)}$$

The post-shock velocity can thus be computed *a priori*

$$u^+ \approx \frac{u_s A^+}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\frac{\partial \eta^+(y^*,z^*)}{\partial y^*}(y-y^*) + \frac{\partial \eta^+(y^*,z^*)}{\partial z^*}(z-z^*)}{\left[(x-\eta^+(y^*,z^*))^2 + (y-y^*)^2 + (z-z^*)^2\right]^{3/2}} \, dy^* \, dz^*$$

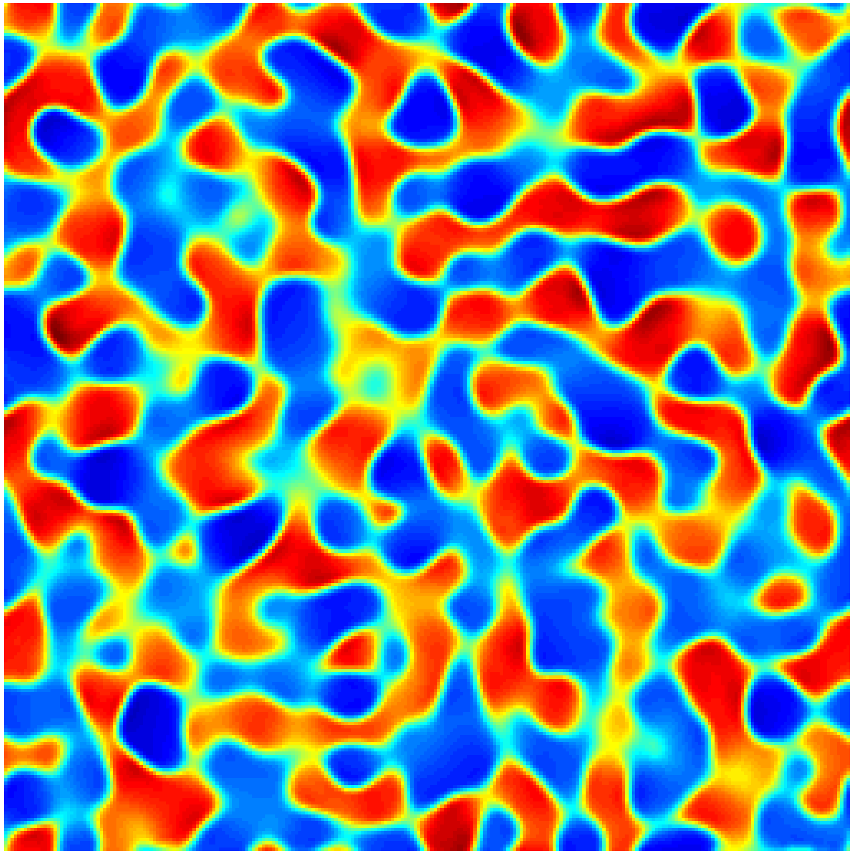
How good is the impulsive-planar-shock approximation?

Mass flux (ρu) on the equimolar plane, $\langle \xi \rangle = 0.5$

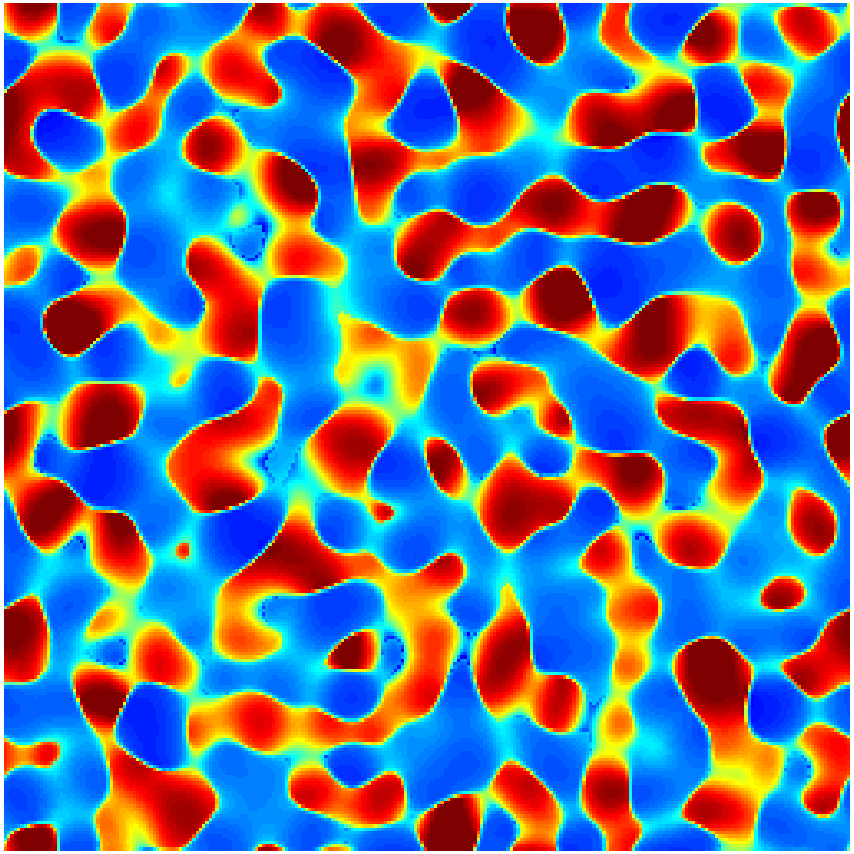
$A = 0.53$

$M_i = 1.1$

$\eta_{rms} / \lambda_o = 0.1$



Simulation



Model

Visualization at nondimensional times

$$t\frac{\dot{h}_o}{\lambda_o} = \quad 0 \quad 1 \quad 2 \quad 5 \quad 30$$

$A = 0.35$

$M_i = 1.5$

$\eta_{rms} / \lambda_o = 0.1$

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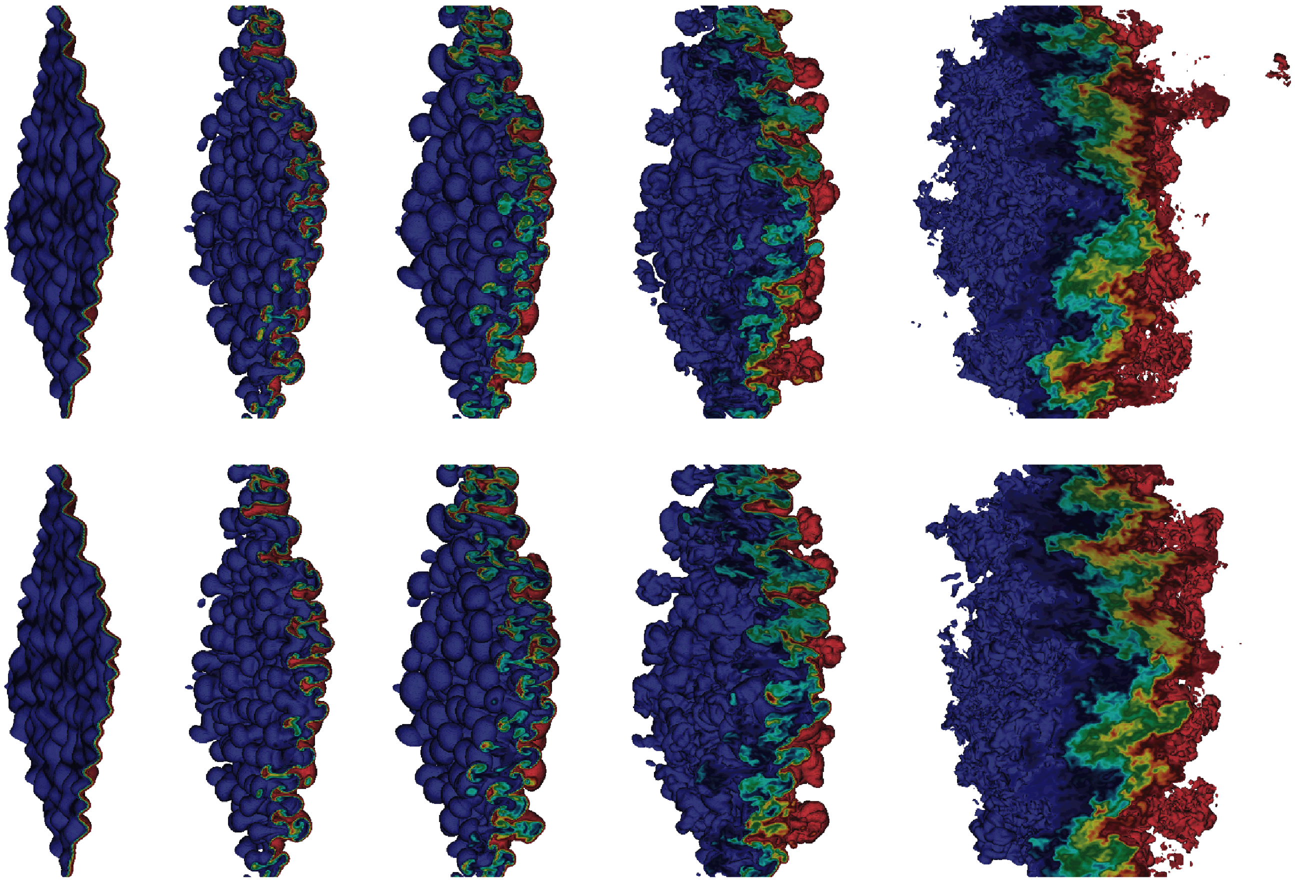
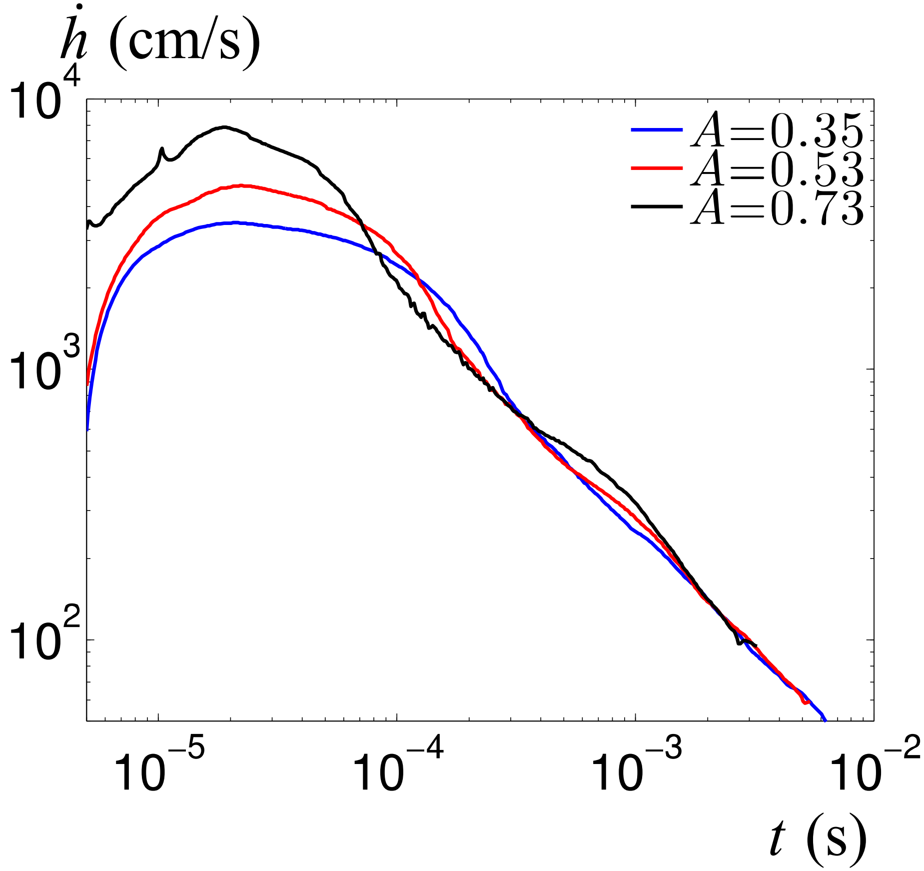


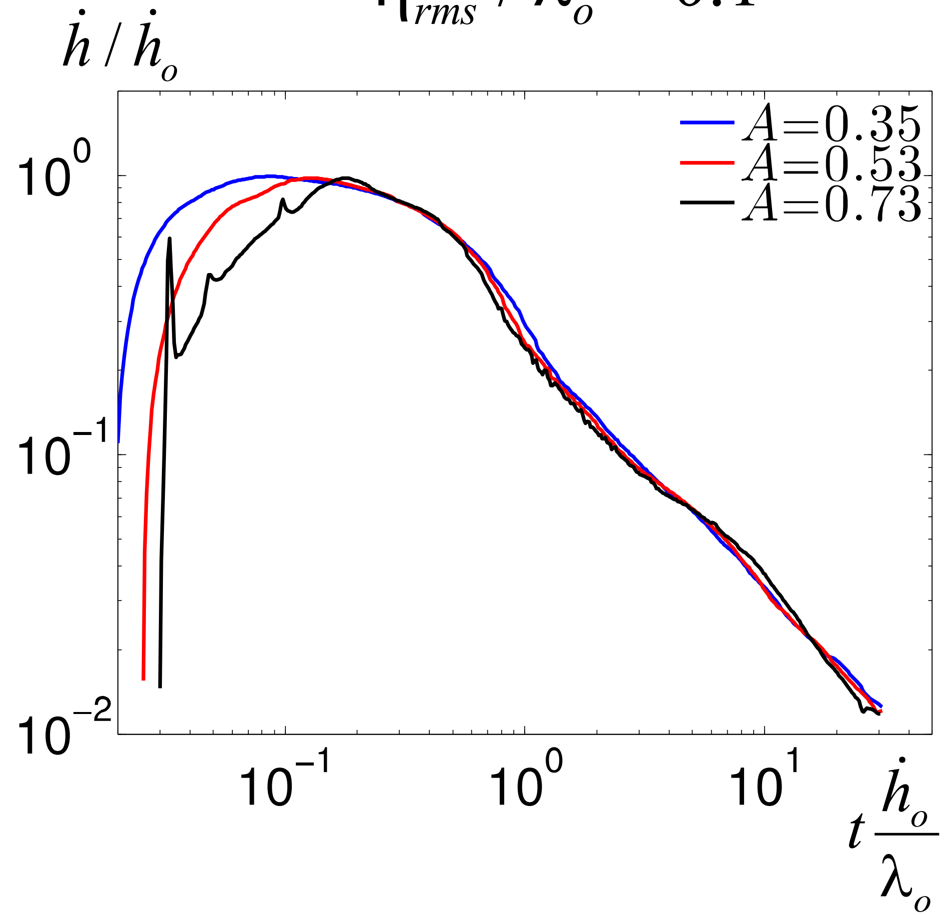
$0.05 < \xi < 0.95$

The scaled growth rates collapse for different Atwood numbers ($A > 0$)

$$M_i = 1.5$$

$$\eta_{rms} / \lambda_o = 0.1$$

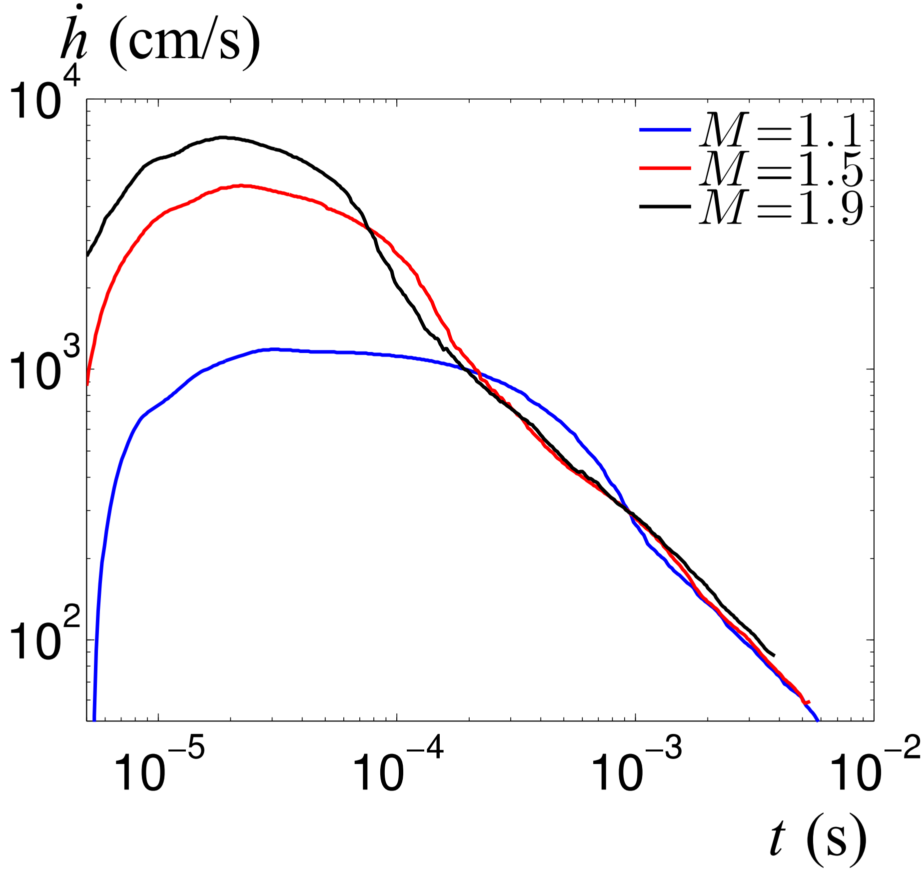


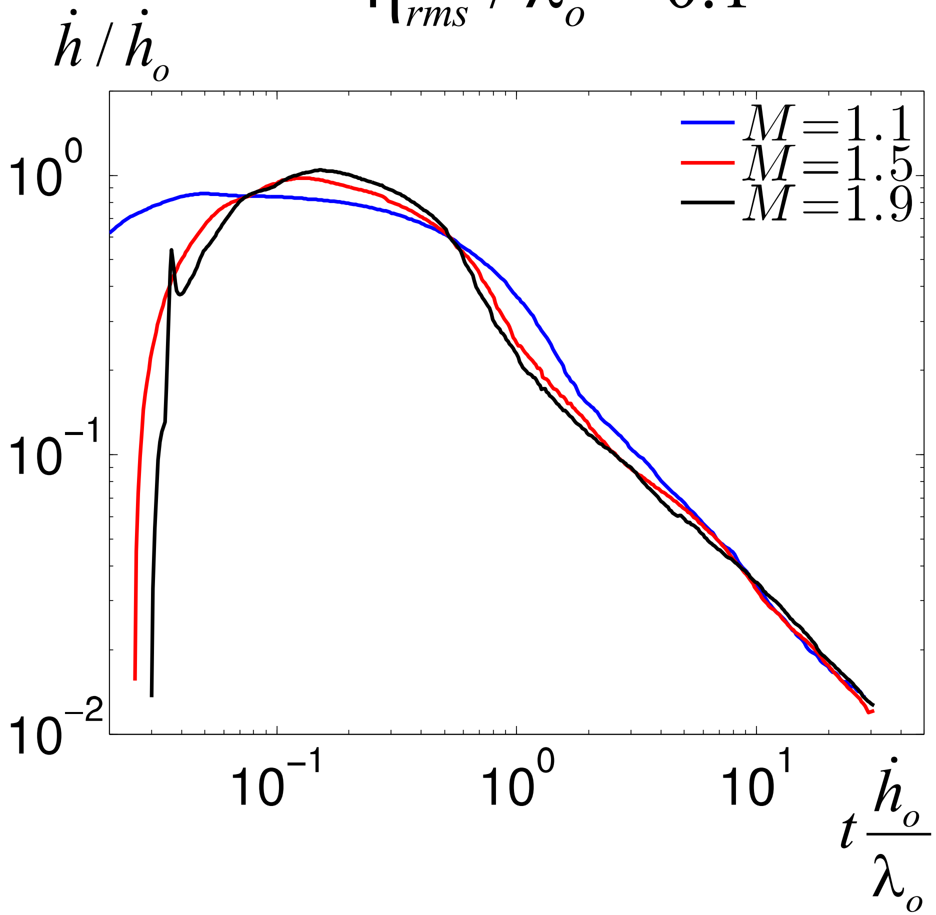


The scaled growth rates collapse for different Mach numbers

$$A = 0.53$$

$$\eta_{rms} / \lambda_o = 0.1$$

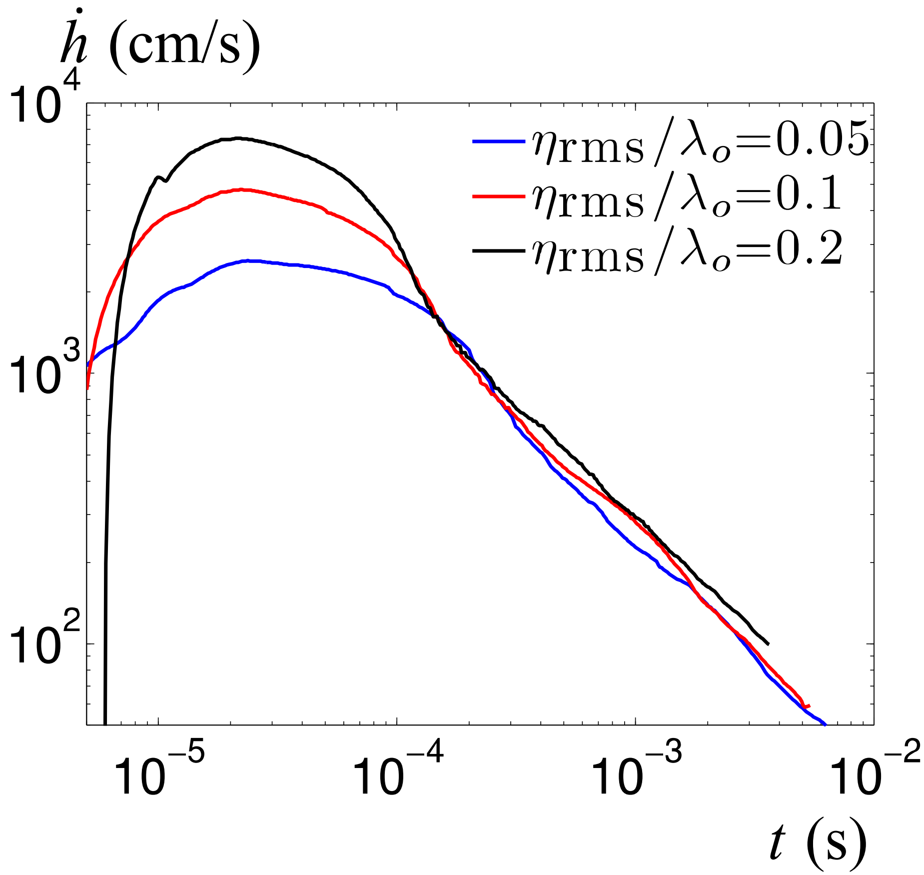


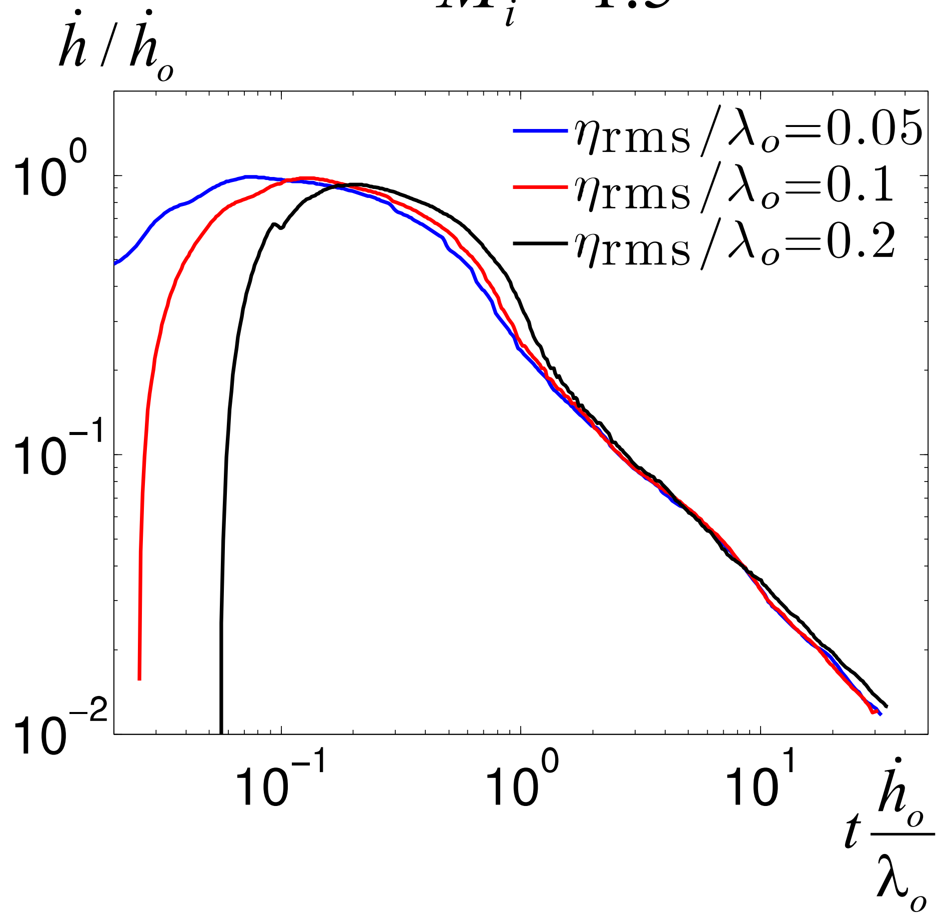


The scaled growth rates collapse for different amplitude/wavelength ratios

$A = 0.53$

$M_i = 1.5$





Time axis shift for heavy-to-light cases ($A < 0$)

Interface thickness:
(note that $\dot{h}_o < 0$ for A < 0)

$$h = \dot{h}_o t + h_o$$

Thickness is minimum at:

$$h \approx 0 \qquad t = \frac{-h_o}{\dot{h}_o}$$

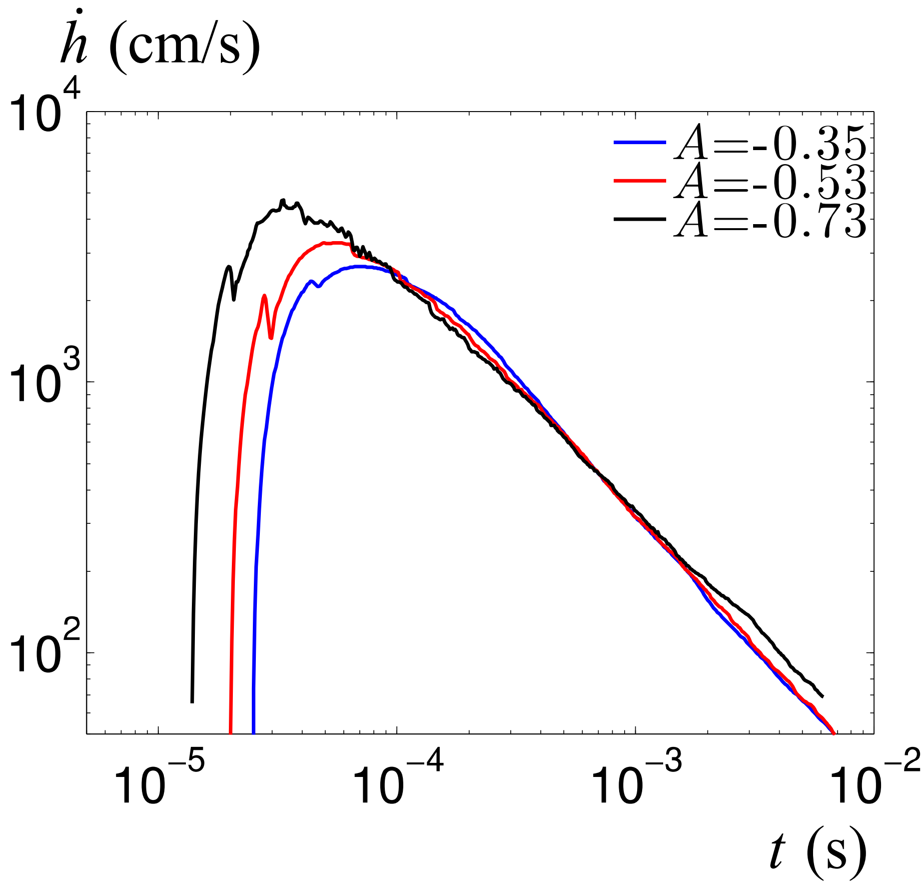
Shift time axis by this amount
(phase inversion time)

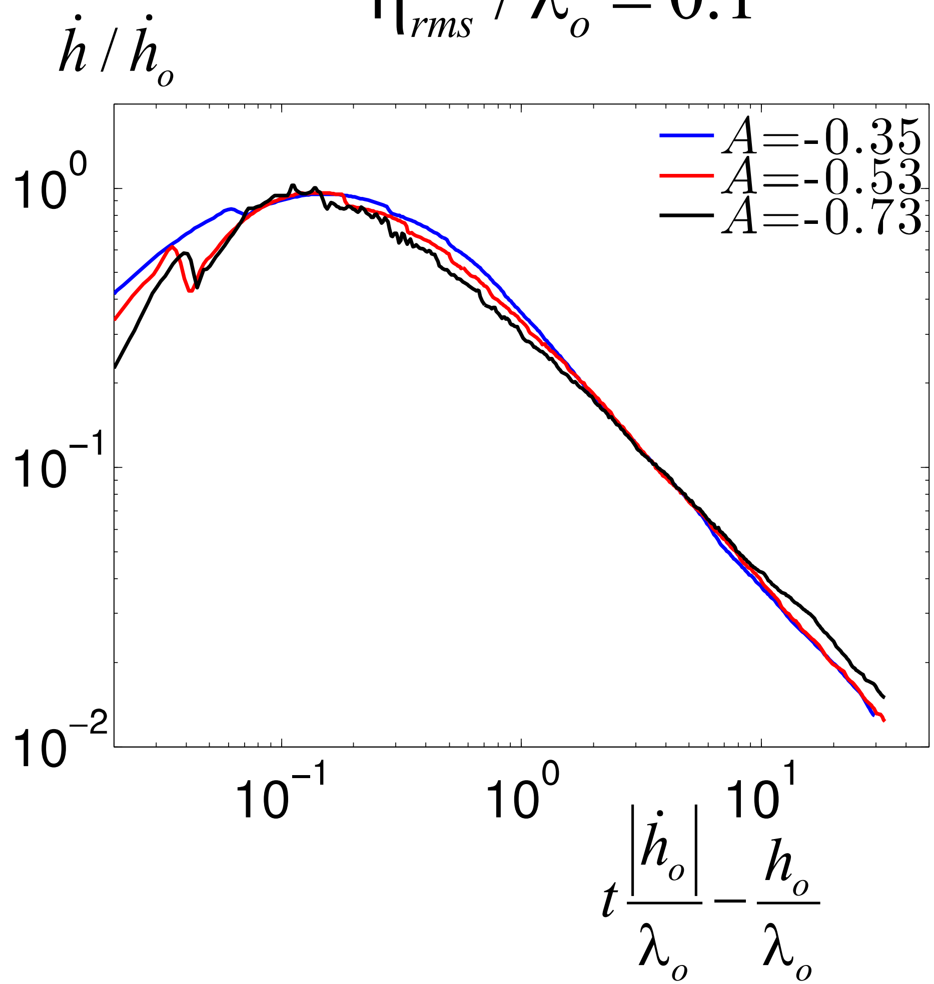
$$t - \frac{-h_o}{\dot{h}_o} \xrightarrow{\text{nondimensional}} t\frac{\left|\dot{h}_o\right|}{\lambda_o} - \frac{h_o}{\lambda_o}$$

The scaled growth rates collapse for different Atwood numbers (A < 0)

$$M_i = 1.5$$

$$\eta_{rms} / \lambda_o = 0.1$$

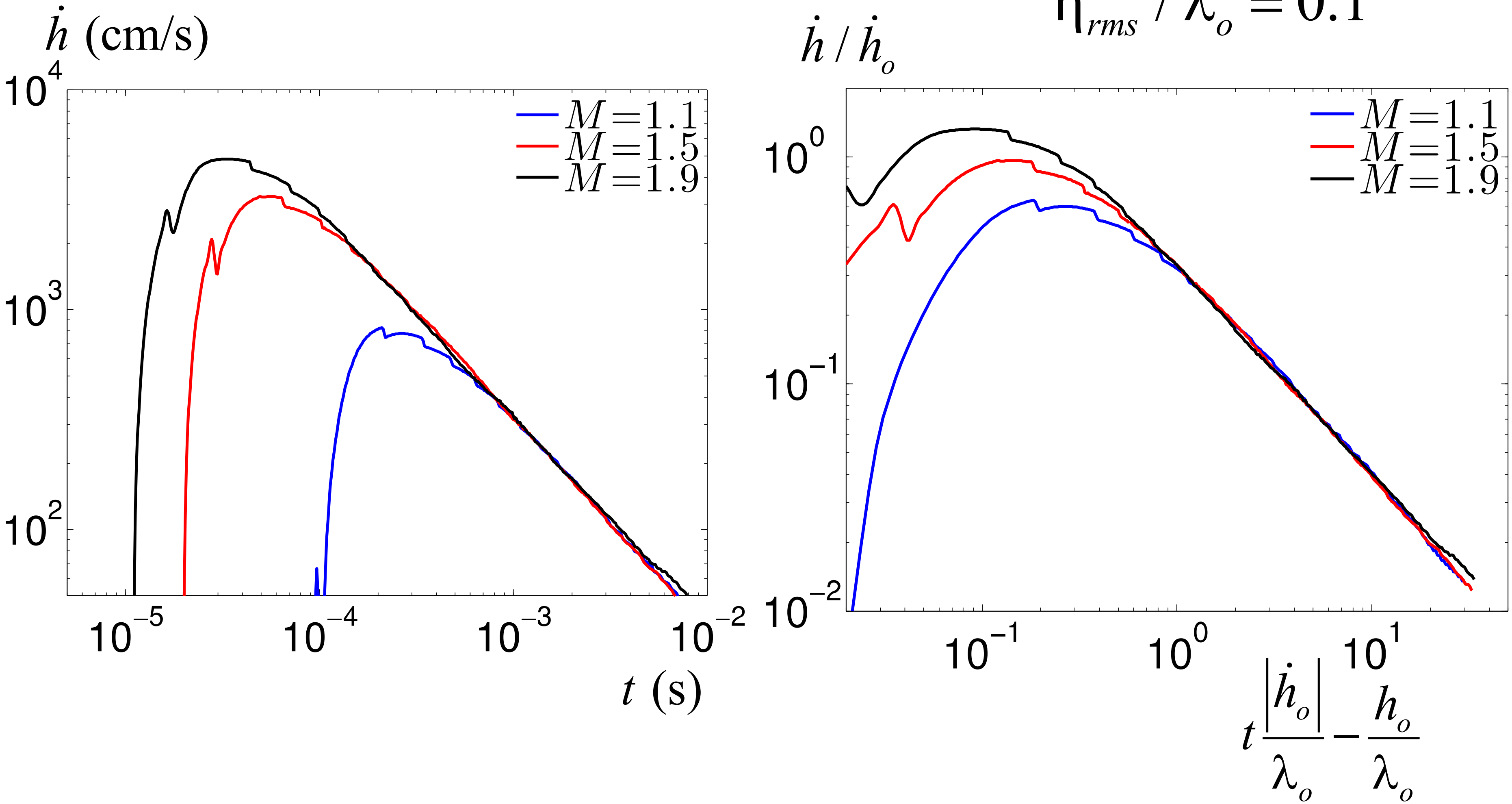




The scaled growth rates collapse for different Mach numbers

$$A = -0.53$$

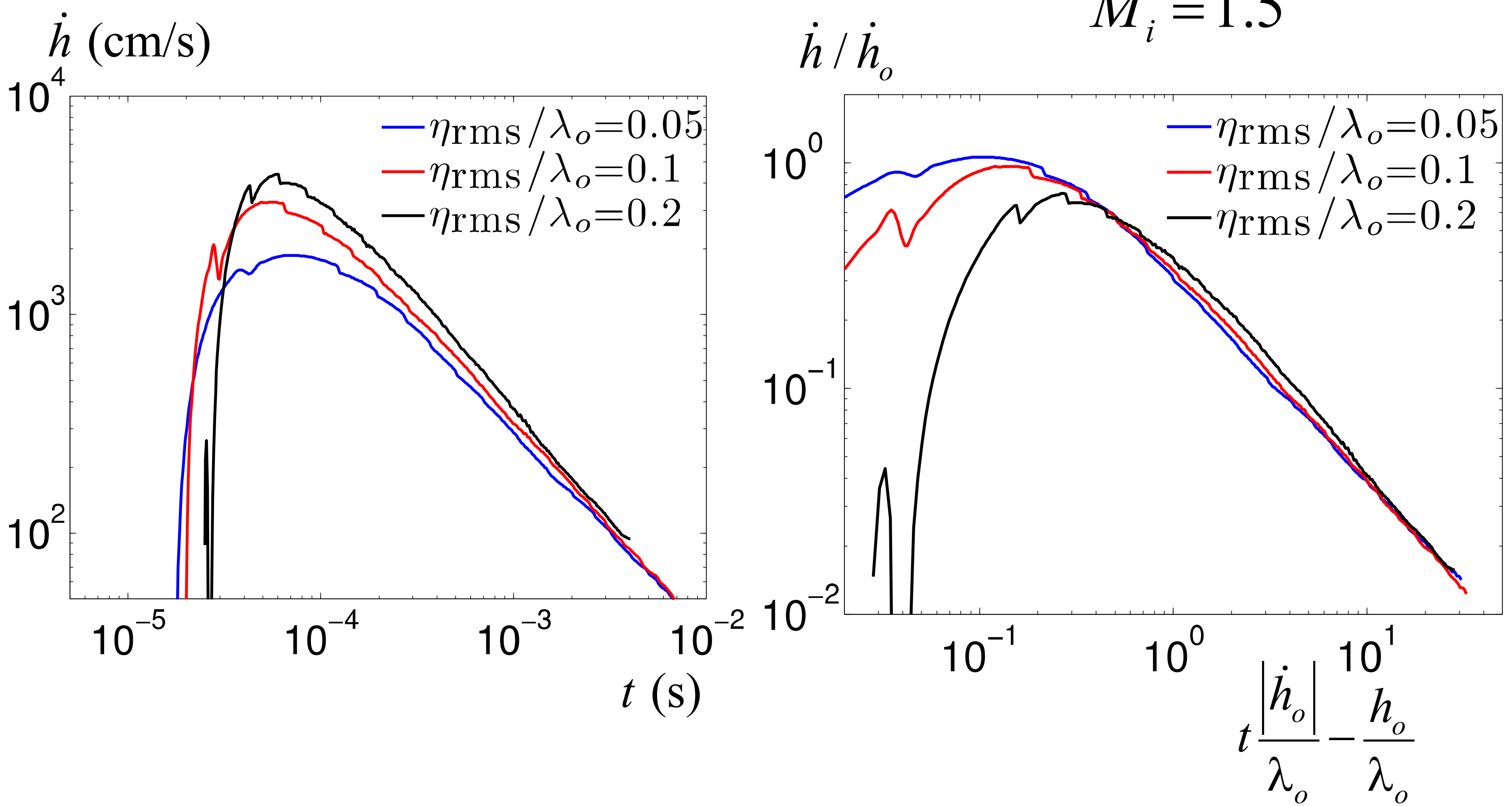
$$\eta_{rms} / \lambda_o = 0.1$$



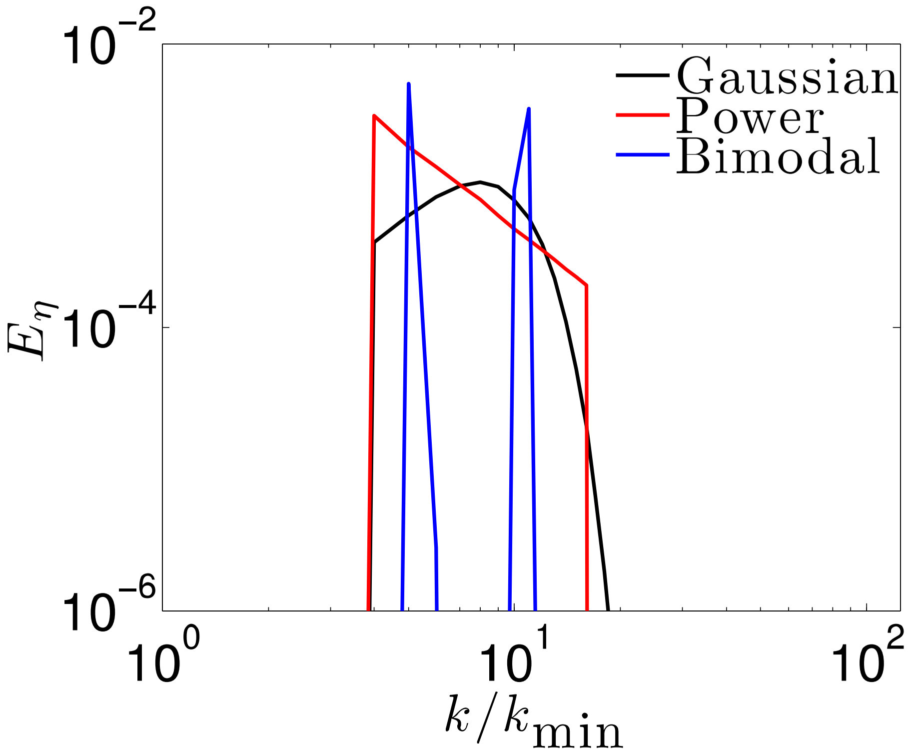
The scaled growth rates collapse for different amplitude/wavelength ratios

$$A = -0.53$$

$$M_i = 1.5$$

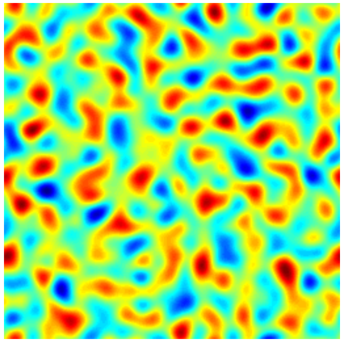


Do the growth rate curves collapse for different spectral shapes?



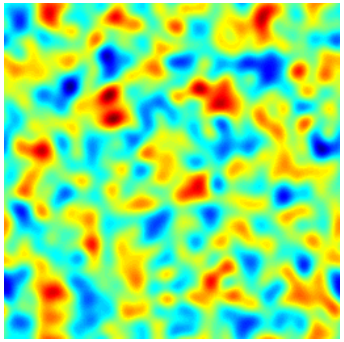
Gaussian

$$k_{\mathrm{peak}} / k_{\mathrm{min}} = 32$$



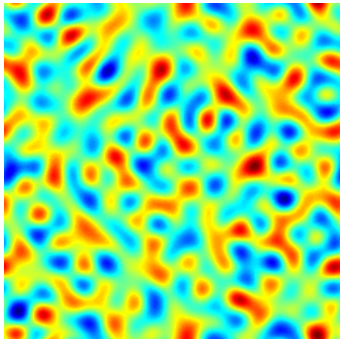
Power Law

$$E_\eta \propto k^{-2}$$



Bimodal

$$k_{\mathrm{peak}} / k_{\mathrm{min}} = 24 \;\&\; 48$$



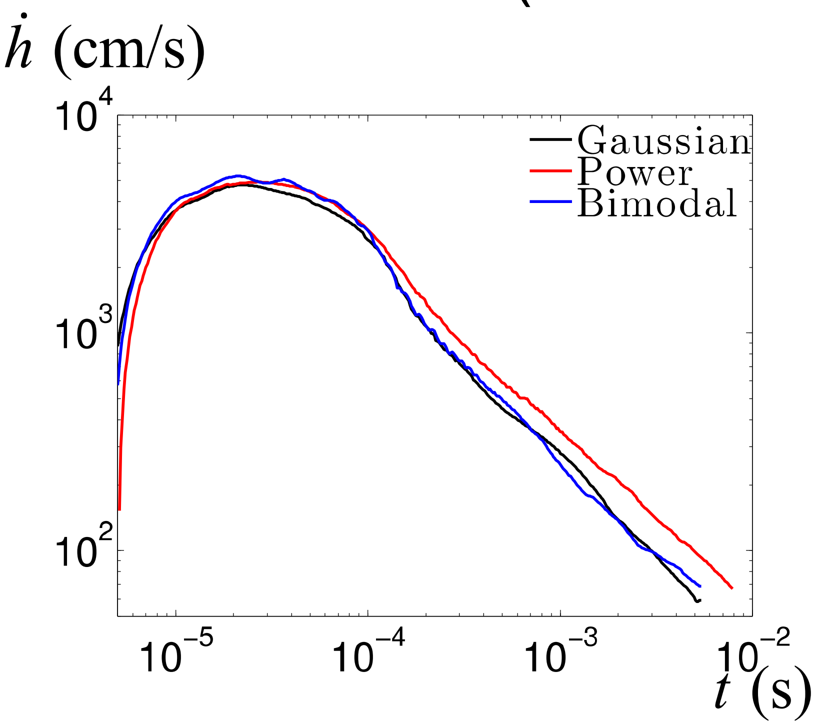
The scaled growth rates collapse for different perturbation spectra

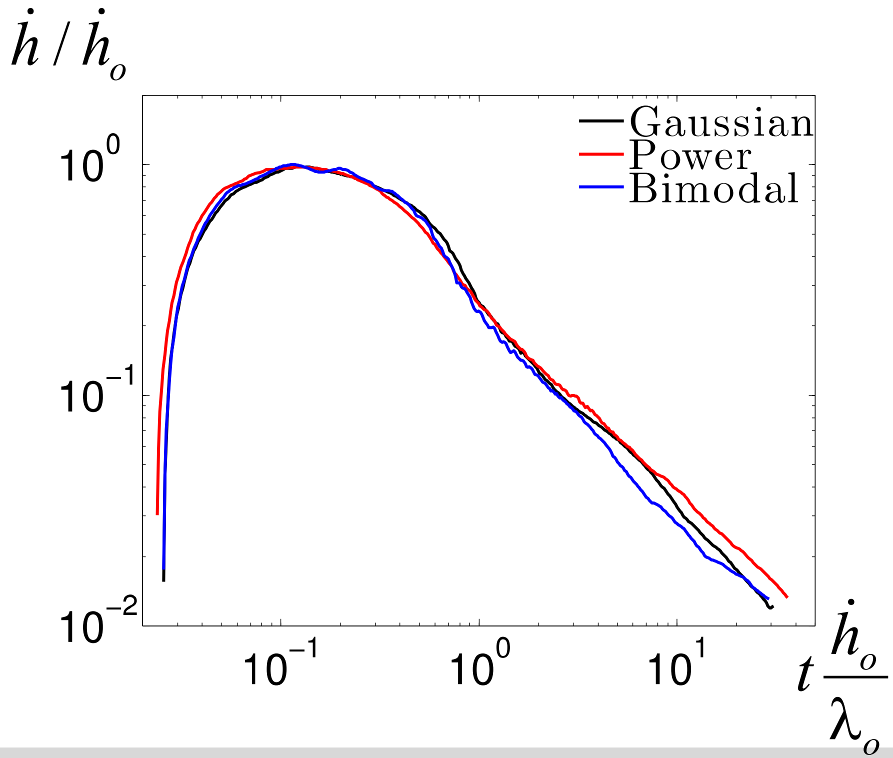
- Possible different behavior at late times
- Initial perturbation spectra widths are all rather narrow (less than a decade)

$$A = 0.53$$

$$M_i = 1.5$$

$$\eta_{rms} / \lambda_o = 0.1$$





Collapse of the growth rate curves suggests the thickness/growth rate history can be represented by a single equation

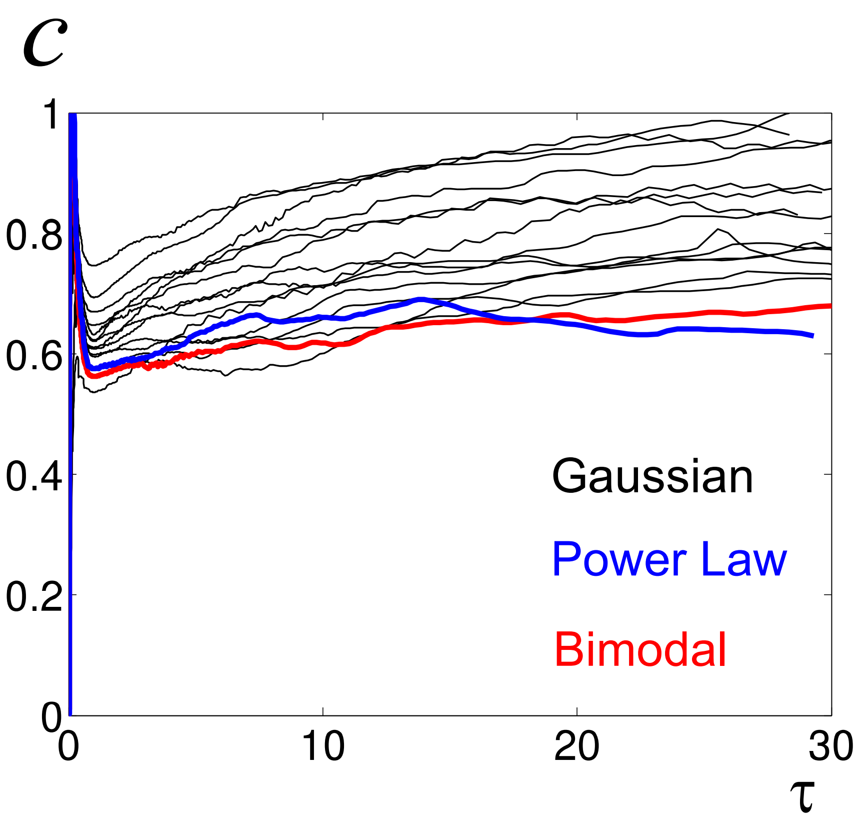
- Growth rate curves at later times fit the form

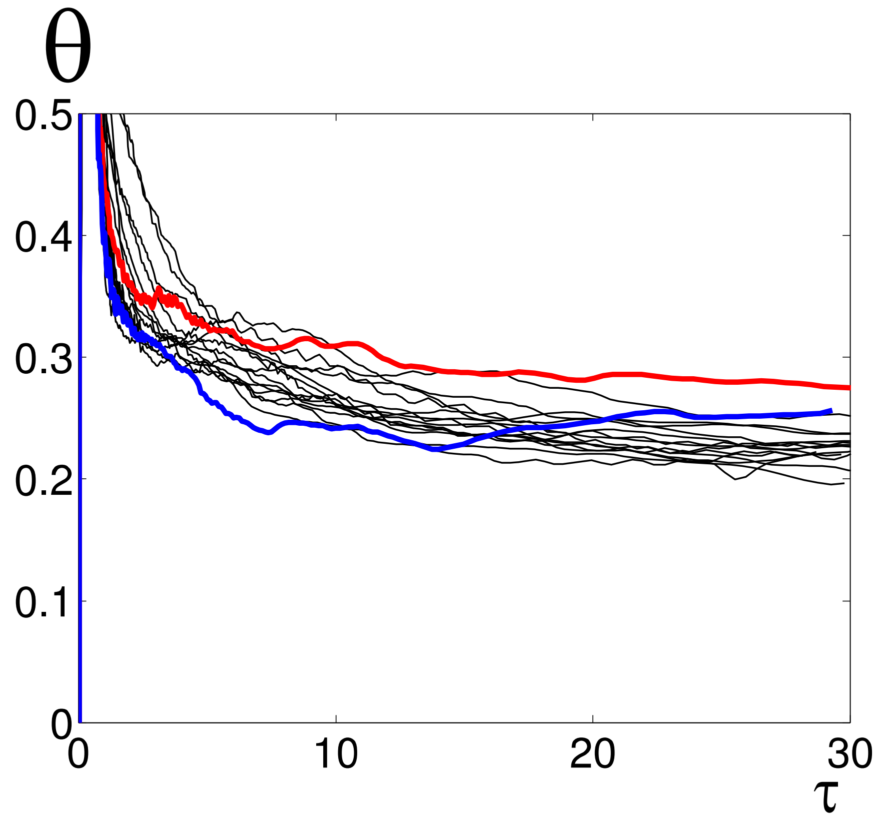
$$\frac{\dot{h}}{\dot{h}_o} = c\theta\tau^{\theta-1} \xrightarrow{\text{integrating}} \frac{h-h_o}{\lambda_o} = c\tau^{\theta} \qquad \tau = \begin{cases} t\dfrac{\dot{h}_o}{\lambda_o} & \text{for } A>0 \\ t\dfrac{\dot{h}_o}{\lambda_o} - \dfrac{h_o}{\lambda_o} & \text{for } A<0 \end{cases}$$

- Solve for the unknowns

$$\theta = \frac{\dot{h}}{\left|\dot{h}_o\right|}\frac{\lambda_o}{h-h_o}\tau \qquad c = \frac{h-h_o}{\lambda_o}\tau^{\left(-\frac{\dot{h}}{\left|\dot{h}_o\right|}\frac{\lambda_o}{h-h_o}\tau\right)}$$

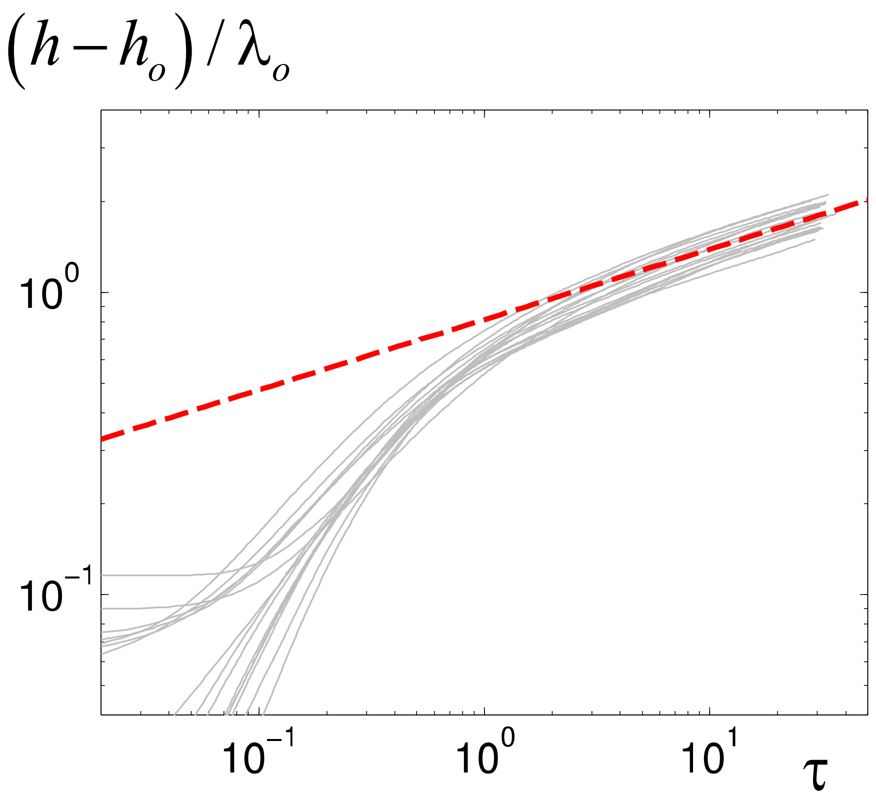
Growth rate coefficients

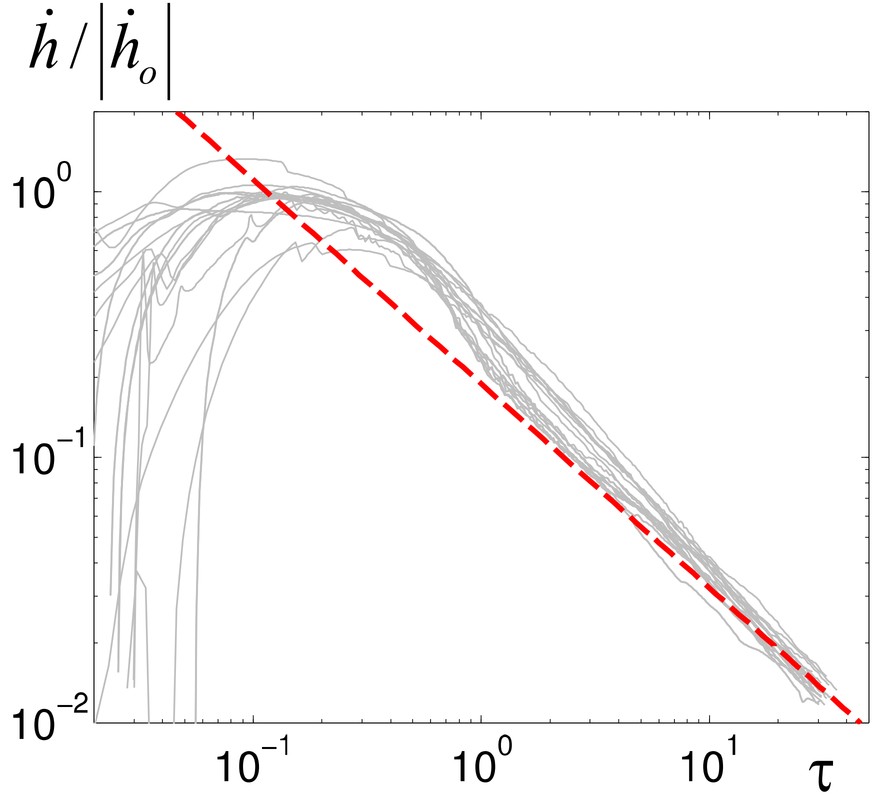




Average beyond $\tau > 20$ $\quad c = 0.813 \quad \theta = 0.233$

Curve fit matches the later time data well





$$\frac{(h - h_o)}{\lambda_o} = 0.812\ \tau^{0.236}$$

Conclusions

- The growth rate of the mixing region is determined solely by the net mass flux through the equimolar plane.
- The post-shock density and velocity fields (and hence the initial mass flux) can be accurately modeled if the interfacial perturbations are known.
- The initial growth rate (computed *a priori*) can be used to collapse the mixing curves for various Atwood numbers, Mach numbers etc.
- The collapse of the growth curves (and hence the universality of the scaling) may depend on whether the initial spectrum is narrow or broadband.
- A universal value of θ may only exist for perturbation spectra of the same form.

Extra Slides

Simulation Setup

Gaussian Perturbation Spectrum

$$E_\eta(k) \propto \exp\left(\frac{-\left(k-k_p\right)^2}{k_b^2}\right)$$

Perturbation energy

$$\eta_{\mathrm{RMS}}^2 = \left\langle \eta^2 \right\rangle = \int_0^\infty E_\eta(k)\, dk$$

Dominant wavelength

$$\lambda_0 \equiv 2\pi \frac{\int_0^\infty E(k)/k \; dk}{\int_0^\infty E(k) \; dk} \; .$$

Interface profile

$$\xi(x,y,z) = \frac{1}{2}\left(1 + \mathrm{Erf}\left(\frac{d(x,y,z)}{\sigma}\right)\right)$$

Distance function

$$d(x,y,z) = \mathrm{sign}(x - \eta(\bar{y},\bar{z}))\, \min_{\bar{y},\bar{z}}\left(\sqrt{(x-\eta(\bar{y},\bar{z}))^2 + (y-\bar{y})^2 + (z-\bar{z})^2}\right)$$

Simulation Resolution

- By 256^2 x 512, the peak growth rate is within 2% of the modeled $\dot{h}_o$
- A $k^{-5/3}$ inertial range develops at the two highest resolutions

$A = 0.53$

$M_i = 1.5$

$\eta_{rms} / \lambda_o = 0.1$