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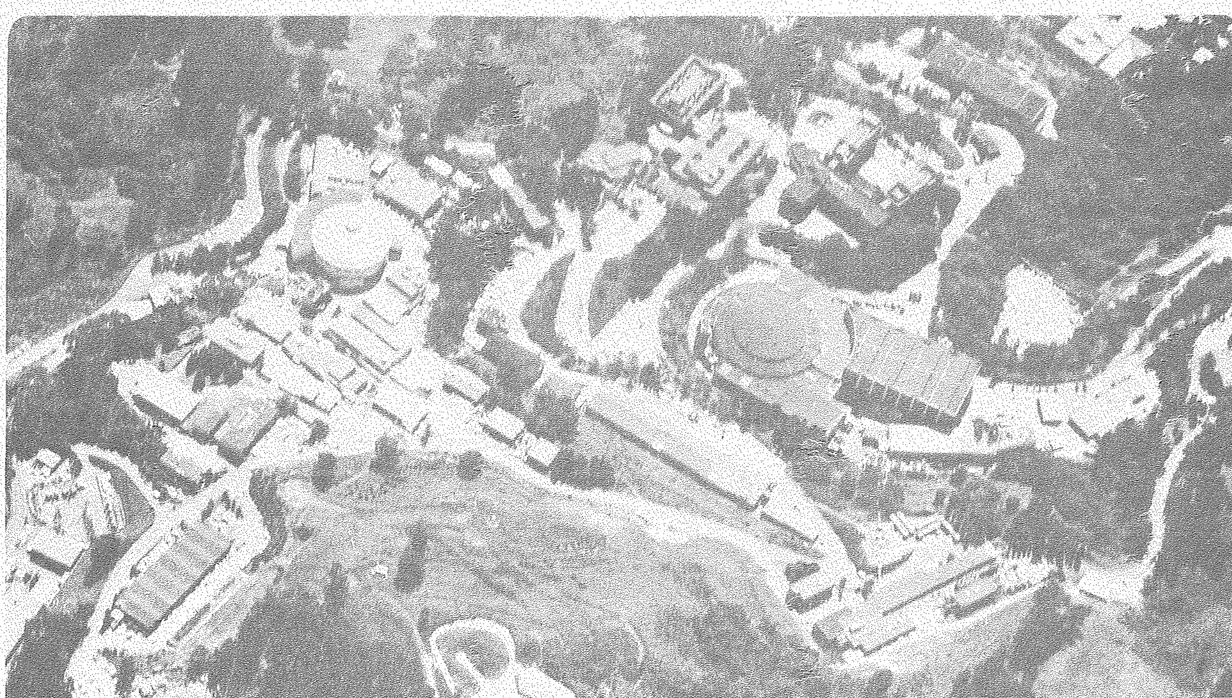

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A GAUGE-INVARIANT MULTIPOLE EXPANSION SCHEME FOR HEAVY-QUARK SYSTEMS IN QUANTUM CHROMODYNAMICS

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ABSTRACT

Separation of short-distance and long-distance dynamics for heavy quark-antiquark systems interacting with color gluons is investigated through a classification of gluons according to their ranges. A gauge-invariant double-multipole expansion scheme is constructed which takes into account color fluctuation of heavy-quark systems. Hadronic transitions between heavy quark-antiquark bound states as well as the static quark-antiquark potential are studied within this framework.

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I. INTRODUCTION

Heavy-quark systems, such as the ψ and Υ families and those of heavier quarks of possible existence, provide useful laboratories[1] for testing the basic structure of quantum chromodynamics (QCD). As the quark mass increases, the size of a heavy-quark system gets smaller. For sufficiently heavy quarks, the system size will eventually become much smaller than some characteristic time scale at which the system interacts with external perturbations. The difference between the system size and the interaction time will then serve as a useful expansion parameter for the description of such an interacting system. Along this line, a multipole expansion[2-8] of the gluon field around a heavy-quark system, originally introduced by Gottfried[2], has extensively been studied and applied to heavy-quark physics[9] such as hadronic transitions[10] in a heavy-quark family.

The purpose of this paper is to develop a systematic classification of gluon interactions according to their ranges and to construct a gauge-invariant multipole expansion scheme for heavy-quark systems. Applications to hadronic transitions between heavy quark-antiquark $(Q\bar{Q})$ bound states and to the heavy-quark potential are studied.

For the construction of the multipole expansion we start by assuming that heavy $Q\bar{Q}$ mesons are color-Coulombic bound states (for which the QCD coupling constant $\alpha_S = g^2/(4\pi) < 1$). Special care, however, will be taken to show that the resulting multipole expansion scheme possesses a wider range of applicability than restricted by this assumption.

The one-Coulomb-gluon-exchange potential[11] is attractive ($-\frac{4}{3}\alpha_S/r$) between a color-singlet $Q\bar{Q}$ pair with separation r while it is repulsive ($+\frac{1}{6}\alpha_S/r$) between a color-octet $Q\bar{Q}$ pair. Owing to this energy difference $\Delta\varepsilon = \frac{3}{2}\alpha_S/r$, a color-octet $Q\bar{Q}$ (scattering) state is unstable and has a lifetime $\tau \sim 1/\Delta\varepsilon \sim r/\alpha_S > r$. With the emission or absorption of a color gluon, a $Q\bar{Q}$ state undergoes a color singlet $\leftrightarrow$ color octet ($\underline{1} \leftrightarrow \underline{8}$) or $\underline{8} \leftrightarrow \underline{8}$ transition. This reminds us of a close analogy between this color fluctuation of the two-body $Q\bar{Q}$ system over a period of order $1/\Delta\varepsilon$ and electric-charge fluctuation of a charged particle over a region of its Compton wavelength. The Foldy-Wouthuysen (FW) transformation,[12] accordingly, is a useful guide for the construction of the QCD multipole expansion. The relevance of color fluctuation (or the binding effect) of the $Q\bar{Q}$ system to the perturbative study of the heavy-quark potential has been pointed out by Appelquist, Dine and Muzinich[13].

A heavy $Q\bar{Q}$ bound state has some characteristic scales; the quark mass M, the Bohr radius r_B, and the binding energy $\sim\Delta\varepsilon$. [In order of magnitude, $M: 1/r_B: \Delta\varepsilon = 1/\alpha_S: 1 : \alpha_S$ for a Coulombic bound state.] Correspondingly, there is a natural divison of the gluons distributed around the $Q\bar{Q}$ bound state, relative to these scales, as illustrated in Fig. 1. (I) The gluons distributed over a region of the Bohr radius (i.e. those with momentum $|\vec{k}| \sim 1/r_B$) predominantly build up $Q\bar{Q}$ binding. Very hard gluons with momentum $|\vec{k}| \gtrsim M$, however, mainly contribute to the renormalization of the one-body structure of the quark rather than to the two-body $Q\bar{Q}$ structure. (II) The gluons distributed over a region of dimensions of order $1/\Delta\varepsilon$ (i.e. with $1/r_B \gtrsim |\vec{k}| \gtrsim \Delta\varepsilon$) are responsible for the

abovementioned color fluctuation of the $Q\bar{Q}$ system. These gluons (which we shall call hard gluons) may be expanded in multipoles with the expansion parameter $\rho = (Q\bar{Q}\text{ separation})/(1/\Delta\varepsilon) = r\Delta\varepsilon$. (III) The gluons softer than the scale $\Delta\varepsilon$ have longer ranges and tend to connect this fluctuating system with external perturbations. These soft gluons ($|\vec{k}| \lesssim \Delta\varepsilon$, or symbolically[14] $gA_\mu \lesssim \Delta\varepsilon$) will be classified into multipoles with the expansion parameter ξ = (gluon momentum or energy) $/\Delta\varepsilon \sim gA_\mu/\Delta\varepsilon$. In this way one is led to the idea of a double-multipole expansion of the gluon field surrounding a $Q\bar{Q}$ system.

In Sec. II, we perform the separation of soft and hard gluons in the QCD Lagrangian. In Secs. III and IV, we construct the multipole expansion scheme for heavy-quark systems. We avoid the problem of gauge invariance[15] by casting the multipole expansion scheme into a gauge-invariant form in the early stage of the construction by use of an appropriate unitary transformation.[7,8] Selection rules for hadronic transitions between heavy $Q\bar{Q}$ bound states are discussed in Sec. IV. Unlike photons which are neutral, gluons carry color themselves. It is, therefore, important to study how soft gluons are coupled to the hard-gluon cloud around the $Q\bar{Q}$ system. The effect of the hard-gluon cloud on hadronic transitions of low multipole-orders turns out nonleading as compared with that of the basic $Q\bar{Q}$ structure. This indicates that the present framework will be applicable to known heavy-quark families by use of some phenomenological $Q\bar{Q}$ potentials in place of the Coulomb potential. In Sec. V, we derive an effective Hamiltonian projected to the color-singlet $Q\bar{Q}$ sector, and briefly study the effect of very soft gluons on heavy $Q\bar{Q}$ mesons.

In Sec. VI, the effect of soft gluons on the static heavy-quark potential is studied perturbatively within the present framework. Sec. VII is devoted to concluding remarks.

II. SEPARATION OF SOFT AND HARD GLUONS

In this section we separate soft-and hard-momentum components of the color-octet gluon field $\mathcal{A}_\mu^a(x)$ interacting with a color-triplet heavy-quark field $\psi^a(x)$.

We adopt the Coulomb gauge $\partial^k \mathcal{A}_k^a(x) = 0$ to quantize this quark-gluon system. The Lagrangian is given by

$$\mathcal{L} = \bar{\psi}\cdot(i\partial\!\!\!/ - M + g\mathcal{A}\!\!\!/^a T_a)\psi - \frac{1}{4}F_{\mu\nu}[\mathcal{A}]^2 + (\mathrm{FP}), \tag{2.1}$$

with

$$F_{\mu\nu}^a[\mathcal{A}] = \partial_\mu \mathcal{A}_\nu^a - \partial_\nu \mathcal{A}_\mu^a + g\, f^{abc}\mathcal{A}_\mu^b \mathcal{A}_\nu^c \ , \tag{2.2}$$

where $T_a = \frac{1}{2}\lambda_a (a = 1,\ldots,8)$ are the color matrices for the quark. It is straightforward to include light quarks, which are omitted here for simplicity. The Faddeev-Popov ghost term (FP), whose explicit form is well-known,[16] will be suppressed in what follows.

In the Coulomb gauge, the free-field propagator $< T^* \mathcal{A}_\mu^a(x)\mathcal{A}_\nu^b(y) >^{\mathrm{free}} = i\delta^{ab}\Delta_{\mu\nu}^C(x-y)$ derived from the free-field Lagrangian (Eq. (2.1) with $g = 0$) is given by

$$\Delta_{00}^C(x-y) = < x|1/\nabla^2|y > ,$$

$$\Delta_{k\ell}^C(x-y) = < x|(-\delta^{k\ell} - \partial^k\partial^\ell/\nabla^2)/(\partial^2 - i0)|y >, \tag{2.3}$$

where $\partial^2 = \partial^\mu\partial_\mu$ and $\nabla^2 = \partial^k\partial_k$. As is well-known,[16] the generating functional of the free-gluon propagator

$W_0[\eta] = < 0|T^*\exp\left(i\int d^4x\eta^\mu(x)\cdot\mathcal{A}_\mu(x)\right)|0 >^{\mathrm{free}}$ is written as

$$W_0[\eta] = \exp\left[-\frac{1}{2}\, i \int d^4x d^4y\, \eta^\mu(x)\cdot\Delta_{\mu\nu}^C(x-y)\, \eta^\nu(y)\right]. \tag{2.4}$$

Let us divide the propagator $\Delta_{\mu\nu}^C$ into soft and hard components relative to a momentum scale Λ: $\Delta_{\mu\nu}^C(x-y) = \Delta_{\mu\nu}^S(x-y) + \Delta_{\mu\nu}^H(x-y)$. This division is rather arbitrary. The choice of the hard component $\Delta_{\mu\nu}^H(x-y)$ which we use in what follows is

$$\Delta_{00}^H(x-y) = < x|1/(\nabla^2 + \Lambda^2)|y > ,$$

$$\Delta_{k\ell}^H(x-y) = < x|(-\delta^{k\ell} - \partial^k\partial^\ell/\nabla^2)/(\partial^2 + \Lambda^2 - i0)|y >. \tag{2.5}$$

This is the propagator for a vector particle of mass Λ in the Coulomb gauge so that its range is of the order of the Compton wavelength $1/\Lambda$.

As verified easily (note Eq. (B.1) in Appendix B), this "hard" propagator and the associated "soft" propagator $\Delta_{\mu\nu}^S = \Delta_{\mu\nu}^C - \Delta_{\mu\nu}^H$ are constructed from the Lagrangian

$$\begin{aligned}\mathcal{L}_0[A,B] = &-\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}\Lambda^2 B_\mu^2 \\ &-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}\frac{1}{\Lambda^2}\left[(\nabla^2 A_0)^2 + (\partial^2 A_k)(\partial^2 A^k)\right]\end{aligned} \tag{2.6}$$

by the standard path-integral method[16] where the fields $A_\mu^a(x)$ and $B_\mu^a(x)$ are treated as independent fields subject to the Coulomb-gauge condition

$$\partial^k A_k^a = \partial^k B_k^a = 0 \ . \qquad (2.7)$$

The generating functional $W_0[\eta]$ is represented by the path-integral

$$W_0[\eta] = \int [dA][dB]\,\delta(\partial^k A_k)\delta(\partial^k B_k)\ \exp\{i\int d^4x[\mathcal{L}_0[A,B] + \eta^\mu\cdot(A_\mu + B_\mu)]\}. \qquad (2.8)$$

Let us denote the interaction part of (2.1) by $\mathcal{L}_{int}[\mathcal{A};\psi,\bar{\psi};\cdots]$. The standard perturbation theory is generated by operating $\exp(i\int d^4x \mathcal{L}_{int}[\delta/i\delta\eta;\ldots])$ on $W_0[\eta]$, where only the gluon part is shown explicitly.

This implies that, in terms of A_μ and B_μ, the present quark-gluon system is described by the effective Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L}_0[A,B;\ldots] + \mathcal{L}_{int}[A + B;\ \psi, \bar{\psi};\ldots] \ , \qquad (2.9)$$

where $\mathcal{L}_0[A, B;\ldots]$ (originally defined in Eq. (2.6)) now involves the free-field Lagrangians for the quark and the FP ghost as well. [It is necessary to decompose the FP-ghost field into soft-and hard-momentum components in the same manner as done for $\mathcal{A}_\mu^a$.] Note the relation $F_{\mu\nu}[A+B] = F_{\mu\nu}[A] + \nabla_\mu[A]B_\nu - \nabla_\nu[A]B_\mu - ig\bar{B}_\mu B_\nu$, where $\nabla_\mu[A]$ is the covariant derivative $\nabla_\mu^{ab}[A] = \delta^{ab}\partial_\mu + g f^{abc}A_\mu^c$,

and $\bar{B}_\mu$ stands for a matrix field $\bar{B}_\mu^{ab} = if^{acb}B_\mu^c$. Then the Lagrangian (2.9) is rewritten in the following compact form

$$\mathcal{L}_{eff} = \bar{\psi}\cdot(i\not{\partial} - M + g\,(\not{A} + \not{B})^a T_a)\psi$$

$$- \frac{1}{4}F_{\mu\nu}[A]^2 - \frac{1}{4}(\nabla_\mu[A]B_\nu - \nabla_\nu[A]B_\mu - ig\,\bar{B}_\mu B_\nu)^2$$

$$+ B^\nu\cdot\{\nabla^\mu[A]F_{\mu\nu}[A] - \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu)\} + \delta\mathcal{L}\ , \qquad (2.10)$$

$$\delta\mathcal{L} = \frac{1}{2}\Lambda^2 B^2 + \frac{1}{2}\frac{1}{\Lambda^2}\{(\nabla^2 A_0)^2 + (\partial^2 A_k)(\partial^2 A^k)\}\ . \qquad (2.11)$$

The Feynman rules are derived from the path-integral representation of the generating functional of the form (2.8) with $\mathcal{L}_0[A,B]$ replaced by the full Lagrangian $\mathcal{L}_{eff}$.

The Lagrangians (2.1) and (2.10) are equivalent in the sense that they lead to the same set of Feynman rules. The original field $\mathcal{A}_\mu^a$ in the former is decomposed into the soft- and hard-momentum components A_μ^a and B_μ^a in the latter. The fact that A_μ and B_μ are treated independently in $\mathcal{L}_{eff}$ plays a key role in the present approach. Note that the gauge-invariant structure of $\mathcal{L}_{eff}$ becomes manifest only for very soft A_μ or very hard B_μ.

It is straightforward to generalize the content of this section to arbitrary gauges.

III. A MULTIPOLE EXPANSION SCHEME FOR HEAVY-QUARK SYSTEMS

In this and next sections we construct a multipole expansion scheme for a heavy quark-antiquark $(Q\bar{Q})$ system.

The physical picture explained in Sec. I and the formalism developed in Sec. II are combined to lead to the following program: At first, we choose the scale Λ to be of the order of $\Delta\varepsilon$ and sum the contribution of the hard-gluon field B_μ in the Lagrangian (2.10). This hard-gluon summation procedure generates $Q\bar{Q}$ binding as well as the multipole expansion of the hard-gluon field (developed in powers of $\rho = r\Delta\varepsilon$). We perform the hard-gluon summation according to the number of hard-gluon loops. The resulting Lagrangian describes how soft gluons are coupled to the $Q\bar{Q}$ system surrounded by the hard-gluon cloud. The multipole expansion of the soft-gluon field is obtained by rearranging the soft-gluon interactions in powers of $\xi = gA_\mu/\Delta\varepsilon$.

The assignment of multipole orders is done in the following way. For a (low-lying) Coulombic bound state with quarks of mass M, the binding energy is, in order of magnitude, given by $M\alpha_S^2 \sim \vec{p}^{\,2}/M \sim \Delta\varepsilon = \frac{3}{2}\alpha_S/r$, where $r = |\vec{r}|$ is the $Q\bar{Q}$ separation and $p^k = -i\partial/\partial r^k$ is the relative $Q\bar{Q}$ momentum. Accordingly, $\rho = r\Delta\varepsilon \sim O(\alpha_S)$, $M \sim O(\rho^{-2}\Delta\varepsilon)$ and $\vec{p} \sim O(\rho^{-1}\Delta\varepsilon)$. [From now on, the multipole order $O(\rho^n\xi^m\Delta\varepsilon)$ will simply be denoted by $O(n,m)$.] By definition, $\Lambda \sim \Delta\varepsilon \sim O(0,0)$ and $g\,A_\mu \sim O(0,1)$. On the other hand, $g\,B_\mu \sim O(0,0)$ since a hard gluon exchanged between Q and $\bar{Q}$ gives rise to the α_S/r potential. We assume that the $Q\bar{Q}$ system is originally at rest so that its center-of-mass (c.m.) momentum P^k is a result of recoil against external perturbations. Hence we assign $\vec{P} \sim O(g\vec{A}) \sim O(0,1)$. The derivatives ∂_μ acting on soft gluons A_ν are of order (0,1): This implies that the soft-gluon sector should be treated as a fully interacting system in the present formalism since

the kinetic part $\partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ and the selfinteraction part $g\,f^{abc}A_\mu^b A_\nu^c$ in $F_{\mu\nu}^a[A]$ are of the same multipole-order. For hard gluons B_μ, $\partial_k \sim \partial/\partial r^k \sim O(-1,0)$.

The double-multipole expansion scheme constructed in this way does not take a manifestly gauge-invariant form. It is, however, possible to cast it into a gauge-invariant form[17] by an appropriate gauge transformation. Actually, it is advantageous to introduce the gauge transformation in the early stage of the construction.

To find a suitable transformation, let us try to express the soft-gluon field $A_\mu(x)$ at position $\vec{x}$ in terms of the field at some fixed position $\vec{u}$ (which will later be chosen to be the c.m. position of the $Q\bar{Q}$ system). We consider the gauge transformation

$$\psi(x) \to \psi'(x) = V(\theta)\psi(x) \ ,$$

$$A_\mu^a(x) \to A'^{\,a}_\mu(x) = U^{ab}(\theta)A_\mu^b(x) + b_\mu^a(\theta) \ , \tag{3.1}$$

where $V(\theta) = \exp(i\theta^a T_a)$, $U(\theta) = \exp(i\bar{\theta})$, $\bar{\theta}^{ab} = if^{acb}\theta^c$ and $\overline{b_\mu(\theta)} = (i/g)\,U(\theta)\left(\partial_\mu U^\dagger(\theta)\right)$. In what follows, unless otherwise stated, all quantities are defined at common time t which will be suppressed. Taylor-expanding $A_\mu^b(\vec{x})$ around $\vec{u}$ in (3.1) gives

$$A'^{\,a}_\mu(x) = U^{ab}(\theta)e^{w\cdot\partial}A_\mu^b(\vec{u}) + b_\mu^a(\theta) \ , \tag{3.2}$$

where $\vec{w} = \vec{x} - \vec{u}$, $w\cdot\partial = w^k\partial_k^{(u)}$ and $\partial_k^{(u)} \equiv \partial/\partial u^k$.

Suppose we fix $\theta(x)$ in such a way that

$$U^{ab}(\theta)e^{w\cdot\partial} = \left[e^{w\cdot\nabla[A(\vec{u})]}\right]^{ab} , \tag{3.3}$$

where $w\cdot\nabla[A(\vec{u})] = w^k\left(\partial_k^{(u)} - ig\overline{A_\mu(\vec{u})}\right)$. It is not difficult to verify that $U(\theta)$ fixed in this way contains no derivatives $\partial/\partial u^k$ as operators and that $V(\theta)$ takes an analogous form[18]

$$V(\theta) = \exp\left[w^k(\partial_k^{(u)} - igA_k^a(\vec{u})T_a)\right]\exp(-w\cdot\partial). \tag{3.4}$$

Then, as shown in detail in Appendix A, the transformed soft-gluon field $A'_\mu(\vec{x})$ is expressed in terms of field tensors $F_{\mu\nu}[A]$ defined at $\vec{u}$ so that

$$A'_0(\vec{x}) = A_0(\vec{u}) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (w\cdot\nabla[A(\vec{u})])^n w^\ell F_{\ell 0}[A(\vec{u})] ,$$

$$A'_k(\vec{x}) = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} \left(w\cdot\nabla[A(\vec{u})]\right)^n w^\ell F_{\ell k}[A(\vec{u})] . \tag{3.5}$$

Notice that $A'_k(\vec{u}) = 0$. For the hard gluon $B_\mu(\vec{x})$ we define the new field by

$$B'^a_\mu(\vec{x}) = U^{ab}(\theta)B^b_\mu(\vec{x}) . \tag{3.6}$$

As is obvious, $gB'_\mu \sim O(gB_\mu) \sim O(0,0)$. With these new fields, the Lagrangian (2.10) is rewritten as

$$\mathcal{L}_{eff} = \overline{\psi}'\cdot[i\not\partial - M + g(\not A' + \not B')\cdot T]\psi' - \frac{1}{4}F_{\mu\nu}[A]^2$$
$$- \frac{1}{4}(\nabla_\mu[A']B'_\nu - \nabla_\nu[A']B'_\mu - ig\overline{B'_\mu B'_\nu})^2 + \frac{1}{2}\Lambda^2 B'^2_\mu$$
$$+ \ldots , \tag{3.7}$$

where only terms relevant in what follows are shown. We may regard ψ', B'_μ and A_μ as fundamental fields in $\mathcal{L}_{eff}$; this means a change of field variables $(\psi, B_\mu, A_\mu) \to (\psi', B'_\mu, A_\mu)$ in the path-integral formalism. The Coulomb-gauge condition $\partial^k B_k = 0$ is translated into

$$(\partial^k - ig\overline{b^k(\theta)})B'_k = 0 . \tag{3.8}$$

For $b^k(\theta) = -A^k(\vec{u}) + \ldots$, defined in (3.1), see Eq. (A.7) in Appendix A.

Having cast the soft-gluon interactions into a gauge-invariant form, let us now study hard-gluon exchanges, as illustrated in Fig. 2. In the zero-hard-gluon-loop approximation which follows from a Gaussian path-integration over the hard-gluon field B'_μ, hard-gluon exchanges between quark color charges are described by the effective Lagrangian

$$\mathcal{L}^{(0)}_{eff} = \overline{\psi}'(x)\cdot[i\not\partial - M + g\not A'\cdot T]\psi'(x)$$
$$- \frac{1}{2}\int d^3\vec{y}dy_0 J^{\mu a}(x)\mathcal{D}^{ab}_{\mu\nu}(x,y;A')J^{\nu b}(y) \tag{3.9}$$
$$- \frac{1}{4}F_{\mu\nu}[A]^2 + \frac{1}{2}\frac{1}{\Lambda^2}\{(\nabla^2 A_0)^2 + (\partial^2 A_k)(\partial^2 A^k)\} + \ldots ,$$

where $J^a_\mu(x) = \overline{\psi}'(x)\gamma_\mu T_a \psi'(x)$ is the color current of the quark. Here $\mathscr{D}^{ab}_{\mu\nu}(x, y; A')$ is the hard-gluon propagator $i^{-1}\langle T^* B'^a_\mu(x) B'^b_\nu(y)\rangle$ in the presence of A_μ. Its explicit form as a functional A_μ is given in Eq. (B.1) of Appendix B. As a consequence of (3.8), $\mathscr{D}^{ab}_{\mu\nu}$ depends upon $b^k(\theta[A])$. As verified readily, when $\mathscr{D}^{ab}_{\mu\nu}(x, y; A')$ is expanded in powers of A'_μ and $b^k(\theta)$, terms involving $b^k(\theta)$ contain long-range propagators such as $1/(\partial^k\partial_k)$. Since we are summing only hard gluons, such long-range terms should be excluded from $\mathscr{D}^{ab}_{\mu\nu}$. This implies that the hard gluon B'_k is effectively subject to the Coulomb-gauge condition $\partial^k B'_k = 0$.

Diagrammatically, $\mathscr{D}^{ab}_{\mu\nu}(x, y; A')$ consists of hard-Coulomb- and transverse-gluon exchanges accompanied by soft-gluon emission, as depicted in Figs. 2 and 3. Because of the retarded nature of the transverse-gluon propagator (Eq. (2.5)), $\mathscr{D}^{ab}_{\mu\nu}(x, y; A')$ is not in general instantaneous. It is, however, well approximated by its instantaneous part. Note that, for a transverse-gluon exchange between a $Q\bar{Q}$ pair, the energy transfer ($\sim \Delta\varepsilon$) is smaller that the momentum transfer ($\sim 1/r$) by a single power of $\rho \sim \alpha_S$. Correspondingly, with the expansion of the propagator

$$(\partial^2 + \Lambda^2)^{-1} = (\nabla^2 + \Lambda^2)^{-1}[1 + \partial_0^2(\nabla^2 + \Lambda^2)^{-1}]^{-1}, \qquad (3.10)$$

one can extract the instantaneous limit of $\mathscr{D}^{ab}_{\mu\nu}(x, y; A')$. We denote the instantaneous part by $\mathscr{D}^{ab}_{\mu\nu}(\vec{x}, \vec{y}; A')$ so that $\mathscr{D}^{ab}_{\mu\nu}(x, y; A') = \mathscr{D}^{ab}_{\mu\nu}(\vec{x}, \vec{y}; A')\delta(x_0 - y_0)$.

Details of the calculation of $\mathscr{D}^{ab}_{\mu\nu}(\vec{x}, \vec{y}; A')$ are given in Appendix B. Here we simply make a remark and list the result.

Remark: Soft gluons tend to be coupled to the hard-gluon cloud distributed close to Q or $\bar{Q}$. This is because the hard-gluon propagator, e.g. $\Delta^H_{00}(\vec{x} - \vec{z}) = \delta(x_0 - z_0)\exp(-\Lambda|\vec{x} - \vec{z}|)/(4\pi|\vec{x} - \vec{z}|)$, blows up as $\vec{z} \to \vec{x}$. This fact leads to the color-dipole (as well as multipole) nature of the hard-gluon cloud around the $Q\bar{Q}$ system.

Figure 3 (a) ~ (e) are the diagrams that contribute to $\mathscr{D}^{ab}_{00}(\vec{x}, \vec{y}; A')$ up to $O(3,4)$. We choose the fixed position $\vec{u}$ to be $\vec{u} = \frac{1}{2}(\vec{x} + \vec{y})$ and express A'_μ in terms of the soft-gluon field defined at $\vec{u}$ (using Eq. (3.5)). The result is

$$\begin{aligned} g^2\mathscr{D}^{ab}_{00}(A') = \frac{\alpha_S}{r}\Big[1 &- \frac{g^2 r}{12\Lambda^3}\Big(\frac{1}{4}\overline{F_{k\ell}}\,\overline{F_{k\ell}} + \overline{F_{k0}}\,\overline{F_{k0}} + O(1,6)\Big) \\ &+ \frac{igr}{12\Lambda}\Big(r^k\overline{\nabla^\ell F_{k\ell}}\Big) + O(2,5) - \frac{igr^2}{12}\Big(r^k\overline{\nabla^\ell F_{k\ell}}\Big) \\ &- \frac{g^2 r}{24\Lambda}\Big(r^k\overline{F_{km}}\,r^\ell\overline{F_{\ell m}} - \frac{r^2}{4}\overline{F_{k\ell}}\,\overline{F_{k\ell}}\Big) \\ &+ \frac{g^2 r}{20\Lambda}\Big(r^2\overline{F_{k0}}\,\overline{F_{k0}} - \frac{1}{2}r^k\overline{F_{k0}}\,r^\ell\overline{F_{\ell 0}}\Big) \\ &+ \frac{g^2 r}{96\Lambda^3}\partial_0\Big(\Big(\frac{1}{\Lambda^2} - \frac{r^2}{6}\Big)\overline{F_{k\ell}}\,\overline{F_{k\ell}} + \frac{1}{3}r^k\overline{F_{km}}\,r^\ell\overline{F_{\ell m}}\Big)\partial_0 \\ &+ O(3,6)\Big]^{ab}, \end{aligned} \qquad (3.11)$$

where $\vec{r} = \vec{x} - \vec{y}$, $\nabla_k = \nabla_k[A(\vec{u})]$, $(\overline{F_{k0}})^{ab} = i\, f^{acb}F^c_{k0}[A(\vec{u})]$, etc.

The first term, the Coulomb potential, is of $O(0,0)$. The contributions of order ρ, ρ^2 and ρ^3 start with $O(1,4)$, $O(2,3)$ and $O(3,3)$, respectively. The last term involving time derivatives[19] represents noninstantaneous interactions of $O(3,4)$ coming from diagram Fig. 3(e).

Transverse gluons are responsible for spin-dependent quark-gluon interactions . The coupling of transverse gluons to a heavy quark of mass M is suppressed by a power of $1/M$. Correspondingly, in order to study the spin-dependent multipole interactions up to order ρ^3, one needs to calculate $g^2 \mathscr{D}_{0k}(\vec{x}, \vec{y}; A')$ and $g^2 \mathscr{D}_{k\ell}(\vec{x}, \vec{y}; A')$ up to order ρ^2 and ρ^1, respectively.

Figure 3 (f) contributes to $g^2 \mathscr{D}_{0k}(\vec{x}, \vec{y}; A')$ up to $O(1, 2)$ and $O(2,2)$:

$$g^2 \mathscr{D}^{ab}_{0k}(A') = ig\alpha_S \Big[-\frac{1}{3\Lambda} \overline{F^{0k}} + O(1, 4) + \frac{r}{8} (3\delta^{k\ell} - \frac{r^k r^\ell}{r^2}) \overline{F^{0\ell}} + \frac{1}{24\Lambda} r^\ell \overline{F^{\ell k}} \partial_0 + O(2,3) \Big]^{ab}, \quad (3.12)$$

Similarly, the evaluation of diagrams (g) and (h) gives

$g^2 \mathscr{D}_{k\ell}(\vec{x}, \vec{y}; A')$ up to $O(1, 2)$:

$$g^2 \mathscr{D}^{ab}_{k\ell}(A') = \alpha_S \Big[-\frac{1}{2r}\left(\delta^{k\ell} + \frac{r^k r^\ell}{r^2}\right) + \frac{2}{3}\Lambda\delta^{k\ell} + \frac{1}{3\Lambda} \{ig\overline{F^{k\ell}} + \delta^{k\ell}\nabla_0\nabla_0\} + \dots \Big]^{ab}, \quad (3.13)$$

where the first two terms follow from the instantaneous part of the free transverse-gluon propagator expanded in powers of $r\Lambda$.

The last term involves time derivatives $\nabla_0 = \partial_0 - ig\overline{A_0(\vec{u})}$: Here we observe that the noninstantaneous terms in Eqs. (3.11) ~ (3.13) contribute to $O(3,2)$ or higher in the present multipole expansion scheme; in what follows, we shall ignore them so that the Lagrangian (3.9) is regarded as local in time.

The nonrelativistic reduction of the Lagrangian (3.9) is most efficiently done by applying the FW transformation to the Lagrangian (3.7) rather that to (3.9). Up to $O(\rho^3)$, the result is cast in the form

$$\begin{aligned} \mathscr{L}_{\text{eff}} = \overline{\psi''} \cdot \Big[& i\gamma^0(\partial_0 - igA'_0\cdot T) - \left(M + \frac{1}{2M} O^2 - \frac{1}{8M^3} O^4\right) \\ & + \frac{g}{8M^2}\gamma_0 \Big((\nabla^k[B' + A']F_{k0}[B' + A'])\cdot T \\ & \qquad - \sigma^{k\ell}\{p_k + g(B'_k + A'_k)\cdot T, F_{\ell 0}[B' + A']\cdot T\} \Big) \\ & + \dots \Big] \psi'' + \dots , \end{aligned} \quad (3.14)$$

where ψ'' is the transformed quark field, $O = -\gamma^0\gamma^k[p_k + g(B'_k + A'_k)\cdot T]$ and $p^k = -i\partial/\partial x^k$. The gluon sectors remain unchanged in the above. Functional integration over B' leads to the nonrelativistic version of Eq. (3.9).

IV. HEAVY QUARK - ANTIQUARK SYSTEMS

The effective Lagrangian (3.9) describes how soft gluons are coupled to heavy-quark systems surrounded by hard gluons. Let us, in what follows, ignore pair creation of heavy quarks and the renormalization of the one-body stucture of the quark, both of which are caused by very hard gluons $(|\vec{k}| > M)$. Then, it is convenient to project the Lagrangian (3.9) or (3.14) onto the two-body $Q\bar{Q}$ subspace. Namely, we replace the quark sector in the Lagrangian integrated over space (i.e. $L = \int d^3x\, \mathcal{L}_{eff}$) by

$$L_{Q\bar{Q}} = \Psi^{\dagger}(\vec{x}_Q,\vec{x}_{\bar{Q}})^{a'b'}[\delta^{a'a}\delta^{b'b} i\partial_0 - \mathcal{H}^{a'b'|ab}]\Psi(\vec{x}_Q,\vec{x}_{\bar{Q}})^{ab}, \tag{4.1}$$

Where $\Psi(\vec{x}_Q, \vec{x}_{\bar{Q}})^{ab}$ is the field operator for the $Q^a\bar{Q}^b$ system with color indices (a, b); spinor indices are suppressed. The two-body Hamiltonian $\mathcal{H}^{a'b'|ab}$ is constructed out of the original Lagrangian.

The projection procedure is straightforward when the quark and antiquark positions $(\vec{x}_Q, \vec{x}_{\bar{Q}})$ are chosen as independent coordinates [We use the charge-conjugated field to denote the antiquark field so that the kinetic term is written as $\bar{\Psi}_{\bar{Q}}\cdot(i\not{\partial} - M + g\not{A}\cdot T^*)\psi_{\bar{Q}}$, where the color matrix T_c^* is the complex conjugate of T_c.]

On the other hand, some care should be taken when the c.m. coordinate $\vec{u} = \frac{1}{2}(\vec{x}_Q + \vec{x}_{\bar{Q}})$ and the relative coordinate $\vec{r} = \vec{x}_Q - \vec{x}_{\bar{Q}}$ are chosen as independent variables. In this case, one has to make the projection onto the $Q\bar{Q}$ sector before the gauge transformation. This is because the c.m. momentum $P^k = -i\partial/\partial u^k$ and the relative momentum $\mathcal{p}^k = -i\partial/\partial r^k$ respond to the gauge transformations (3.3) for the quark and the antiquark sectors. As shown in Appendix A,

under the combined gauge transformations (3.3) for the quark and the antiquark sectors, the covariant derivatives $(\partial_k - igA_k\cdot T)_Q$ and $(\partial_k + igA_k\cdot T^*)_{\bar{Q}}$ undergo the change

$$i(\partial_k - igA_k\cdot T)_Q \to \frac{1}{2}(P_k + g\Omega_k) + (\mathcal{p}_k + g\phi_k)\ ,$$

$$i(\partial_k + igA_k\cdot T^*)_{\bar{Q}} \to \frac{1}{2}(P_k + g\Omega_k) - (\mathcal{p}_k + g\phi_k) \tag{4.2}$$

with

$$\Omega_k = A_k^{(u)}(\vec{x}_Q)\cdot T - A_k^{(u)}(\vec{x}_{\bar{Q}})\cdot T^*$$

$$\phi_k = \frac{1}{2}\left(A_k'(\vec{x}_Q)\cdot T - A_k'(\vec{x}_{\bar{Q}})\cdot T^*\right)\ , \tag{4.3}$$

where $A_k'(x)$ is defined by Eq. (3.5) whereas $A_k^{(u)}(x)$ is given by $A_0'(x)$ in Eq. (3.5) with the Lorentz index 0 replaced by k. As before, $T_c = (T_c)^{a'a}\delta^{b'b}$, $T_c^* = \delta^{a'a}(T_c^*)^{b'b}$ and $P_k = P_k\delta^{a'a}\delta^{b'b}$, etc; the fields Ω_k and ϕ_k are matrices in color space. Color indices will be suppressed in this fashion when we work in the two-body $Q\bar{Q}$ sector. The projection operators $\mathcal{P}_1^{a'b'|ab} = \delta^{a'b'}\delta^{ab}/N$ and $\mathcal{P}_8^{a'b'|ab} = \delta^{a'a}\delta^{b'b} - \mathcal{P}_1^{a'b'|ab}$ (N for color SU(N)) serve to extract color-singlet and color-octet $Q\bar{Q}$ states, respectively. We shall later use the following combination of color matrices: $(T_\pm)_c = T_c \pm T_c^*$. Note that T_- is nonvanishing only between color-octet $Q\bar{Q}$ states while T_+ induces a $\underline{1} \leftrightarrow \underline{8}$ or $\underline{8} \leftrightarrow \underline{8}$ transition of the $Q\bar{Q}$ system. [This follows from the relation $\mathcal{P}_1 T_- = T_-\mathcal{P}_1 = \mathcal{P}_1 T_+ \mathcal{P}_1 = 0$.]

Obviously, $(P_k + g\Omega_k)$ and $(\not{p}_k + g\phi_k)$ are the covariant derivatives associated with the c.m. and relative motions of the $Q\bar{Q}$ system, respectively. Under the present gauge tranformation, the hard-gluon field B_μ is transformed in the same way as before (Eq. (3.6)) and $\mathscr{D}^{ab}_{\mu\nu}(x, y; A)$ is converted to $\mathscr{D}^{ab}_{\mu\nu}(x, y; A')$.

With these remarks taken into account, it is now a simple task to derive the gauge-transformed form of the two-body Hamiltonian. Its relativistic form is obtained simply by the substitution (4.2). To derive the nonrelativistic form, we start with the nonrelativistic Lagrangian (3.14) <u>without</u> primes on the fields and make the gauge transformation using (4.2). The resulting Hamiltonian is written as

$$\mathscr{H}^{new} = \mathscr{H}_{Q\bar{Q}} + \mathscr{H}^{hard} , \tag{4.4}$$

where $\mathscr{H}^{hard}$ consists of hard-gluon exchanges. Up to order ρ^3, $\mathscr{H}_{Q\bar{Q}}$ is given by

$$\begin{aligned}\mathscr{H}_{Q\bar{Q}} = {} & 2M - g(A_0'(\vec{x}_Q)\cdot T - A_0'(\vec{x}_{\bar{Q}})\cdot T^*) + \frac{1}{M}(\vec{\not{p}} + g\vec{\phi})^2 \\ & - \frac{1}{4M^3}(\not{p}^2)^2 + \frac{1}{4M}(\vec{P} + g\vec{\Omega})^2 \\ & - \frac{g}{4M}\left(\sigma_Q^{k\ell} F_{k\ell}[A'(\vec{x}_Q)]\cdot T - \sigma_{\bar{Q}}^{k\ell} F_{k\ell}[A'(\vec{x}_{\bar{Q}})]\cdot T^*\right) \\ & - \frac{g}{4M^2}\not{p}_k F^c_{\ell 0}[A(\vec{u})]\left(\sigma_Q^{k\ell} T_c + \sigma_{\bar{Q}}^{k\ell} T_c^*\right) ,\end{aligned} \tag{4.5}$$

where the spin matrices $\sigma_Q^{k\ell}$ and $\sigma_{\bar{Q}}^{k\ell}$ $\left(\sigma_Q^{k\ell} = \varepsilon^{k\ell m}\sigma_Q^m , \text{etc.}\right)$ act on two-component spinors of the quark and antiquark.

To construct $\mathscr{H}^{hard}$, we first extract the hard-gluon interactions up to order ρ^3 out of the gauge-transformed form of the Lagrangian (3.14):

$$\begin{aligned}\mathscr{H}_B = {} & -gB_0'\cdot T + \frac{g}{2M}\left[\left\{\not{p}^k + \frac{1}{2}\left(P^k + gA^k(\vec{u})\cdot T_-\right), B'^k\cdot T\right\} - \sigma_Q^{k\ell}(\partial_k B_\ell')\cdot T\right] \\ & + \frac{g}{8M^2}\left[(\partial^k\partial_k B_0')\cdot T + \sigma_Q^{k\ell}\left\{\not{p}_\ell + \frac{1}{2}(P_\ell + gA_\ell(\vec{u})\cdot T_-), (\partial_k B_0')\cdot T\right\}\right] \\ & + \frac{g^2}{2M}\left[(B_k'\cdot T)^2 + \frac{1}{2}\, i\sigma_Q^{k\ell}(\overline{B_k' B_\ell'})\cdot T\right] \\ & + (\text{antiquark sector}) + \ldots ,\end{aligned} \tag{4.6}$$

where $B_k'\cdot T = B_k'(\vec{x}_Q)\cdot T$, etc. The antiquark sector is obtained from the quark sector through the replacement $B_k'(\vec{x}_Q) \to B_k'(\vec{x}_{\bar{Q}})$, $T_c \to -T_c^*$ and $\sigma_Q^{k\ell} \to \sigma_{\bar{Q}}^{k\ell}$. On integrating over the hard-gluon field, as done in (3.9), one gets the hard-gluon exchange Hamiltonian $\mathscr{H}^{hard}$.

The part of $\mathscr{H}^{hard}$, which involves $\mathscr{D}_{00}(\vec{x}, \vec{y}; A')$, is given by

$$\begin{aligned}\mathscr{H}_1^{hard} = {} & -g^2\Big[T_c T_e^* \mathscr{D}^{ce}_{00}(\vec{x}_Q,\vec{x}_{\bar{Q}}; A') \\ & - \frac{1}{2}(T_c T_e)\mathscr{D}^{ce}_{00}(\vec{x}_Q,\vec{x}_Q; A') - \frac{1}{2}(T_c^* T_e^*)\mathscr{D}^{ce}_{00}(\vec{x}_{\bar{Q}},\vec{x}_{\bar{Q}}; A')\Big] \\ & + (\alpha_S\pi/M^2)T_c T_c^*\delta^3(\vec{r}) \\ & + \frac{\alpha_S}{4M^2 r^3}\, r^k\left[\sigma_Q^{k\ell}\left\{T_c, \left\{\not{p}^\ell + \frac{1}{2}(P^\ell + gA^\ell\cdot T_-), T_c^*\right\}\right\}\right.\end{aligned}$$

(4.7) continued on next page

$$+ \sigma_{\bar{Q}}^{k\ell}\left\{T_c^*, \left\{-\mathscr{p}^\ell + \tfrac{1}{2}(P^\ell + gA^\ell\cdot T_-),\ T_c\right\}\right\}\Bigg]$$

$$+ \ldots \tag{4.7}$$

The δ-function term which follows from $\partial^k\partial_k(4\pi r)^{-1} = \delta^3(\vec{r})$ is of $O(2,0)$. The $\vec{A}$-dependent part [of $O(3,2)$] of the last term acts only between color-octet $Q\bar{Q}$ states since $\left\{T_c, \left\{A\cdot T_-, T_c^*\right\}\right\} = \left\{T_c^*, \left\{A\cdot T_-, T_c\right\}\right\} = -(2/N)\,A\cdot T_-$, as verified easily. Note the structure of the first three terms: The second and third terms represent hard-gluon exchanges by the same quark (or antiquark),[20] and involve ultraviolet-divergent selfenergy corrections which are removed by mass renormalization. (See Fig. 3 (b$'$) ~ (e$'$).) They, as a matter of fact, play an important role: In the $\underline{1}\leftrightarrow\underline{1}$ and $\underline{1}\leftrightarrow\underline{8}$ channels, i.e. in $\mathscr{P}_1\mathscr{H}_1^{hard}$ or $\mathscr{H}_1^{hard}\mathscr{P}_1$, the Coulomb- exchange terms are proportional to

$$\mathscr{D}_{reg}^{ce}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A') = \mathscr{D}_{00}^{ce}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A') - \tfrac{1}{2}\mathscr{D}_{00}^{ce}(\vec{x}_{\bar{Q}}, \vec{x}_{\bar{Q}}; A')$$

$$- \tfrac{1}{2}\mathscr{D}_{00}^{ce}(\vec{x}_Q, \vec{x}_Q; A')\ . \tag{4.8}$$

Owing to this stucture, terms independent of $\vec{r} = \vec{x}_Q - \vec{x}_{\bar{Q}}$ vanish in $\mathscr{D}_{reg}$. Consequently, $\mathscr{H}_1^{hard}$ starts with $O(2,0)$; the multipole interactions of order ρ which have undesirable direct $\underline{1}\leftrightarrow\underline{1}$ components are removed.[21] (Note that $\mathscr{P}_1(T_cT_e^*)\mathscr{P}_1 = (2N)^{-1}\delta^{ce}\mathscr{P}_1$.) These order-$\rho$ terms constitute the leading infrared structure of $\mathscr{D}_{00}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A')$; they become singular as $\Lambda \to 0$. The removal of these terms makes $\mathscr{H}_1^{hard}$ less sensitive to the structure of the hard-gluon cloud. Note that there is no such cancellation for the $\underline{8}\leftrightarrow\underline{8}$ component of $\mathscr{H}_1^{hard}$.

The part of $\mathscr{H}^{hard}$, that involves $\mathscr{D}_{0k}$, starts with $O(2,2)$ and $O(3,2)$:

$$\mathscr{H}_2^{hard} = \frac{ig\alpha_S}{M} T_cT_e^*\Big[\frac{2}{3\Lambda}\vec{\bar{E}}\cdot\vec{\mathscr{p}} + O(2,4) - \frac{r}{4}\Big\{3\vec{\bar{E}} - \frac{1}{r^2}(\vec{\bar{E}}\cdot\vec{r})\vec{r}\Big\}\cdot\vec{\mathscr{p}}$$

$$- \frac{1}{2r}\vec{S}\cdot(\vec{r}\times\vec{\bar{E}}) + \ldots\Big]^{ce}, \tag{4.9}$$

where $E^k = F^{k0}[A(\vec{u})]$, $\vec{E}\cdot\vec{\mathscr{p}} = E^k\mathscr{p}^k$, etc., and $\vec{S} = \frac{1}{2}(\vec{\sigma}_Q + \vec{\sigma}_{\bar{Q}})$ is the total spin of the $Q\bar{Q}$ system. Some remarks are in order as to the general structure of $\mathscr{H}_2^{hard}$. (i) The order-ρ^2 terms in $\mathscr{H}_2^{hard}$ are given by the $\vec{r} = \vec{x}_Q - \vec{x}_{\bar{Q}} \to 0$ limit of

$$(g^2/M)T_cT_e^*\Big[\mathscr{D}_{0k}^{ce}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A') - \mathscr{D}_{k0}^{ce}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A')\Big]\mathscr{p}^k. \tag{4.10}$$

It follows from this and the Bose symmetry $\mathscr{D}_{0k}^{ce}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A') = \mathscr{D}_{k0}^{ec}(\vec{x}_{\bar{Q}}, \vec{x}_Q; A')$ that $\mathscr{H}_2^{hard}$ has no $\underline{1}\leftrightarrow\underline{1}$ components of order ρ^2. (ii) The spin-dependent part of $\mathscr{H}_2^{hard}$ is proportional to

$$\sigma_Q^{k\ell}\,\partial_k\mathscr{D}_{\ell 0}^{ce}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A') + \sigma_{\bar{Q}}^{k\ell}\,\partial_k\mathscr{D}_{\ell 0}^{ec}(\vec{x}_{\bar{Q}}, \vec{x}_Q; A')\ , \tag{4.11}$$

where ∂_k acts on $\vec{x}_Q$ in the first term and on $\vec{x}_{\bar{Q}}$ in the second term. It is easy to see from this that $\mathscr{H}_2^{hard}$ has no $\Delta S = 1$, $\underline{1}\leftrightarrow\underline{1}$ interactions (i.e. those proportional to $\vec{\sigma}_- = \vec{\sigma}_Q - \vec{\sigma}_{\bar{Q}}$) of order ρ^3.

The part of $\mathscr{H}^{hard}$, that involves $\mathscr{D}_{k\ell}$, is given by

$$\mathscr{H}_3^{hard} = (g^2/M)\, T_c T_e^* \mathscr{D}_{k\ell}^{ce}(\vec{x}_Q, \vec{x}_{\bar{Q}}; A')\, p^k p^\ell$$

$$+ \frac{\alpha_S}{2M^2 r^3}\, r^k \Big[\sigma_Q^{k\ell}\Big\{T_c, \Big\{p^\ell - \frac{1}{2}(P^\ell + gA^\ell\cdot T_-),\ T_c^*\Big\}\Big\}$$

$$+ \sigma_{\bar{Q}}^{k\ell}\Big\{T_c^*, \Big\{p^\ell + \frac{1}{2}(P^\ell + gA^\ell\cdot T_-), T_c\Big\}\Big\}\Big]$$

$$+ \frac{\alpha_S}{4M^2}\, \sigma_Q^k \sigma_{\bar{Q}}^\ell \left[\frac{1}{r^3}\left(-\delta^{k\ell} + 3\,\frac{r^k r^\ell}{r^2}\right) + \frac{8\pi}{3}\,\delta^{k\ell}\delta^3(\vec{r})\right]$$

$$+ \dots \qquad , \qquad (4.12)$$

where the δ-function comes from $\partial_k \partial_{k'} \Delta_{\ell\ell'}(\vec{r})$. Note that hard-transverse-gluon exchanges mainly produce the higher-order structure of $Q\bar{Q}$ binding.

After some manipulation, $\mathscr{H}^{new} = \mathscr{H}_{Q\bar{Q}} + \mathscr{H}^{hard}$ is rewritten as

$$\mathscr{H}^{new} = \mathscr{H}_0 + \mathscr{H}_V + \mathscr{H}_E + \mathscr{H}_H + \mathscr{H}_g \qquad (4.13)$$

with

$$\mathscr{H}_0 = 2M + \frac{1}{M}\vec{p}^{\,2} - \frac{\alpha_S}{r}(T_c T_c^*) - \frac{1}{4M^3}(\vec{p}^{\,2})^2 + \frac{1}{4M}(\vec{P} + g\vec{A}\cdot T_-)^2,$$

$$\mathscr{H}_V = \frac{\alpha_S}{2M^2}(T_c T_c^*)\left[-\frac{1}{r}\left(\vec{p}^{\,2} + \frac{1}{r^2}\,\vec{r}(\vec{r}\cdot\vec{p})\vec{p}\right) + \frac{3}{r^3}\,\vec{S}\cdot\vec{L}\right.$$

$$\left. - \frac{1}{r^3}\left(\vec{S}^2 - \frac{3}{r^2}(\vec{r}\cdot\vec{S})^2\right) + \frac{8\pi}{3}\left(\vec{S}^2 - \frac{3}{4}\right)\delta^3(\vec{r})\right]$$

$$- \frac{\alpha_S}{8M^2 r^3}\cdot\frac{1}{4}\Big\{T_c, \Big\{\vec{\sigma}_-\cdot(\vec{r}\times(\vec{P} + g\vec{A}\cdot T_-))\ ,\ T_c^*\Big\}\Big\} + \dots ,$$

$$\mathscr{H}_E = -\, gA_0\cdot T_- + \frac{1}{2}\, g(\vec{r}\cdot\vec{E})\cdot T_+ + \frac{1}{8}g(r\cdot\nabla\vec{r}\cdot\vec{E})T_- + \frac{1}{48}g\left((r\cdot\nabla)^2\vec{r}\cdot\vec{E}\right)T_+$$

$$+ \dots ,$$

$$\mathscr{H}_H = \frac{g}{4M}\left[\vec{\sigma}_-^k\vec{H}^k\cdot T_+ + (\vec{L} + 2\vec{S})^k\vec{H}^k\cdot T_-\right]$$

$$- \frac{g}{4M}\left[\vec{S}^k(\vec{p}\times\vec{E})^k\cdot T_+ + \frac{1}{2}\vec{\sigma}_-^k(\vec{p}\times\vec{E})^k\cdot T_-\right]$$

$$+ \frac{g}{4M}\left[\vec{S}^k(r\cdot\nabla\vec{H}^k)\cdot T_+ + \frac{1}{2}\vec{\sigma}_-^k(r\cdot\nabla\vec{H}^k)\cdot T_-\right]$$

$$+ \frac{g}{24M}\left\{\vec{L}^k,\ (r\cdot\nabla\vec{H}^k)\cdot T_+\right\} - \frac{g}{8M}\left\{P^k + gA^k\cdot T_-, (\vec{r}\times\vec{H})^k\cdot T_+\right\}$$

$$+ \dots \qquad , \qquad (4.14)$$

where $(E^k)^c$ and $(H^k)^c$ are the color-electric and color-magnetic fields defined at $\vec{u}$, respectively, and $\vec{L} = \vec{r}\times\vec{p}$ is the angular momentum of relative $Q\bar{Q}$ motion. As defined before, $T_\pm = T \pm T^*$, $\vec{S} = \frac{1}{2}(\vec{\sigma}_Q + \vec{\sigma}_{\bar{Q}})$, $\vec{\sigma}_- = \vec{\sigma}_Q - \vec{\sigma}_{\bar{Q}}$, and $r\cdot\nabla = r^k\nabla_k[A(\vec{u})]$.

The first three terms in $\mathscr{H}_0$ represent the relative $Q\bar{Q}$ motion (of $O(0,0)$) and are regarded as the unperturbed Hamiltonian for perturbative calculations in the present multipole expansion scheme. Note that the Coulomb potential is rewritten as $-(T_c T_c^*)\alpha_S/r = -\,C_F\alpha_S/r + \Delta\varepsilon\mathscr{P}_8$, where $C_F = \frac{1}{2}(N^2 - 1)/N$ and $\Delta\varepsilon = \frac{1}{2}N\alpha_S/r$ ($N = 3$ for color $SU(3)$). The last term in $\mathscr{H}_0$ represents the c.m. motion as well as the recoil effect of the $Q\bar{Q}$ system [of $O(2,2)$]. The $\mathscr{H}_V$ represents the $Q\bar{Q}$ potential of $O(2,0)$, apart from the last term of $O(3,1)$. In the Abelian limit it is reduced to the Breit-Fermi Hamiltonian.[22] The $\mathscr{H}_E$ and $\mathscr{H}_H$ describe soft gluons coupled to the quark and antiquark. The $g\,A_0\cdot T_-$ term represents the static Coulomb interaction of $O(0,1)$, which acts only between colored ($\underline{8}$) $Q\bar{Q}$ states. The color-electric dipole (color-E1) interaction $\frac{1}{2}g(\vec{r}\cdot\vec{E})\cdot T_+$ induces a $\Delta S = 0$, $\Delta L = 1$, $\underline{1} \leftrightarrow \underline{8}$ or $\underline{8} \leftrightarrow \underline{8}$ transition [of $O(1,2)$] of the $Q\bar{Q}$ system. The remaining terms in $\mathscr{H}_E$ are color-E2 and color-E3 interactions. The color-magnetic dipole (color-M1) interaction $\frac{1}{4}(g/M)\vec{\sigma}^k_- \vec{H}^k \cdot T_-$ causes a parity-conserving, $\Delta S = 1$, $\underline{1} \leftrightarrow \underline{8}$ or $\underline{8} \leftrightarrow \underline{8}$ transition of $O(2,2)$. Another color-M1 interaction is proportional to the "intrinsic" color-magnetic moment $\frac{1}{4}(g/M)(\vec{L} + 2\vec{S})$ of the colored $Q\bar{Q}$ system.

The last term $\mathscr{H}_g$ describes the coupling of soft gluons to the hard-gluon cloud:

$$
\begin{aligned}
\mathscr{H}_g = \alpha_S T_c T_e^* \Bigg[& \frac{ig}{3\Lambda}\left\{\frac{1}{M}\,\overline{\vec{E}}\cdot\vec{\not{p}} - \frac{1}{4}\, r^{\ell}\,\overline{(\nabla^k F_{\ell k})}\right\} + O(2,4) \\
& - \frac{g^2 r}{48\Lambda}\left\{\overline{\vec{H}}\cdot\overline{\vec{H}} - \frac{2}{r^2}\,(\vec{r}\times\overline{\vec{H}})\cdot(\vec{r}\times\overline{\vec{H}})\right\} \\
& - \frac{g^2 r}{20\Lambda}\left\{\overline{\vec{E}}\cdot\overline{\vec{E}} - \frac{1}{2r^2}\,(\vec{r}\cdot\overline{\vec{E}})(\vec{r}\cdot\overline{\vec{E}})\right\} \\
& + \frac{2}{3}\,\frac{\not{p}^2}{M^2}\left(\Lambda + \frac{1}{2\Lambda}\nabla_0\nabla_0\right) + \cdots \Bigg],
\end{aligned}
\tag{4.15}
$$

where the last term is included here for convenience. As for terms of order ρ^3, only those that have $\underline{1} \leftrightarrow \underline{1}$ components are shown. The first term represents a color-E1 interaction of $O(2,2)$, which is nonleading. It is important to note that the effect of soft gluons coupled to the hard-gluon cloud is in general nonleading as compared with that of soft gluons coupled to Q and $\bar{Q}$.

Intuitively, this is because the color charge associated with the gluon cloud is induced by the quark and antiquark color charges so that its effect is in general smaller, by a single power of $\rho \sim \alpha_S$, than that caused by Q and $\bar{Q}$. Unlike $\mathscr{H}_E$ or $\mathscr{H}_H$, $\mathscr{H}^{\text{hard}}$ possesses direct $\underline{1} \leftrightarrow \underline{1}$ components starting with order ρ^3. Fortunately, they turn out less dominant than those generated by combining the low-order interactions in $\mathscr{H}_E$ and $\mathscr{H}_H$, as is seen from Eq. (4.15) and Fig. 4. In particular, $\mathscr{H}^{\text{hard}}$ involves no $\Delta S = 1$, $\underline{1} \leftrightarrow \underline{1}$ interactions of order ρ^3, as noted earlier.

Hadronic transitions[23] between color-singlet $Q\bar{Q}$ bound states proceed via the emission of gluons. Hadronic transitions of low multipole-orders are listed and schematically illustrated in Fig. 4. Note that any number of color-Coulomb ($\underline{8} \leftrightarrow \underline{8}$) interactions $gA_0\cdot T_-$ can be inserted between a pair of ($\underline{1} \leftrightarrow \underline{8}$) and ($\underline{8} \leftrightarrow \underline{1}$) interactions without increasing a power of ρ.

V. EFFECTIVE $\underline{1} \leftrightarrow \underline{1}$ INTERACTION

The present formalism based on $\mathscr{H}^{\text{new}}$ (Eq. (4.13)) is applicable to reactions involving colored as well as colorless $Q\bar{Q}$ states. It is, nonetheless, instructive to derive an effective $\underline{1} \leftrightarrow \underline{1}$ (nonlocal) interaction out of the local Hamiltonian $\mathscr{H}^{\text{new}}$. A systematic procedure for achieving this is to use a unitary transformation which removes the $\underline{1} \leftrightarrow \underline{8}$ components of $\mathscr{H}^{\text{new}}$, in exact analogy with the FW transformation which removes transitions between positive- and negative-energy states of a Dirac particle.

Let us denote the transformed Hamiltonian by

$$
\mathscr{H}' = e^{iV}(\mathscr{H}^{\text{new}} - i\partial/\partial t)e^{-iV} .
\tag{5.1}
$$

Appendix C outlines the construction of the unitary transformation e^{iV}. We denote the color-singlet piece of the unperturbed $Q\bar{Q}$ Hamiltonian by $\mathscr{H}_1 = 2M + \vec{p}^{\,2}/M^2 - C_F\alpha_S/r$, and the color-octet part by $\mathscr{H}_8 = \mathscr{H}_1 + \Delta\varepsilon$. Then the $\underline{1} \leftrightarrow \underline{1}$ component of the Hamiltonian $\mathscr{H}'$ is given by

$$(\mathscr{H}')_{11} = (\mathscr{H}^{new})_{11} + \mathscr{H}_s \tag{5.2}$$

with

$$\begin{aligned}
<\varepsilon'|\mathscr{H}_s|\varepsilon> = {} & \frac{g^2}{4N} < \varepsilon'|(\vec{r}\cdot\vec{E})[\varepsilon - \mathscr{H}_8 + i\nabla_0]^{-1}(\vec{r}\cdot\vec{E}) + h.c.|\varepsilon> \\
& + \frac{g^2}{4} <\varepsilon'|\left\{\frac{1}{NM}\vec{\sigma}_-\cdot\vec{H} - i\frac{\alpha_S}{3\Lambda}\left(\frac{1}{M}\vec{E}\cdot\vec{p} - \frac{1}{4}r^\ell(\nabla^k F_{\ell k})\right)\right\} \\
& \times[\varepsilon - \mathscr{H}_8 + i\nabla_0]^{-1}(\vec{r}\cdot\vec{E}) + h.c.|\varepsilon> \\
& + \frac{g^3}{8N} <\varepsilon'|(\vec{r}\cdot\vec{E})[\varepsilon - \mathscr{H}_8 + i\nabla_0]^{-1}(\vec{r}\cdot\tilde{\vec{E}})[\varepsilon - \mathscr{H}_8 + i\nabla_0]^{-1}(\vec{r}\cdot\vec{E}) \\
& + h.c.|\varepsilon> + \ldots \quad ,
\end{aligned} \tag{5.3}$$

where $|\varepsilon>$ and $|\varepsilon'>$ are the eigenstates of the color-singlet Hamiltonian $\mathscr{H}_1$, $\mathscr{H}_1|\varepsilon> = \varepsilon|\varepsilon>$, etc. $\tilde{E}$ stands for the matrix field $(\tilde{E})^{ab} = d^{abc}E^c$, where d^{abc} are the totally symmetric coefficients of $SU(N)$. The energy denominator

$$[\varepsilon - \mathscr{H}_8 + i\nabla_0]^{-1} = \sum_{n=0}^{\infty}\left\{-i(\varepsilon - \mathscr{H}_8)^{-1}\nabla_0\right\}^n(\varepsilon - \mathscr{H}_8)^{-1}$$

is a matrix in color indices; the time derivative in $\nabla_0 = \nabla_0[A(\vec{u})]$ acts on all the gluon fields to its right.

The $\mathscr{H}_s$ is the effective Hamiltonian which describes hadronic transitions in a heavy-quark family discussed in the previous section. The first through third terms in $\mathscr{H}_s$ begin with multipole-order (2,4), (3,4) and (3,6), respectively. The first term agrees with the result obtained by Peskin.[6]

The diagonal component $<\varepsilon|\mathscr{H}_s|\varepsilon>$ may be regarded as representing the effect of soft gluons on a heavy $Q\bar{Q}$ bound state $|\varepsilon>$. Let us, following Voloshin,[4] take the expectation value of $<\varepsilon|\mathscr{H}_s|\varepsilon>$ between the gluonic vacuum $|0)$ which is the lowest energy eigenstate in the pure soft-gluon sector. Then the effect of very soft gluons residing in the gluonic vacuum is factorized from the localized $Q\bar{Q}$ system and is represented by the vacuum expectation values of gauge-invariant operators such as $(0|E_k\cdot(\nabla_0)^n E_\ell|0)$, as seen from (5.3).

VI. STATIC QUARK-ANTIQUARK POTENTIAL

In the present multipole expansion scheme the soft-gluon sector (as well as light-quark sectors in case they are included) is treated as a fully interacting system although the hard-gluon sector is treated perturbatively. The standard weak-coupling expansion is inappropriate[13] for the study of soft gluons. The multipole expansion scheme, on the other hand, has control over soft gluons surrounding a $Q\bar{Q}$ system: The binding energy of a $Q\bar{Q}$ bound state provides natural suppression of soft-gluon emission. In this section the multipole expansion scheme is applied to the perturbative study of the effect of soft gluons on the $Q\bar{Q}$ potential.

For this purpose we treat the soft-gluon sector perturbatively in the effective Hamiltonian (5.3). For simplicity we take the static limit $M \to \infty$. The order-ρ^2 term in (5.3) leads to the following correction to the Coulomb potential:

$$\varepsilon(2) = - ig^2/(2N) \int_{-\infty}^{t} dt' e^{-i(t-t')\Delta\varepsilon} r^k r^\ell < T^* E_k^c(t) E_\ell^c(t') > , \tag{6.1}$$

where $[\varepsilon - \mathscr{H}_8 + i\nabla_0]^{-1}$ has been replaced by the causal propagator[25] $< t|[i\partial_0 - \Delta\varepsilon + i0] |0 > = - i\theta(t)\exp(- it\Delta\varepsilon)$.

In coordinate space, the Coulomb-gauge soft-gluon propagator $\Delta^S_{k\ell}(x)$ is, for $x_0^2 >> \vec{x}^2$, well approximated by

$$\delta^{k\ell}(4\pi^2)^{-1} \frac{2}{3} (x_\mu x^\mu - i0)^{-1} . \tag{6.2}$$

With this simplified propagator, (6.1) is rewritten as

$$\varepsilon(2) \simeq (g^2/\pi^2)\, C_F\, r^2 \int_0^\infty d\tau\, \tau^{-4} e^{-i\tau\Delta\varepsilon}. \tag{6.3}$$

The lower end in the relative-time (τ) integration should be cut at some time scale $\tau_0 \sim \Lambda$ which distinguishes between soft- and hard-gluon contributions. Long-time color fluctuation of the $Q\bar{Q}$ system corresponding to the upper end of the τ integration gives rise to some nonanalytic terms in $\Delta\varepsilon$ (hence in α_S):

$$\varepsilon(2) \simeq \frac{2}{3\pi} (C_F \frac{\alpha_S}{r}) [(r\Delta\varepsilon)^3 \ell n(r\Delta\varepsilon) + \ldots] , \tag{6.4}$$

where the omitted terms are regular in $\Delta\varepsilon$. Such nonanalytic terms have been obtained by Appelquist, Dine and Muzinich[13] by selective resummation of the weak-coupling expansion. In the multipole expansion scheme, such rearrangement is done at the Hamiltonian level. In general, higher-multipole-order corrections to the $Q\bar{Q}$ potential consist of higher powers of $\rho = r\Delta\varepsilon$ as well as logarithms of $\Delta\varepsilon$ (which originate from the energy denominator).

VII. CONCLUDING REMARKS

In this paper we have derived a gauge-invariant multipole expansion scheme for heavy quark-antiquark systems and studied some of its applications. A generalization to the case of heavy baryons is straightforward.

The inclusion of hard-gluon-loop corrections, as illustrated in Fig. 2, leads to the successive improvement of the present double-multipole expansion scheme which is still at the tree level as to the treatment of hard gluons. These loop corrections consist of quantum corrections to the $Q\bar{Q}$ potential as well as the soft-gluon interactions coupled to them.

In the present framework no perturbative treatments are assumed for the soft-gluon sector as well as light-quark sectors. The long-distance dynamics should in principle describe how soft gluons turn into hadrons. Practically, one may have to treat hadron formation phenomenologically while treating soft-gluon emission perturbatively. In Sec. IV, we have noted that hadronic transistions of low multipole-orders are rather insensitive to the hard-gluon cloud induced by quark charges. In order to widen the range of applicability of the multipole expansion to the ψ and Υ families, it is necessary to examine to what extent this feature is inherited by phenomenological heavy-quark potentials.

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APPENDIX A

In this appendix we study how the soft-gluon field $A_\mu^a(x)$ is transformed under the gauge transformation (3.1) defined by $U(\theta)$ (Eq. (3.3)).

As preliminaries, let us first evaluate

$$I_k^{(w)} = e^{w\cdot\nabla}\partial_k^{(w)}(e^{-w\cdot\nabla}) , \tag{A.1}$$

where $\nabla_k = \nabla_k[A] = \partial_k^{(u)} - ig\overline{A_k(\vec{u})}$ and $\partial_k^{(w)} = \partial/\partial w^k$. With the aid of the formula

$$\delta e^F = \int_0^1 d\beta \; e^{(1-\beta)F}\delta F \; e^{\beta F} \tag{A.2}$$

for a derivative of an exponential operator, we get

$$I_k^{(w)} = - \int_0^1 d\beta \; e^{\beta w\cdot\nabla}\nabla_k[A] \; e^{-\beta w\cdot\nabla} . \tag{A.3}$$

We expand the integrand in powers of β, noting the formulae $[\nabla_\mu[A] , \nabla_\nu[A]] = - ig \overline{F_{\mu\nu}[A]}$ and $[\nabla_\mu[A] , \overline{G}] = \overline{(\nabla_\mu[A]G)}$, where $\overline{G}^{ab} = i f^{acb}G^c$, etc. The result is

$$I_k^{(w)} = - \nabla_k[A] + ig\sum_{n=0}^{\infty} \frac{1}{(n+2)!} \overline{(w\cdot\nabla)^n w^\ell F_{\ell k}} . \tag{A.4}$$

Here and in what follows, fields with unspecified arguments are defined at position $\vec{u}$. In the same way, it is now straightforward to verify that

$$I_k^{(u)} \equiv e^{w\cdot\nabla}\partial_k^{(u)}(e^{-w\cdot\nabla})$$

$$= ig \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \overline{(w\cdot\nabla)^n w^\ell F_{\ell k}} \ . \qquad (A.5)$$

In Sec. III, an arbitrary position $\vec{x} = \vec{w} + \vec{u}$ is parametrized around some fixed position $\vec{u}$. In this case, $\partial/\partial x^k$ in $b_k(\theta)$ (Eq. (3.1) implies $\partial_k^{(w)}$. Note that $U(\theta)\left(\partial_k^{(w)}U^\dagger(\theta)\right)$ $= e^{w\cdot\nabla}e^{-w\cdot\partial}\partial_k^{(w)}(e^{w\cdot\partial}e^{-w\cdot\nabla})$ is rewritten as

$$\partial_k^{(u)} + I_k^{(u)} + I_k^{(w)} \ . \qquad (A.6)$$

Consequently,

$$b_k(\theta) = -\left[A_k + \sum_{n=0}^{\infty}\frac{1}{(n+2)!}(w\cdot\nabla)^n w^\ell F_{\ell k} + \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(w\cdot\nabla)^n\partial_k^{(u)}(w^\ell A_\ell)\right]. \qquad (A.7)$$

The evaluation of $U(\theta)\ (\partial_0 U^\dagger(\theta))$ is analogous to that of $I_k^{(u)}$, yielding the result

$$b_0(\theta) = -\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left((w\cdot\nabla)^n w^\ell F_{\ell 0}\right) \ . \qquad (A.8)$$

Substitution of (A.7) and (A.8) into Eq. (3.2) leads to the desired result (3.5).

In Sec. IV, two-body variables $\vec{u} = \frac{1}{2}(\vec{x}_Q + \vec{x}_{\bar{Q}})$ and $\vec{r} = \vec{x}_Q - \vec{x}_{\bar{Q}}$ are chosen as independent coordinates instead of $(\vec{x}_Q, \vec{x}_{\bar{Q}})$. In this case, $\vec{u}$ is no longer a fixed position. Expressed in terms of $(\vec{u}, \vec{r})$, the covariant derivative for the quark field is decomposed into those associated with the c.m. and relative motions of the $Q\bar{Q}$ system:

$$(\partial_k - igA_k\cdot T)_{\vec{x}_Q} = \frac{1}{2}\left(\partial_k^{(u)} - igA_k(\vec{x}_Q)\cdot T\right) + \frac{1}{2}\left(\partial_k^{(w)} - igA_k(\vec{x}_Q)\cdot T\right), \qquad (A.9)$$

where $\vec{w} = \frac{1}{2}\vec{r}$. Let us now make the gauge transformation (3.4) in the quark sector $\left(\text{i.e. } \psi_Q(x_Q) \to V_Q(\theta)\psi(x_Q)\right)$. Then, the quark covarient derivative (A.9) is converted to

$$\frac{1}{2}(\partial_k^{(u)} - igA_k^{(u)}(\vec{x}_Q)\cdot T) + \frac{1}{2}(\partial_k^{(w)} - igA_k'(\vec{x}_Q)\cdot T) \ , \qquad (A.10)$$

where $A_k^{(u)}(\vec{x}_Q)$ and $A_k'(\vec{x}_Q)$ are given by Eq. (3.1) with the derivative ∂_k in $b_k(\theta)$ replaced by $\partial_k^{(u)}$ and $\partial_k^{(w)}$, respectively. Obviously, $A_k'(\vec{x}_Q)$, the gauge field associated with relative $Q\bar{Q}$ motion, is given by Eq. (3.5). A simple calculation shows that $A_k^{(u)}(\vec{x}_Q)$ is given by $A_0'(\vec{x}_Q)$ in Eq. (3.5) with the Lorentz index 0 replaced by k.

The subsequent gauge transformation in the antiquark sector $\left(\psi_{\bar{Q}}(\vec{x}_{\bar{Q}}) \to V_{\bar{Q}}\psi_{\bar{Q}}(\vec{x}_{\bar{Q}})\right)$ with

$$V_{\bar{Q}} = \exp\left[-\frac{1}{2}r^k(\partial_k^{(u)} + igA_k(\vec{x}_{\bar{Q}})\cdot T^*)\right]\exp\left[\frac{1}{2}r\cdot\partial\right]$$

leaves $A_k^{(u)}\cdot T$ and $A'_k\cdot T$ in (A.10) unaffected. However, the derivative $\frac{1}{2}\partial_k^{(u)} + \partial_k^{(r)} = \frac{1}{2}\left(\partial_k^{(u)} + ig\, A_k(\vec{x}_{\bar{Q}})\cdot T^*\right) + \left(\partial_k^{(r)} - \frac{1}{2} igA_k(\vec{x}_{\bar{Q}})\cdot T^*\right)$ undergoes the transformation. It is now a simple excercise to verify that the effect of the combined transformation $V_{\bar{Q}}V_Q$ (or $V_QV_{\bar{Q}}$) on the covariant derivatives for the quark and the antiquark is summarized as in Eq. (4.2).

APPENDIX B

In this appendix we construct the hard-gluon propagator $\mathscr{D}_{\mu\nu}^{ab}(\vec{x}, \vec{y}; A')$ in the presence of soft gluons.

A Gaussian integration over the hard gluon field leads to

$$\mathscr{D}_{\mu\nu}^{ab}(x,y;A') = D_{\mu\nu}^{ab}(x,y) - \int d^4z d^4v\, D_{\mu k}^{ac}(x,z)\,(W^{k\ell}(z,v))^{ce} D_{\ell\nu}^{eb}(v,y),$$

$$W^{k\ell}(z,v) = d^k[b]_z \left[d^m[b]\, D_{mn}\, d^n[b]\right]_{z,v}^{-1} d^\ell[b]_v, \tag{B.1}$$

where $d^k[b] = \partial^k - ig\overline{b^k(\theta)}$, and $D_{\mu\nu}^{ab}(x,y)$ is defined as the inverse of

$$<x, a|g_{\mu\nu}(\nabla_\rho[A']\nabla^\rho[A'] + \Lambda^2) - \nabla_\nu[A']\nabla_\mu[A']\,|y, b>. \tag{B.2}$$

As explained in Sec. III, $b^k(\theta)$ is set equal to zero in what follows; then (B.1) is reduced to the standard set of Feynman rules in the Coulomb gauge.

Let us recall the multipole-order assignment in Sec. III. The leading terms in (B.2) are of order (-2, 0) and yield the instantaneous propagators

$$\Delta_{00}(x - y) = \langle\vec{x}|\ 1/\nabla^2|\vec{y}\rangle\ \delta(x_0 - y_0),$$

$$\Delta_{k\ell}(x - y) = \langle\vec{x}|(-\delta^{k\ell} - \partial^k\partial^\ell/\nabla^2)/(\nabla^2 + \Lambda^2)|\vec{y}\rangle\ \delta(x_0 - y_0). \tag{B.3}$$

We denote the rest of (B.2) by $\Gamma_{\mu\nu}$:

$$\Gamma_{00} = -ig\{\overline{A'_k}, \partial^k\} - g^2\overline{A'_k}\overline{A'^k},$$

$$\Gamma_{0k} = -ig\left(\overline{F_{0k}[A']} - \nabla_0[A']A'_k\right) + \ldots ,$$

$$\Gamma_{k\ell} = -ig\overline{F_{k\ell}[A']} + g_{k\ell}\nabla_0[A']\nabla_0[A'] + g_{k\ell}\Gamma_{00} + g^2\overline{A'^j}\,\overline{A'_j} + \ldots . \tag{B.4}$$

In Γ_{0k} and $\Gamma_{k\ell}$, we have used the relation $[\nabla_\mu, \nabla_\nu] = -ig\overline{F_{\mu\nu}}$ and omitted terms that vanish when the transversality of $\Delta_{k\ell}$ is taken into account. The full propagator $\mathscr{D}_{\mu\nu}^{ab}(x, y; A')$ is expanded in powers of $\Gamma_{\mu\nu}$: For example,

$$\mathscr{D}_{00} = \Delta_{00} + \Delta_{00}\left[-\Gamma^{00} + \Gamma^{00}\Delta_{00}\Gamma^{00} + \Gamma^{0k}\Delta_{k\ell}\Gamma^{\ell 0}\right]\Delta_{00} + \ldots , \tag{B.5}$$

where integration symbols are suppressed in an obvious fashion. (See Fig. 2(a).)

Some remarks as to general structures of the above expansions are in order. (i) Consider n space derivatives acting on a product of m $A'_k(\vec{u})$'s, i.e. symbolically, $(\vec{\partial})^n(g\vec{A}')^m\big|_{\vec{x}=\vec{u}}$. [Note the definition of differentiation: e.g., $\partial_\ell A'_k(\vec{u})$ stands for

$$\partial_\ell A'_k(\vec{x})\Big|_{\vec{x}=\vec{u}} = \partial^{(w)}_\ell A'_k(\vec{w}+\vec{u})\Big|_{\vec{w}=0} = \frac{1}{2}F_{\ell k}[A(\vec{u})]\,.\Big]$$

As seen from Eq. (3.5), such an expression vanishes for $n < m$; it is, however, in general nonvanishing and is $O(0, n+m)$ for $n \geqslant m$. This fact implies that $g A'_k$ in $\Gamma_{\mu\nu}$ can effectively be regarded as $O(0,2)$. (ii) Power-counting tells us that, in $\mathcal{D}_{\mu\nu}$, terms with two or more A'_μ are both ultraviolet and infrared covergent.

Let us first consider Fig. 3(b). In momentum space, its contribution to $g^2\,\mathcal{D}_{00}(\vec{x},\vec{y};A')$ is written as

$$\frac{2ig^3}{(2\pi)^6}\frac{\partial}{\partial r^j}\int d^3k\,d^3q\;e^{-i\vec{k}\cdot\vec{u}+i\vec{q}\cdot\vec{r}}\Delta(\vec{q}-\tfrac{1}{2}\vec{k})\Delta(\vec{q}+\tfrac{1}{2}\vec{k})\overline{A'^{\,j}(\vec{k})}\,, \tag{B.6}$$

where $\Delta(\vec{p}) = 1/(\vec{p}^{\,2}+\Lambda^2)$, $\vec{r}=\vec{x}-\vec{y}$, $\vec{u}=\frac{1}{2}(\vec{x}+\vec{y})$, and $A'_j(\vec{k})$ is the Fourier transform of $A'_j(\vec{z})$. We combine the two denominators, integrate over $\vec{q}$ and differentiate with respect to $\vec{r}$. The result is

$$-ig\alpha_S\int_0^1 d\beta\left[\frac{r^j}{r}+i(\beta-\tfrac{1}{2})\frac{k^j}{\boldsymbol{m}(\Lambda,\vec{k}^2)}\right]e^{-r\boldsymbol{m}(\Lambda,\vec{k}^2)-i(\beta-\frac{1}{2})\vec{k}\cdot\vec{r}}\,\overline{A'^{\,j}(\vec{u})}, \tag{B.7}$$

where $\boldsymbol{m}(\Lambda,\vec{k}^2) = (\Lambda^2+\beta(1-\beta)\vec{k}^2)^{\frac{1}{2}}$ and k^j stands for the

derivative $k^j = i\partial/\partial u^i$ acting on $A'(\vec{u})$. As expected from power-counting, (B.7) possesses the $\vec{r}/r$ structure which becomes singular as $\vec{r}\to 0$; its contribution, however, vanishes because $\vec{A}'(\vec{u}) = 0$. We expand (B.7) in powers of $\vec{r}$ and express $\vec{A}'$ in terms of $\vec{A}$ using Eq. (3.5) and the Jacobi identity $\nabla_{[\rho}F_{\mu\nu]} = 0$. The final expression is

$$\frac{1}{12}\,ig\alpha_S\;(1/\Lambda - r)\;r^k\overline{(\nabla^\ell F_{k\ell})} + O(2,5) + O(3,5)\,. \tag{B.8}$$

Figure 3 (c), which represents the emission of two $\vec{A}'$'s, is similar in structure to diagram (b). Its contribution is given by

$$-\frac{1}{2}\,\alpha_S g^2\int_0^1 d\beta\left[\boldsymbol{m}^{-1} - r + \frac{1}{2}r^2\boldsymbol{m} - \frac{1}{2}(\beta-\tfrac{1}{2})^2(\vec{k}\cdot\vec{r})^2\boldsymbol{m}^{-1}+\ldots\right]\overline{A'_j(\vec{u})}\,\overline{A'_j(\vec{u})}\,, \tag{B.9}$$

where $\boldsymbol{m} = \boldsymbol{m}(\Lambda,\vec{k}^2)$ defined in (B.7). The $(\overline{F_{k\ell}})^2$ term in the first bracket of Eq. (3.11) follows from the first term in (B.9). The order-(3,4) terms in (B.9) are combined with an analogous term in (B.12) below to give the two terms in the fourth bracket in Eq. (3.11).

The evaluation of diagram (d) is rather involved. Using two Feynman parameters (ζ,β), we first combine the two Coulomb-gluon propagators attached to $\vec{x}$ and $\vec{y}$, and subsequently include the middle one. A momentum integration yields the following contribution to $g^2\,\mathcal{D}_{00}$:

$$\alpha_S \int_0^1 d\zeta \int_0^1 d\beta\beta\, e^{-r\mathcal{N} - i\vec{h}\cdot\vec{r}} \left\{ \frac{\delta^{mn}}{\mathcal{N}} - \frac{r^m r^n}{r} - \frac{i}{\mathcal{N}} \left[r^m (h^n - \tfrac{1}{2}k^n) + r^n (h^m + \tfrac{1}{2}\ell^m) \right] \right.$$

$$\left. + \mathcal{N}^{-3/2}(1 + r\mathcal{N})(h^m + \tfrac{1}{2}\ell^m)(h^n - \tfrac{1}{2}k^n) \right\} \overline{A'_m(\vec{u}')}\,\overline{A'_n(\vec{u}'')}, \quad \text{(B.10)}$$

where $\vec{h} = \left(\frac{1}{2} - \beta(1-\zeta)\right)\vec{k} - \left(\frac{1}{2} - \beta\zeta\right)\vec{\ell}$,

$$\mathcal{N} = \left[\Lambda^2 + \beta(1-\beta)\ (1-\zeta)\vec{k}^2 + \zeta\vec{\ell}^2 + \beta^2\zeta(1-\zeta)(\vec{k}+\vec{\ell})^2 \right]^{1/2}, \quad \text{(B.11)}$$

and (k^j, ℓ^j) stand for $(k^j = i\partial/\partial u'^j,\ \ell^j = i\partial/\partial u''^j)$; $\vec{u}'$ and $\vec{u}''$ are set equal to $\vec{u}$ after these derivatives are taken. After some manipulation, (B.11) is rewritten as

$$O(1,6) + O(2,6) - \frac{1}{32}(g^2\alpha_S/\Lambda)\,(r^m\overline{F_{mj}})^2 + O(3,6)\ . \quad \text{(B.12)}$$

Diagram (e) contains a hard-transverse-gluon exchange. A somewhat lengthy calculation leads to the result (up to $O(1,4)$ and $O(3,4)$)

$$\frac{\alpha_S}{12\Lambda^3} \left[\delta^{mn} + \frac{3}{10}\Lambda^2 (r^m r^n - 2r^2\delta^{mn}) - \frac{1}{8}\left\{ \left(\frac{3}{\Lambda^2} - \frac{r^2}{2}\right)\delta^{mn} \right. \right.$$

$$\left. \left. + r^m r^n \right\} \vec{k}\cdot\vec{\ell} \right] \overline{F^m}\,\overline{G^n}, \quad \text{(B.13)}$$

where $F^m = ig(F^{m0}[A'(u')] - A'^m(\vec{u}')\partial_0)$, $G^n = ig(F^{0n}[A'(u'')] - \partial_0 A'^n(\vec{u}''))$ and $(k^j, \ell^j) = (i\partial/\partial u'^j, i\partial/\partial u''^j)$. Equation (B.13) leads to the $\overline{F_{k0}}\,\overline{F_{k0}}$ term [of $O(1,4)$] as well as the terms in the last two brackets [of $O(3,4)$] in Eq. (3.11).

Figure 3 (b') ~ (e') represent $\mathcal{D}_{00}(\vec{x}, \vec{x}; A')$ and $\mathcal{D}_{00}(\vec{y}, \vec{y}; A')$, which appear in $\mathcal{D}_{reg}(\vec{x}, \vec{y}; A')$ (Eq. (4.8)). Note that, e.g., $\mathcal{D}_{00}(\vec{x}, \vec{x}; A')$ is obtained from $\mathcal{D}_{00}(\vec{x}, \vec{y}; A')$ by letting $\vec{y} \to \vec{x}$, i.e. by first letting $\vec{r} \to 0$ and then replacing $\vec{u}$ by $\vec{x} = \vec{u} + \frac{1}{2}\vec{r}$. Diagram (b') has a vanishing contribution. A direct calculation shows that terms of $O(\rho^3)$ in diagrams (c') and (d') are cancelled in the sum. The order-(3,4) term coming from diagram (e') involves time derivatives ∂_0.

Let us next calculate $g^2\mathcal{D}^{ab}_{0k}(\vec{x}, \vec{y}; A')$ up to $O(2,2)$. The evaluation of diagram Fig. 3 (f) leads to the result

$$\alpha_S \left[-\frac{1}{3\Lambda}\delta^{\ell k} + \frac{r}{8}\left(3\delta^{\ell k} - \frac{r^\ell r^k}{r^2}\right) \right.$$

$$\left. + \frac{i}{30\Lambda}\left\{ -\delta^{\ell k} r^m q^m + \frac{3}{2}(r^\ell q^k + r^k q^\ell) \right\} + \ldots \right] \overline{G^\ell}, \quad \text{(B.14)}$$

where $q^\ell \equiv i\partial/\partial u^\ell$ and G^ℓ is defined in (B.13). Substitution of Eq. (3.5) into (B.14) gives the result (3.12).

Figure 3 (h_1) and (h_2) with two hard-transverse-gluon exchanges contribute to $g^2\mathcal{D}_{k\ell}(\vec{x}, \vec{y}; A')$ up to $O(1,2)$. As in $\mathcal{D}_{00}$, diagram (h_1) has an $\vec{r}/r$ structure, which vanishes upon acting on $\vec{A}'(\vec{u})$. Terms involving one $\partial/\partial u^k$ are extracted after somewhat lengthy calculation:

$$\frac{1}{6}(ig\alpha_S/\Lambda)(\partial_k\overline{A'_\ell} - \partial_\ell\overline{A'_k}) + O(1,3)\ . \quad \text{(B.15)}$$

For diagram (h_2), it is sufficient to extract terms with no derivatives:

$$\frac{1}{5}(\alpha_S/\Lambda)\left\{\delta^{kk'}\delta^{\ell\ell'} + \frac{1}{6}\left(\delta^{\ell'}{}_{k_\delta}k'{}_\ell + \delta^{k'\ell'}\delta^{k\ell}\right)\right\}$$

$$\times\left(ig\,\overline{F^{k'\ell'}} + \delta^{k'\ell'}\,\nabla_0[A]\nabla_0[A]\right) . \qquad (B.16)$$

Combining (B.15) and (B,16), one is led to Eq. (3.13).

APPENDIX C

In this appendix, we determine the unitary transformation e^{iV} so that $\mathscr{H}' = e^{iV}(\mathscr{H}^{new} - i\partial_0)e^{-iV}$ (Eq. (5.1) has no $\underline{1}\leftrightarrow\underline{8}$ ("odd") components.

For this purpose we expand $\mathscr{H}^{new}$ and V in powers of ρ: $\mathscr{H}^{new} = \sum_{n=0}^{\infty}\mathscr{H}^{(n)}$ and $V = \sum_{n=1}^{\infty}V^{(n)}$. For convenience, the rest mass 2M is included in $\mathscr{H}^{(0)}$, which has no "odd" components. We assume that V has only $\underline{1}\leftrightarrow\underline{8}$ components and fix it to each order in ρ:

$$\left\{\mathscr{H}^{(1)} - \partial_0 V^{(1)} + i\left[V^{(1)}, \mathscr{H}^{(0)}\right]\right\}_{odd} = 0,$$

$$\left\{\mathscr{H}^{(2)} - \partial_0 V^{(2)} + i\left[V^{(2)}, \mathscr{H}^{(0)}\right] + i\left[V^{(1)}, \mathscr{H}^{(1)}\right]\right\}_{odd} = 0, \qquad (C.1)$$

etc. With this choice of V, the $\underline{1}\leftrightarrow\underline{1}$ component of the transformed Hamiltonian $(\mathscr{H}')_{11} = (\mathscr{H}^{(0)})_{11} + (\mathscr{H}'^{(2)})_{11} + (\mathscr{H}'^{(3)})_{11} + \cdots$ is given by

$$(\mathscr{H}'^{(2)})_{11} = (\mathscr{H}^{(2)})_{11} + i\,\frac{1}{2}\left[V^{(1)}, \mathscr{H}^{(1)}\right]_{11} ,$$

$$(\mathscr{H}'^{(3)})_{11} = (\mathscr{H}^{(3)})_{11} + i\,\frac{1}{2}\left[V^{(2)}, \mathscr{H}^{(1)}\right]_{11} + i\,\frac{1}{2}\left[V^{(1)}, \mathscr{H}^{(2)}\right]_{11} \qquad (C.2)$$

Let us denote $\mathscr{H}_1 = (\mathscr{H}^{(0)})_{11}$ and $\mathscr{H}_8 = \mathscr{H}_1 + \Delta\varepsilon$ so that $(\mathscr{H}^{(0)})_{88} = \mathscr{H}_8 - gA_0\cdot T_-$. To solve (C.1) for V, we operate (C.1) on the eigenstate $|\varepsilon>$ of $\mathscr{H}_1$, and use the formulae $(T_-)_a(T_+)_b\mathcal{P}_1 = i\,f^{abc}(T_+)_c\mathcal{P}_1$ and $\mathcal{P}_8(T_+)_a(T_-)_b\mathcal{P}_1 = d^{abc}(T_+)_c\mathcal{P}_1$.
The result is

$$(V^{(i)})_{81} = W^{(i)}\cdot(T_+)_{81} \qquad (i = 1, 2) \qquad (C.3)$$

with

$$W^{(1)}|\varepsilon> = \frac{1}{2}\,ig(\varepsilon - \mathscr{H}_8 + i\nabla_0)^{-1}(\vec{r}\cdot\vec{E})|\varepsilon> ,$$

$$W^{(2)}|\varepsilon> = (\varepsilon - \mathscr{H}_8 + i\nabla_0)^{-1}\left[iF + \frac{g}{2}(\vec{r}\cdot\vec{\tilde{E}})W^{(1)}\right]|\varepsilon> , \qquad (C.4)$$

where $(\tilde{E})^{ab} \equiv d^{abc}E^c$ and F^c is defined by $(\mathscr{H}^{(2)})_{81} = F\cdot(T_+)_{81}$. Substitution of these expressions into (C.2) leads to the $\underline{1}\leftrightarrow\underline{1}$ Hamiltonian (5.2).

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14. It is the combination gA_k that causes a momentum variation of the quark motion, as seen from the covariant derivative $\partial_k - igA_k^a T_a$.
15. The question gauge invariance of the multipole expansion has been examined by Peskin, Ref. 6.
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17. By a "gauge-invariant" form it is meant that the soft-gluon interactions are expressed in terms of $F_{\mu\nu}[A]$ and covariant derivatives only.
18. This is the closed expression for the unitary transformation previously found in Ref.[7].
19. Time derivatives ∂_0 acting on the quark (or antiquark) field are taken to be $O(\Delta\varepsilon) \sim O(0,0)$.
20. For the rest of the terms in $\mathscr{H}^{hard}$, hard-gluon exchanges by the some quark are neglected since the nonrelativistic Hamiltonian $\mathscr{H}_B$ which ignores very hard transverse gluons ($g\vec{B} \gtrsim M$) is not adequate for extracting them.
21. This cancellation phenomenon has been noted in some earlier calculations in Refs. [5, 6].
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23. The group-theoretical structure of hadronic transitions has recently been studied by Yan, Ref.[8], within a gauge-invariant multipole expansion scheme.
24. In Eq. (5.3), the hermitian conjugate of $[\varepsilon - \mathscr{H}_q + i\nabla_0]^{-1}$ is given by $[\varepsilon' - \mathscr{H}_q + i\overleftarrow{\nabla}_0]^{-1}$, where $\overleftarrow{\nabla}_0 = -\overleftarrow{\partial}_0 - ig\overline{A_0}$

($\overleftarrow{\partial}_0$ acts to the left).

25. A comparison with the standard perturbation theory tells us that $[-\Delta\varepsilon + i\partial_0]^{-1}$ is defined as the causal propagator.

FIGURE CAPTIONS

Fig. 1 Classification of gluons surrounding a heavy quark-antiquark system.

Fig. 2 Diagrammatic representation of hard-gluon exchanges between qaurk color charges.

(a) Zero-hard-gluon-loop approximation. Dashed lines are hard (Coulomb or transverse) gluons. Shaded blobs represent the coupling of soft gluons to the hard gluons.

(b) One-hard-gluon-loop approximation. The hard-gluon loop in diagram (b_1) includes a hard FP-ghost loop. Small black blobs at the triple hard-gluon vertex represent the coupling to soft gluons.

Fig. 3 Detailed structures of hard-gluon exchanges. Dotted lines are hard Coulomb gluons while zigzag lines are hard transverse gluons. Wavy lines represent soft (Coulomb or transverse) gluons.

Fig. 4 Hadronic transitions of low multipole-orders between heavy $Q\bar{Q}$ bound states. Here E1 and M1 imply color-E1 and color-M1 transitions, respectively. PP' stands for parity change.

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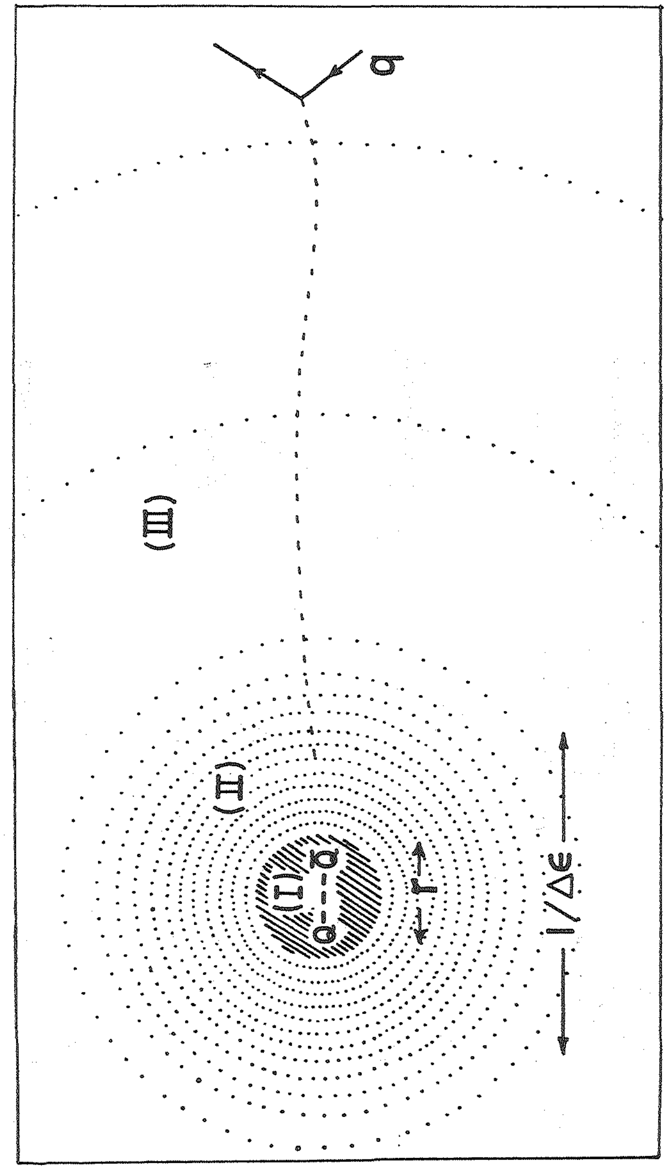

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Fig. 1

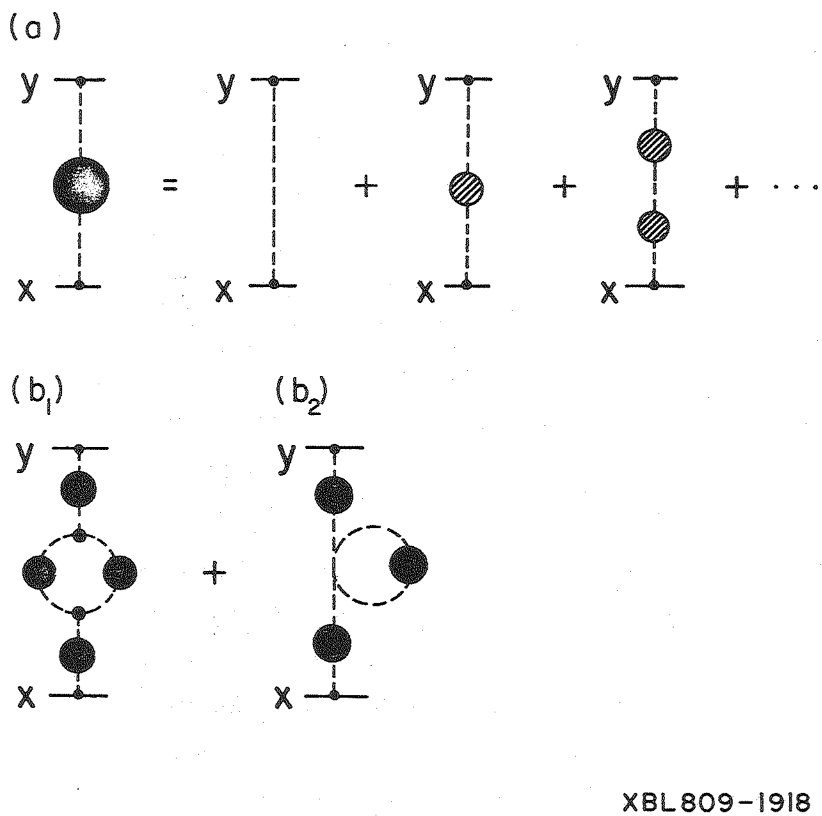

Fig. 2

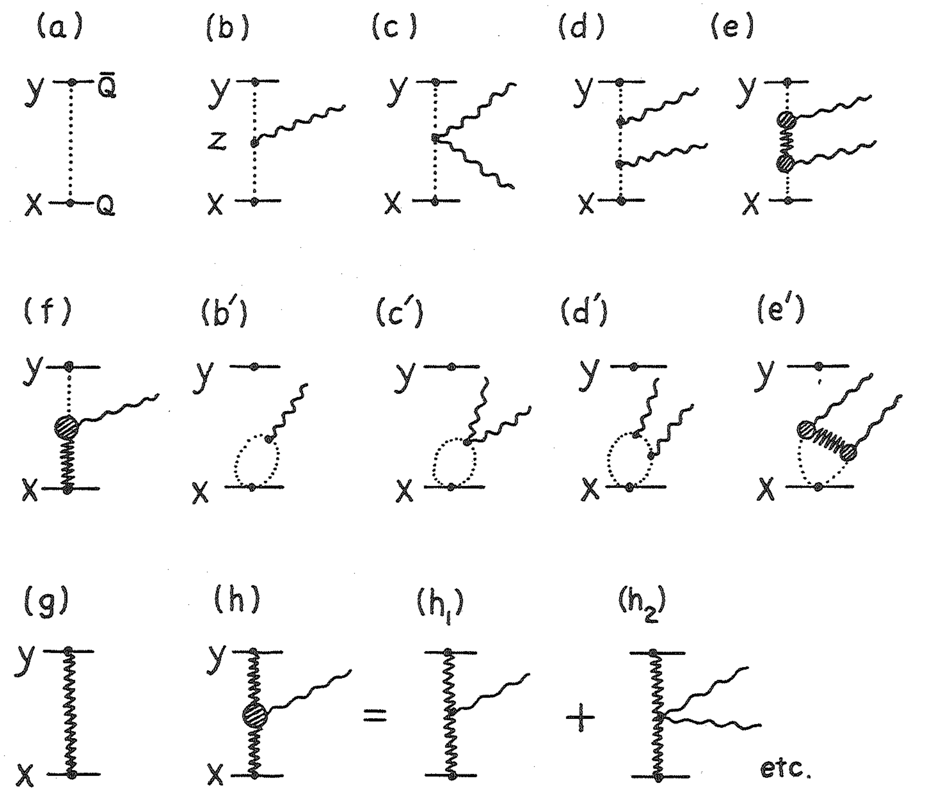

Fig. 3

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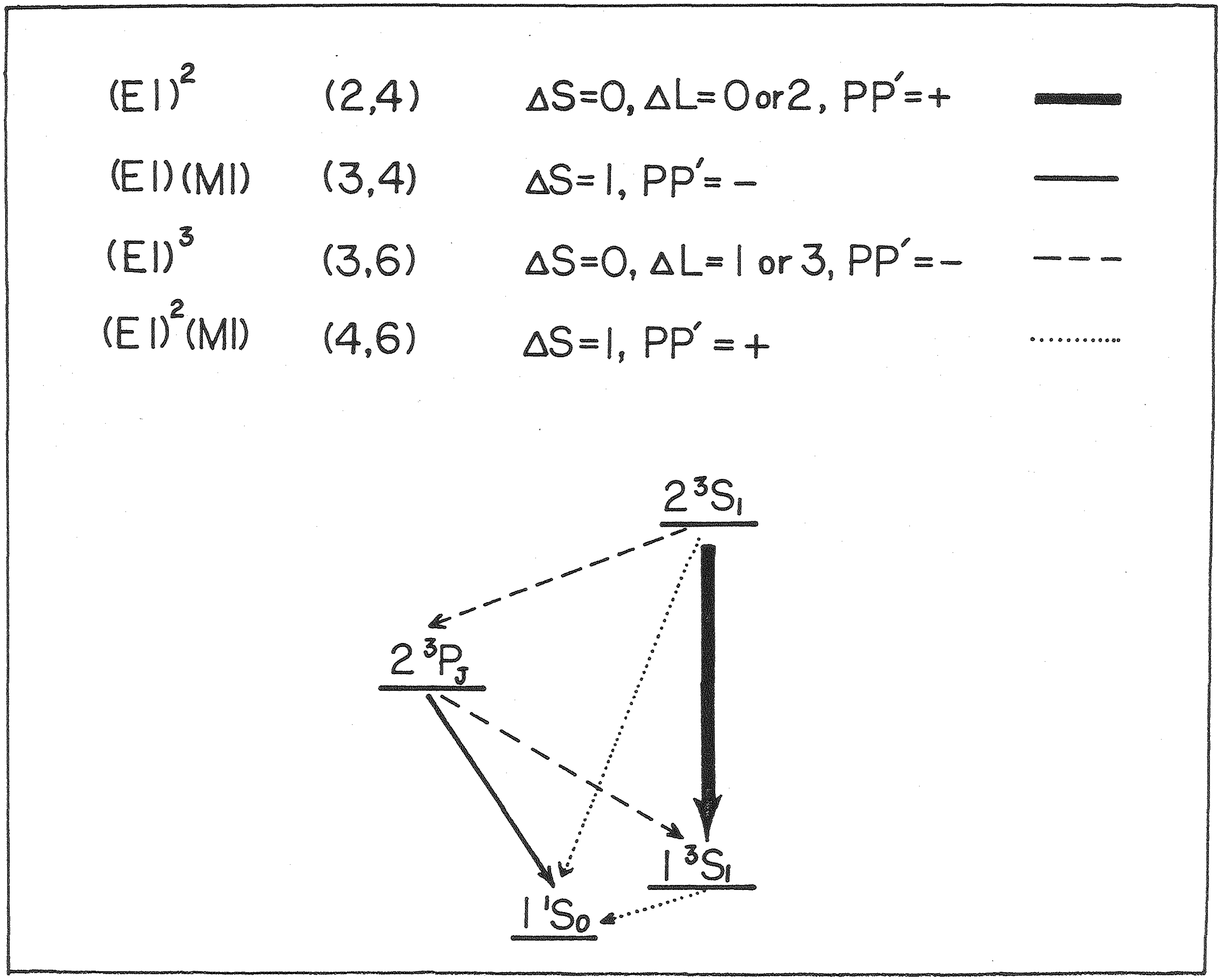
(E1)^2 (2,4) ΔS=0, ΔL=0 or 2, PP′=+
(E1)(M1) (3,4) ΔS=1, PP′=−
(E1)^3 (3,6) ΔS=0, ΔL=1 or 3, PP′=−
(E1)^2(M1) (4,6) ΔS=1, PP′=+
$2\,^3S_1$
$2\,^3P_J$
$1\,^3S_1$
$1\,^1S_0$

Fig. 4