Orbit-averaged guiding-center Fokker–Planck operator

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A general orbit-averaged guiding-center Fokker–Planck operator suitable for the numerical analysis of transport processes in axisymmetric magnetized plasmas is presented. The orbit-averaged guiding-center operator describes transport processes in a three-dimensional guiding-center invariant space: the orbit-averaged magnetic-flux invariant $\bar{\psi}$, the minimum-B pitch-angle coordinate ξ_0, and the momentum magnitude p. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Transport processes play a dominant role in the long-time behavior of strongly magnetized plasmas. In the absence of wave-induced (or turbulent) transport, the long-time magnetic confinement of plasmas is based on the small dimensionless parameter $\epsilon_B \equiv \rho / L_B \ll 1$ (the ratio of the characteristic gyroradius ρ and the magnetic nonuniformity length scale L_B). For such plasmas, the dimensionless parameter $\epsilon_\nu \equiv L_B / \lambda_\nu$ (λ_ν being the mean free path) can be used to describe different classes of collisional transport processes, such as classical collisional transport[1] ($\lambda_\nu \ll L_B$) and neoclassical "collisionless" transport[2] ($\lambda_\nu \gg L_B$). The quasilinear (wave-induced) transport processes associated with rf-induced heating and current drive, on the other hand, cause a slow time evolution of the background plasma distribution (as a result of one or more wave-particle resonances) with a time-scale ordering that is quadratic in a small parameter $\epsilon_w = |\mathbf{B}_w| / |\mathbf{B}_0| \ll 1$ associated with the rf-wave amplitude.[3,4]

Because the long-time behavior of the plasma distribution function for each particle species depends on competing collisional and quasilinear transport processes, an accurate treatment of both transport processes in realistic magnetic geometry is a crucial element in determining the equilibrium and behavior of fusion plasmas. For this purpose, the use of dynamical-reduction methods (e.g., guiding-center transformation[5,6]) can yield reduced transport operators (in phase space) that possess attractive numerical properties in addition to accurately representing collisional[7] and/or quasilinear[4] transport processes of interest.

The purpose of the present paper is to present a brief derivation of a general orbit-averaged guiding-center Fokker–Planck operator suitable for numerical studies of transport processes in general axisymmetric magnetic geometry. This reduced Fokker–Planck operator represents drag and diffusion processes in a three-dimensional space composed of guiding-center invariants. In this reduced formulation, the conjugate orbital angles have either been eliminated from the guiding-center Fokker–Planck operator by averaging or are absent by axisymmetry.

The remainder of the paper is organized as follows. In Sec. II, we discuss the guiding-center Hamiltonian dynamics of charged particles in unperturbed axisymmetric magnetic geometry. In Sec. III, we introduce the guiding-center Fokker–Planck operator previously derived for general magnetic geometry and arbitrary guiding-center orbit topology.[7] In Sec. IV, we first introduce the orbit-averaging operation for standard and nonstandard guiding-center orbits in axisymmetric tokamak geometry. Next, we present the orbit-averaged guiding-center Fokker–Planck operator and discuss its properties. In Sec. V, we briefly discuss the derivation of the bounce-center Fokker–Planck operator obtained by performing the bounce-center phase-space transformation[8,9] on the guiding-center Fokker–Planck operator and discuss its connection to the orbit-averaged guiding-center Fokker–Planck operator derived in the previous section. Lastly, we summarize our work in Sec. VI and discuss its applications.

II. GUIDING-CENTER DYNAMICS IN AXISYMMETRIC MAGNETIC GEOMETRY

The existence of the small parameter $\epsilon_B \ll 1$ in magnetically confined plasmas forms the basis of the unperturbed guiding-center dynamical reduction,[6] in which the fast gyromotion time scale associated with the gyroangle ζ_g (with $\dot{\zeta}_g \equiv \epsilon_B^{-1} \Omega$), with the gyroaction $J_g \equiv \mu B / \Omega$ acting as its canonically conjugate (adiabatic) invariant, is asymptotically decoupled from the parallel and cross-field motions of a guiding-center particle. The unperturbed guiding-center dynamics is expressed in terms of the guiding-center position $\mathbf{X}$ and the parallel velocity $v_\parallel$, where the guiding-center velocity

$$\dot{\mathbf{X}} = v_\parallel \hat{\mathbf{b}} + \epsilon_B \frac{\hat{\mathbf{b}}}{m\Omega} \times (\mu \nabla B + m v_\parallel^2 \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \equiv v_\parallel \hat{\mathbf{b}} + \epsilon_B \mathbf{v}_B \tag{1}$$

is decomposed in terms of a parallel velocity $v_\parallel \equiv \hat{\mathbf{b}} \cdot \dot{\mathbf{X}}$ along a field line and a slower cross-field drift velocity $\mathbf{v}_B$ due to weak magnetic-field nonuniformity. The guiding-center parallel acceleration,

$$\dot{v}_\parallel = -\frac{\mu}{m} \left(\hat{\mathbf{b}} + \epsilon_B \frac{\mathbf{v}_B}{v_\parallel} \right) \cdot \nabla B, \tag{2}$$

guarantees that the guiding-center kinetic energy,

$$\mathcal{E} = \frac{m}{2}v_\parallel^2 + \mu B \equiv \frac{p^2}{2m}, \tag{3}$$

is a constant of motion ($\dot{\mathcal{E}} \equiv 0$ for a time-independent magnetic field and in the absence of an electric field). In Eqs. (1) and (2), the guiding-center magnetic-moment invariant μ and the coordinates $(\mathbf{X}, v_\parallel)$ in the reduced four-dimensional guiding-center phase space are expressed as asymptotic expansions in powers of the small parameter ϵ_B, where first-order corrections explicitly take into account magnetic-field nonuniformity.[5]

We note that, while a nonrelativistic guiding-center formulation is considered here, its generalization to a relativistic formulation (appropriate for fast electrons) can easily be accommodated.[10] The kinetic energy (3) is thus replaced with $\mathcal{E} = (\gamma - 1)mc^2$, the parallel velocity $v_\parallel$ is replaced with the relativistic parallel momentum $p_\parallel = \gamma m v_\parallel$, and the magnetic moment μ is replaced with the relativistic magnetic moment $\mu = |\mathbf{p}_\perp|^2 / 2mB$, where the relativistic factor is $\gamma = (1 + 2\mu B/mc^2 + p_\parallel^2/m^2c^2)^{1/2}$. Additional details on relativistic guiding-center dynamics in axisymmetric magnetic geometry can be found in Ref. 11.

A. Axisymmetric magnetic geometry

The general axisymmetric magnetic field considered in the present paper is expressed in terms of three equivalent representations:[12]

$$\mathbf{B} = \begin{cases} \nabla\phi \times \nabla\psi + q(\psi)\,\nabla\psi \times \nabla\theta, \\ B_\phi \nabla\phi + B_\theta \nabla\theta + B_\psi \nabla\psi, \\ B^\phi \,\partial\mathbf{X}/\partial\phi + B^\theta \,\partial\mathbf{X}/\partial\theta, \end{cases} \tag{4}$$

where the two-covariant, covariant, and contravariant representations (from top to bottom, respectively) are expressed in terms of the (poloidal) magnetic flux ψ, which satisfies the condition $\mathbf{B} \cdot \nabla\psi = 0$ (i.e., magnetic field lines lie entirely on a constant-ψ surface), and the poloidal and toroidal angles θ and ϕ. The toroidal and poloidal components of the magnetic field (4) are $B_{\text{tor}} = B_\phi/R = B^\phi R$, where $R \equiv |\partial\mathbf{X}/\partial\phi| = |\nabla\phi|^{-1}$, and $B_{\text{pol}} = |\nabla\psi|/R = B^\theta |\partial\mathbf{X}/\partial\theta|$. The safety factor $q(\psi)$ appearing in the two-covariant representation is defined as[13]

$$q(\psi) \equiv \frac{\mathbf{B} \cdot \nabla\phi}{\mathbf{B} \cdot \nabla\theta} = \frac{B^\phi}{B^\theta}. \tag{5}$$

We note that, because of the magnetic-flux condition $\mathbf{B} \cdot \nabla\psi = 0$, the covariant component $B_\psi \equiv -B_\theta(\nabla\theta \cdot \nabla\psi)/|\nabla\psi|^2$ in Eq. (4) vanishes only if the coordinates ψ and θ are orthogonal. The spatial Jacobian $\mathcal{V}$ associated with the coordinates (ψ, θ, ϕ) is

$$\mathcal{V} \equiv (\nabla\psi \times \nabla\theta \cdot \nabla\phi)^{-1} = (\mathbf{B} \cdot \nabla\theta)^{-1} = (B^\theta)^{-1}, \tag{6}$$

where B^θ is assumed to be positive. Lastly, the infinitesimal length element along a magnetic field line is

$$ds \equiv \frac{B}{B^\theta}d\theta. \tag{7}$$

Note that, according to the standard axisymmetric tokamak

ordering (with $B \simeq B_{\text{tor}}$), the ratio B/B^θ can also be expressed as $B/B^\theta = B/[B_{\text{tor}}/(qR)] \simeq qR$, so that we recover the standard approximation $ds \simeq qRd\theta$.

B. Guiding-center motion in axisymmetric magnetic geometry

The guiding-center motion in arbitrary magnetic geometry described by Eqs. (1) and (2) possesses two constants of the motion: the total guiding-center energy (3) and the guiding-center magnetic moment μ. Guiding-center motion in general axisymmetric magnetic geometry (4) is also characterized, according to Noether's theorem, by a third constant of the motion: the toroidal canonical guiding-center momentum

$$P_\phi \equiv \frac{\partial\mathbf{X}}{\partial\phi} \cdot \left(\frac{e}{c}\mathbf{A} + mv_\parallel \hat{\mathbf{b}}\right) = -\frac{e}{c}(\psi - \rho_\parallel B_\phi). \tag{8}$$

Here, the vector potential $\mathbf{A} = -\psi\nabla\phi + \psi_{\text{tor}}(\psi)\nabla\theta$ (with $q \equiv d\psi_{\text{tor}}/d\psi$ defined in terms of the toroidal magnetic flux ψ_{tor}) was obtained from the two-covariant representation in Eq. (4), where the coordinates (ψ, θ, ϕ) now describe the guiding-center position $\mathbf{X}$, $\rho_\parallel \equiv v_\parallel/\Omega$ denotes the parallel gyroradius, and $B_\phi \equiv \mathbf{B} \cdot \partial\mathbf{X}/\partial\phi$ denotes the covariant component of the axisymmetric magnetic field.

The projection of the two-dimensional drift surface,[2]

$$\psi - \rho_\parallel B_\phi \equiv \bar{\psi}, \tag{9}$$

onto the poloidal plane (X, Z) generates a closed curve $\psi = \tilde{\psi}(\theta)$ parametrized by the poloidal angle θ and labeled by the guiding-center invariants $(\bar{\psi}, \mathcal{E}, \mu)$:

$$\tilde{\psi}(\theta, \sigma; \bar{\psi}, \mathcal{E}, \mu) \equiv \bar{\psi} + \delta\psi(\theta, \sigma; \bar{\psi}, \mathcal{E}, \mu), \tag{10}$$

where the "bounce-radius" $\delta\psi = \tilde{\psi} - \bar{\psi}$ represents the departure of the drift surface (10), labeled by $\bar{\psi}$, from a magnetic surface ψ and $\sigma = \pm 1$ denotes the sign of $v_\parallel$. A guiding-center orbit $\mathcal{O}$ is obtained either by integrating the guiding-center equations of motion,

$$\dot{\psi} \equiv \dot{\mathbf{X}} \cdot \nabla\psi = v_\parallel \hat{\mathbf{b}} \cdot \nabla\psi + \epsilon_B \mathbf{v}_B \cdot \nabla\psi \equiv \epsilon_B \dot{\psi}_B$$

$$\dot{\theta} \equiv \dot{\mathbf{X}} \cdot \nabla\theta = v_\parallel \hat{\mathbf{b}} \cdot \nabla\theta + \epsilon_B \mathbf{v}_B \cdot \nabla\theta \equiv v_\parallel B^\theta/B + \epsilon_B \dot{\theta}_B, \tag{11}$$

for a given set of guiding-center invariants $(\bar{\psi}, \mathcal{E}, \mu)$ or generating the orbit directly by the constant-of-motion method.[14] For each *generic* set of guiding-center invariants $(\bar{\psi}, \mathcal{E}, \mu)$, there corresponds a unique guiding-center orbit $\mathcal{O}$, which is either a *trapped-particle* orbit (if the bounce-radius $\delta\psi$ vanishes along the orbit), or a *passing-particle* orbit (if the bounce-radius $\delta\psi$ does not vanish). Nongeneric orbits[15] include the *stagnation* orbits (where $\dot{\theta} = 0$ or $v_\parallel B^\theta/B = -\epsilon_B \dot{\theta}_B \neq 0$) and the *pinch* orbits (barely trapped orbits in the zero-banana-width limit).

On each flux-surface ψ, the magnetic field amplitude is assumed to vary monotonically between a minimum value $B_0(\psi)$ and a maximum value $B_1(\psi)$, located at the poloidal locations $\theta_0(\psi)$ and $\theta_1(\psi)$, respectively. The turning points

$(\psi_b, \theta_b^{\pm})$ of trapped-particle orbits $(\bar{\psi}, \mathcal{E}, \mu)$ are located on the drift-surface label $\psi_b = \bar{\psi}$, where the magnetic field reaches its maximum value along the orbit and $\delta\psi(\theta_b^{\pm}, \pm 1; \psi_b, \mathcal{E}, \mu) \equiv 0$. Consequently, the **X** point separating trapped-particle and passing-particle orbits parametrized by the drift-surface label $\bar{\psi}$ is located at the position $[\bar{\psi}, \theta_1(\bar{\psi})]$.

While the magnetic moment μ is an important invariant for guiding-center dynamics, the pitch-angle coordinate

$$\xi(\tilde{\psi}, \sigma; \mathcal{E}, \mu) = v_{\parallel}/v = \sigma\sqrt{1 - \mu B(\tilde{\psi})/\mathcal{E}} \qquad (12)$$

is better suited in describing the transition between trapped-particle and passing-particle orbits. In order to convert the pitch-angle coordinate into a suitable guiding-center invariant, we replace the guiding-center magnetic-moment invariant μ with[16]

$$
\xi_0(\tilde{\psi}, \sigma; \mathcal{E}, \mu)
$$
$$
\equiv \begin{cases} \sqrt{1 - \mu B_0(\bar{\psi})/\mathcal{E}} & \text{(for trapped-particle orbits)}, \\ \sigma\sqrt{1 - \mu B_0(\bar{\psi})/\mathcal{E}} & \text{(for passing-particle orbits)}. \end{cases}
$$
$$(13)$$

The physical interpretation of the pitch-angle invariant ξ_0 can be given in terms of its connection with the bounce-action invariant J_b. With this definition, the trapped-passing separatrix given by the relation

$$1 - (1 - \xi_0^2)B_1(\bar{\psi})/B_0(\bar{\psi}) = 0,$$

which is an even function of ξ_0 and does not depend upon the energy $\mathcal{E}$ (important for numerical applications).

The guiding-center Jacobian in coordinates $(\mathbf{X}, p, \xi, \zeta_g)$ is $\mathcal{J}_{gc} \equiv p^2$, while in terms of the coordinates $(\mathbf{X}, p, \xi_0, \zeta_g)$, it is $\mathcal{J}_{gc0} \equiv p^2|\partial\xi/\partial\xi_0| = p^2\Psi|\xi_0/\xi|$, where we used the definition $\Psi(\tilde{\psi}, \theta) \equiv B(\tilde{\psi}, \theta)/B_0(\bar{\psi})$ with the relation

$$|\xi| = \sqrt{1 - \Psi(1 - \xi_0^2)}. \qquad (14)$$

Next, the spatial Jacobian in coordinates $\mathbf{X} \equiv (\bar{\psi}, \theta, \phi)$ is $\mathcal{V} = [B^{\theta}(\tilde{\psi})]^{-1}$, so that the total guiding-center Jacobian in coordinates $(\bar{\psi}, \theta, \phi; p, \xi_0, \zeta_g)$ is

$$\mathcal{J} \equiv \mathcal{V}\mathcal{J}_{gc0} = p^2\frac{\Psi}{B^{\theta}}\left|\frac{\xi_0}{\xi}\right| = \frac{p^2}{B_0}\left(\frac{B}{B^{\theta}}\left|\frac{\xi_0}{\xi}\right|\right). \qquad (15)$$

In the next section, we express a general guiding-center Fokker–Planck operator in terms of the guiding-center invariants $(\bar{\psi}, p, \xi_0)$ in a form that will be suitable for orbit averaging in Sec. IV.

III. GENERAL GUIDING-CENTER FOKKER–PLANCK EQUATION

The Fokker–Planck equation[2,3,17] forms a paradigm for the investigation of classical, neoclassical, and quasilinear transport processes in plasmas. When written in terms of the test-particle phase-space coordinates $(\mathbf{x}, \mathbf{p})$, the general Fokker–Planck operator is expressed as a local partial-differential operator in momentum space:[2]

$$\mathcal{C}[f](\mathbf{x}, \mathbf{p}) = -\frac{\partial}{\partial\mathbf{p}} \cdot \left(\mathbf{K}f(\mathbf{x}, \mathbf{p}) - \mathbf{D} \cdot \frac{\partial f(\mathbf{x}, \mathbf{p})}{\partial\mathbf{p}}\right), \qquad (16)$$

where the Fokker–Planck momentum-friction vector **K** and the momentum-diffusion tensor **D** are functions of $(\mathbf{x}, \mathbf{p})$. When representing collisional transport processes, for example, these Fokker–Planck coefficients $\mathbf{K} \equiv \Sigma'\mathbf{K}[f']$ and $\mathbf{D} \equiv \Sigma'\mathbf{D}[f']$ are expressed as integral operators acting on the field-particle distribution f' (where the field-particle species may coincide with the test-particle species). Hence, the Fokker–Planck operator (16) may either be a linear, bilinear, or nonlinear operator acting f, depending on the type of transport problem one wishes to investigate.

While collisions and wave-particle interactions take place locally in particle phase space $\mathbf{z} = (\mathbf{x}, \mathbf{p})$, the transformation to reduced phase-space coordinates $\bar{\mathbf{z}}$ (e.g., guiding-center coordinates) will generically introduce transport coefficients in the full reduced phase space. This implies that the Fokker–Planck coefficients (K^i, D^{ij}) in three-dimensional particle-momentum space are replaced with six-dimensional reduced phase-space Fokker-Planck coefficients $(\bar{K}^{\alpha}, \bar{D}^{\alpha\beta})$, defined as

$$\bar{K}^{\alpha}(\bar{\mathbf{z}}) \equiv \left[K^i(\mathbf{z})\frac{\partial\bar{z}^{\alpha}(\mathbf{z})}{\partial p^i}\right]_{\mathbf{z}=\mathbf{z}(\bar{\mathbf{z}})} \quad \text{and}$$
$$(17)$$
$$\bar{D}^{\alpha\beta}(\bar{\mathbf{z}}) \equiv \left[\frac{\partial\bar{z}^{\alpha}(\mathbf{z})}{\partial p^i}D^{ij}(\mathbf{z})\frac{\partial\bar{z}^{\beta}(\mathbf{z})}{\partial p^j}\right]_{\mathbf{z}=\mathbf{z}(\bar{\mathbf{z}})},$$

and the Fokker–Planck operator (16) transforms to

$$\overline{\mathcal{C}}[\bar{f}](\bar{\mathbf{z}}) = -\frac{1}{\overline{\mathcal{J}}}\frac{\partial}{\partial\bar{z}^{\alpha}}\left[\overline{\mathcal{J}}\left(\bar{K}^{\alpha}\bar{f} - \bar{D}^{\alpha\beta}\frac{\partial\bar{f}}{\partial\bar{z}^{\beta}}\right)\right], \qquad (18)$$

where $\bar{f}(\bar{\mathbf{z}}) \equiv f(\mathbf{z})$ is the particle distribution expressed in terms of the reduced phase-space coordinates and $\overline{\mathcal{J}}$ is the Jacobian for the transformation $\mathbf{z} \to \bar{\mathbf{z}}$. The most important aspect of the transformation to Eq. (18) involves the choice of the new phase-space coordinates $\bar{\mathbf{z}}$. Note that the definitions (17) for the reduced phase-space Fokker–Planck coefficients require that the transformation $\mathbf{z} \to \bar{\mathbf{z}}(\mathbf{z})$ and its inverse $\bar{\mathbf{z}} \to \mathbf{z}(\bar{\mathbf{z}})$ must be known up to the desired order in magnetic-field nonuniformity.

One possible choice is to adopt a canonical action-angle formulation,[18] where $\bar{\mathbf{z}} = (\bar{\mathbf{J}}, \bar{\boldsymbol{\theta}})$ includes the three-dimensional action invariants $\bar{\mathbf{J}}$ for magnetically confined particles and their canonically conjugate angles $\bar{\boldsymbol{\theta}}$ (which are ignorable coordinates by construction $\partial\bar{f}/\partial\bar{\boldsymbol{\theta}} \equiv 0$). While the three-dimensional action-space Fokker–Planck operator derived by Bernstein and Molvig[18] formally describes classical and neoclassical transport processes in axisymmetric magnetic geometry, it is not suitable for numerical implementation since some of the action coordinates (e.g., the bounce action J_b) are not *local* coordinates. Hence, the action-space Fokker–Planck operator does not describe local transport processes, which makes the transport analysis of its results difficult to interpret.

Another choice is to adopt *local* noncanonical guiding-center coordinates[7,16] leading to the construction of a re-

duced guiding-center Fokker–Planck operator. In a weakly nonuniform plasma with a strong axisymmetric magnetic field, for example, the reduced guiding-center Fokker–Planck equation describes transport processes in a four-dimensional space: the poloidal-flux and poloidal-angle coordinates (ψ, θ) in physical space and the energy and magnetic moment coordinates $(\mathcal{E}, \mu)$ in velocity space, which are invariants for the guiding-center motion. Here, the reduction from six to four dimensions is associated with the fact that the guiding-center plasma distribution is independent of the gyroangle ζ_g (by definition) and the toroidal angle ϕ (by axisymmetry), so that transport along these ignorable angles is irrelevant. While Zaitsev *et al.*[16] considered the lowest-order definitions (in magnetic-field nonuniformity) for their choice of guiding-center coordinates, Brizard[7] considered first-order corrections ($\epsilon_B = \rho/L_B \ll 1$) as well. In both works, classical collisional transport appears in Eq. (17) as a result of the lowest-order relation $\partial \mathbf{X}/\partial \mathbf{p} = -\hat{\mathbf{b}} \times \mathbf{I}/m\Omega$ for the guiding-center position $\mathbf{X}(\mathbf{x}, \mathbf{p}) \equiv \mathbf{x} - \boldsymbol{\rho}_0(\mathbf{x}, \mathbf{p})$, where $\boldsymbol{\rho}_0 \equiv \hat{\mathbf{b}} \times \mathbf{p}/m\Omega$ represents the lowest-order gyroradius. The higher-order guiding-center corrections kept by Brizard[7] are consistent with the low-collisionality regime, where the mean-free-path $\lambda_\nu > L_B$ is longer than the magnetic-nonuniformity length scale L_B.

A. Guiding-center Fokker–Planck operator

The derivation of a reduced Fokker–Planck operator that is suitable for numerical implementation must begin with finding local invariant coordinates that also allow relative computational simplicity for realistic magnetic geometries. In the present work, the exact invariants associated with the time-independent axisymmetric magnetic geometry are the kinetic momentum $p = \sqrt{2m\mathcal{E}}$ and the drift-surface label $\bar{\psi} = \psi_b$ (for trapped particles) or ψ_t (for passing particles). As our third invariant coordinate, we use the pitch-angle invariant ξ_0, defined in Eq. (13), which allows an explicit representation of the trapping and detrapping transport processes.

Once a set of invariant guiding-center coordinates $I^a = (\bar{\psi}, p, \xi_0)$ is chosen (not necessarily action coordinates), the transformation of the Fokker–Planck operator (16) can be greatly simplified by writing it in Poisson-bracket form as

$$\mathcal{C}[f] = \equiv -\{x^i, (K^i f - D^{ij}\{x^j, f\})\}, \tag{19}$$

where the noncanonical Poisson bracket $\{\cdots\}$ is used to replace momentum partial derivatives $\partial g/\partial p_i \equiv \{x^i, g\}$. The significant computational advantage of this Poisson-bracket formulation (19) is based on the fact that Poisson brackets transform naturally under the type of phase-space transformations considered in the present work (i.e., those generated by Lie-transform methods).

The general guiding-center Fokker–Planck operator was derived by Brizard[7] as a result of the guiding-center dynamical reduction of the Fokker–Planck operator (16):

$$\mathcal{C}_{gc}[F] \equiv \langle \mathbf{T}_{gc}^{-1}\mathcal{C}[\mathbf{T}_{gc}F]\rangle_g$$
$$= -\langle \{X^i + \rho_\epsilon^i, (\mathbf{T}_{gc}^{-1}K^i F - \mathbf{T}_{gc}^{-1}D^{ij}\{X^j + \rho_\epsilon^j, F\}_{gc})\}_{gc}\rangle_g, \tag{20}$$

where $\mathbf{T}_{gc}$ and $\mathbf{T}_{gc}^{-1}$ denote the pull-back and push-forward (Lie-transform) operators associated with the guiding-center transformation for a nonuniform magnetic field, $\langle \cdots \rangle_g$ denotes an average with respect to the guiding-center gyroangle ζ_g, and $\{\cdots\}_{gc}$ denotes the guiding-center Poisson bracket. In Eq. (20), the generalized gyroradius vector $\boldsymbol{\rho}_\epsilon$ contains first-order corrections associated with magnetic-field nonuniformity.

When we use the phase-space-divergence property of Poisson bracket

$$\{F, G\}_{gc} \equiv \frac{1}{\mathcal{J}}\frac{\partial}{\partial Z^\alpha}(\mathcal{J}\{F, Z^\alpha\}_{gc}G),$$

where F and G are two arbitrary guiding-center phase-space functions, we can write the divergence form of the guiding-center Fokker–Planck operator (20):

$$\mathcal{C}_{gc}[F] \equiv -\frac{1}{\mathcal{J}}\frac{\partial}{\partial Z^\alpha}\left[\mathcal{J}\left(\mathcal{K}_{gc}^\alpha F - \mathcal{D}_{gc}^{\alpha\beta}\frac{\partial F}{\partial Z^\beta}\right)\right], \tag{21}$$

where $\mathcal{J}$ is the total Jacobian (15). The guiding-center Fokker–Planck friction components,

$$\mathcal{K}_{gc}^\alpha \equiv \langle \mathbf{T}_{gc}^{-1}\mathbf{K} \cdot \boldsymbol{\Delta}^\alpha \rangle_g, \tag{22}$$

and the guiding-center Fokker–Planck diffusion components,

$$\mathcal{D}_{gc}^{\alpha\beta} \equiv \langle (\boldsymbol{\Delta}^\alpha)^{\mathsf{T}} \cdot \mathbf{T}_{gc}^{-1}\mathbf{D} \cdot \boldsymbol{\Delta}^\beta \rangle_g, \tag{23}$$

are expressed in terms of the guiding-center push-forward expressions $\mathbf{T}_{gc}^{-1}\mathbf{K}$ and $\mathbf{T}_{gc}^{-1}\mathbf{D}$ of the particle momentum-space Fokker–Planck friction vector $\mathbf{K}$ and diffusion tensor $\mathbf{D}$, and the vector-valued *projection* coefficients

$$\boldsymbol{\Delta}^\alpha \equiv \{\mathbf{X} + \boldsymbol{\rho}_\epsilon, Z^\alpha\}_{gc} = \boldsymbol{\Delta}^{\mathbf{X}} \cdot \nabla Z^\alpha + \boldsymbol{\Delta}^{\mathcal{E}}\frac{\partial Z^\alpha}{\partial \mathcal{E}} + \boldsymbol{\Delta}^\mu\frac{\partial Z^\alpha}{\partial \mu}. \tag{24}$$

Here, the guiding-center vector-valued projection coefficients in $(\mathbf{X}, \mathcal{E}, \mu)$ space,[7]

$$\boldsymbol{\Delta}^{\mathbf{X}} = \hat{\mathbf{b}} \times \mathbf{I}/(m\Omega) = -(\boldsymbol{\Delta}^{\mathbf{X}})^{\mathsf{T}}$$

$$\boldsymbol{\Delta}^{\mathcal{E}} = \mathbf{p}_\epsilon/m \tag{25}$$

$$\boldsymbol{\Delta}^\mu = (\Omega/B)\,\partial\boldsymbol{\rho}_\epsilon/\partial\zeta_g,$$

are constructed from the guiding-center Poisson-bracket expressions involving the guiding-center push-forward of the particle position $\mathbf{T}_{gc}^{-1}\mathbf{x} \equiv \mathbf{X} + \boldsymbol{\rho}_\epsilon$, where the generalized gyroradius $\boldsymbol{\rho}_\epsilon \equiv \mathbf{T}_{gc}^{-1}\boldsymbol{\rho}$ and the generalized guiding-center momentum

$$\mathbf{p}_\epsilon \equiv \mathbf{T}_{gc}^{-1}\mathbf{p} = m\left(\frac{d_{gc}\mathbf{X}}{dt} + \frac{d_{gc}\boldsymbol{\rho}_\epsilon}{dt}\right) \tag{26}$$

contain first-order corrections associated with the guiding-center transformation. Note that, while the projection coefficients $\boldsymbol{\Delta}^{\mathcal{E}}$ and $\boldsymbol{\Delta}^\mu$ in Eq. (25) retain first-order corrections in magnetic-field nonuniformity, we omitted first-order correc-

tions in $\mathbf{\Delta^X}$ since they yield second-order contributions in the final expressions for Eq. (21).

B. Guiding-center projection vectors

Using the definitions for the invariant-space coordinates $I^a = (\bar{\psi}, p, \xi_0)$, the guiding-center vector-valued projection coefficients $\mathbf{\Delta}^a = (\mathbf{\Delta}^{\bar{\psi}}, \mathbf{\Delta}^p, \mathbf{\Delta}^{\xi_0})$ are expressed as[7]

$$\mathbf{\Delta}^{\bar{\psi}} \equiv \mathbf{\Delta^X} \cdot \nabla \bar{\psi} = \frac{\hat{\mathbf{b}}}{m\Omega} \times \nabla \bar{\psi}, \tag{27}$$

$$\mathbf{\Delta}^p \equiv \mathbf{\Delta}^{\mathcal{E}} \frac{\partial p}{\partial \mathcal{E}} = \frac{\mathbf{p}_\epsilon}{|\mathbf{p}_\epsilon|}, \tag{28}$$

$$\mathbf{\Delta}^{\xi_0} \equiv \mathbf{\Delta^X} \cdot \nabla \xi_0 + \mathbf{\Delta}^{\mathcal{E}} \frac{\partial \xi_0}{\partial \mathcal{E}} + \mathbf{\Delta}^\mu \frac{\partial \xi_0}{\partial \mu}$$
$$= \left(\frac{1 - \xi_0^2}{2\xi_0} \right) \left[\left(\frac{\mathbf{p}_\epsilon}{m\mathcal{E}} - \frac{\Omega}{\mu B} \frac{\partial \boldsymbol{\rho}_\epsilon}{\partial \zeta_g} \right) - \mathbf{\Delta}^{\bar{\psi}} \frac{d \ln B_0}{d\bar{\psi}} \right]. \tag{29}$$

We now discuss the physical nature of the projection vectors (27)–(29).

1. Guiding-center radial projection

Using the magnetic representation (4), the guiding-center radial projection vector (27) becomes

$$\mathbf{\Delta}^{\bar{\psi}} = \frac{\hat{\mathbf{b}}}{m\Omega} \times \nabla \bar{\psi}$$
$$= -\frac{c}{eB^2} [|\nabla \bar{\psi}|^2 \nabla \phi - q(\nabla \bar{\psi} \times \nabla \theta) \times \nabla \bar{\psi}]$$
$$\equiv -\frac{c}{e} \frac{|\nabla \bar{\psi}|^2}{B^2} \nabla_\psi \chi, \tag{30}$$

where we introduced the anglelike coordinate $\chi \equiv \phi - q(\bar{\psi})\theta$, such that the two-covariant representation of the axisymmetric magnetic field (4) becomes $\mathbf{B} \equiv \nabla \chi \times \nabla \bar{\psi}$. Here, $\nabla_\psi \theta \equiv (\nabla \bar{\psi} \times \nabla \theta) \times \nabla \bar{\psi} / |\nabla \bar{\psi}|^2$ denotes the projection of the gradient of θ that is perpendicular to $\nabla \bar{\psi}$, while $\nabla_\psi \phi \equiv \nabla \phi$ (since $\nabla \phi \cdot \nabla \bar{\psi} \equiv 0$). Hence, the guiding-center radial projection vector (30) projects momentum-space transport processes onto a magnetic-flux surface in a direction that is locally perpendicular to $\hat{\mathbf{b}}$.

The guiding-center radial projection vector $\mathbf{\Delta}^{\bar{\psi}}$, whose magnitude is

$$|\mathbf{\Delta}^{\bar{\psi}}| \equiv \frac{|\nabla \bar{\psi}|}{m\Omega} = \frac{cB_{\text{pol}}}{eB} R,$$

therefore generates the local projection of the guiding-center push-forward of the momentum-space Fokker–Planck friction,

$$\mathcal{K}_{gc}^{\bar{\psi}} = \langle \mathbf{T}_{gc}^{-1} \mathbf{K} \rangle_g \cdot \frac{\hat{\mathbf{b}}}{m\Omega} \times \nabla \bar{\psi}, \tag{31}$$

and the momentum-space Fokker–Planck diffusion

$$\mathcal{D}_{gc}^{a\bar{\psi}} = \langle \mathbf{\Delta}^a \cdot \mathbf{T}_{gc}^{-1} \mathbf{D} \rangle_g \cdot \frac{\hat{\mathbf{b}}}{m\Omega} \times \nabla \bar{\psi}, \tag{32}$$

$$\mathcal{D}_{gc}^{\bar{\psi}\bar{\psi}} = \frac{\hat{\mathbf{b}}}{m\Omega} \times \nabla \bar{\psi} \cdot \langle \mathbf{T}_{gc}^{-1} \mathbf{D} \rangle_g \cdot \frac{\hat{\mathbf{b}}}{m\Omega} \times \nabla \bar{\psi}, \tag{33}$$

where $a \neq \bar{\psi}$ in Eq. (32). These projections only take into account transport processes occurring within a magnetic-flux surface. The connection between radial transport and toroidal and/or poloidal rotation is therefore naturally contained within this projection.

2. Guiding-center energy projection

The guiding-center energy projection vector (28) is expressed as

$$\mathbf{\Delta}^p \equiv \frac{\mathbf{p}_\epsilon}{|\mathbf{p}_\epsilon|} = \hat{\mathbf{p}} + \epsilon \mathbf{\Delta}_1^p + \cdots, \tag{34}$$

where $\mathbf{\Delta}_1^p$ denotes the first-order correction of $\mathbf{\Delta}^p$. It can be explicitly expressed in terms of the gyroangle-independent first-order gyroradius $\bar{\boldsymbol{\rho}}_{gc} \equiv (\hat{\mathbf{b}}/\Omega) \times \mathbf{v}_B$ and $\boldsymbol{\rho}_\epsilon \equiv \boldsymbol{\rho}_0 + \epsilon \boldsymbol{\rho}_1$ as[7]

$$\mathbf{\Delta}_1^p = \frac{1}{\rho} \left(\bar{\boldsymbol{\rho}}_{gc} \times \hat{\mathbf{b}} + \frac{\partial \boldsymbol{\rho}_1}{\partial \zeta_g} \right) + \xi \hat{\mathbf{b}} \cdot \nabla^* \boldsymbol{\rho}_0 \equiv \Delta_{1\xi}^p \frac{\partial \hat{\mathbf{p}}}{\partial \xi} + \Delta_{1\zeta}^p \frac{\partial \hat{\mathbf{p}}}{\partial \zeta_g}, \tag{35}$$

where $\rho \equiv p/(m\Omega)$ is the gyroradius magnitude for a deeply trapped particle (i.e., for $\xi_0 = 0$) and

$$\frac{1}{\rho} \bar{\boldsymbol{\rho}}_{gc} \times \hat{\mathbf{b}} = \rho \hat{\mathbf{b}} \times \left[\frac{1}{2}(1 - \xi^2) \nabla \ln B + \xi^2 \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right].$$

It can be shown explicitly, however, that $\mathbf{\Delta}_1^p$ does not have any component directed along the zeroth-order unit vector $\hat{\mathbf{p}}$ (because the guiding-center kinetic energy is identical to the particle kinetic energy). Hence, the magnitude of the guiding-center energy projection vector (28) is $|\mathbf{\Delta}^p| \equiv 1$ up to second order in magnetic-field nonuniformity.

3. Guiding-center pitch-angle projection

The guiding-center pitch-angle projection vector (29) is expressed as

$$\mathbf{\Delta}^{\xi_0} = \frac{\sqrt{1 - \xi_0^2}}{p\xi_0} \left[\frac{\xi}{\sqrt{\Psi}} \hat{\xi} + \epsilon \sqrt{1 - \xi_0^2} \left(\mathbf{\Delta}_1^p - \frac{(\partial \boldsymbol{\rho}_1 / \partial \zeta_g)}{\rho_0(1 - \xi_0^2)} \right. \right.$$
$$\left. \left. - \frac{\rho}{2} \hat{\mathbf{b}} \times \nabla \ln B_0 \right) \right] \equiv \frac{\sqrt{1 - \xi_0^2}}{p\xi_0} \frac{\xi \hat{\xi}}{\sqrt{\Psi}} + \epsilon \mathbf{\Delta}_1^{\xi_0}, \tag{36}$$

where $\mathbf{\Delta}_1^{\xi_0}$ denotes the first-order correction of $\mathbf{\Delta}^{\xi_0}$. To lowest order in magnetic-field nonuniformity, we find $|\mathbf{\Delta}^{\xi_0}| = \sqrt{1 - \xi_0^2} \xi / (p \sqrt{\Psi} \xi_0)$.

C. Guiding-center Fokker–Planck equation

Now that the guiding-center Fokker–Planck components are expressed in terms of the invariant-space coordinates $I^a = (\bar{\psi}, p, \xi_0)$, we turn our attention to the guiding-center Fokker–Planck kinetic equation. When the original guiding-center coordinates $(\psi, \theta, \phi; \mathcal{E}, \mu, \zeta_g)$ are used in describing guiding-center dynamics in axisymmetric magnetic geometry (where we replace the guiding-center parallel velocity $v_\parallel$ with the guiding-center energy $\mathcal{E}$ and the gyroangle ζ_g is an ignorable coordinate),[19] the guiding-center Vlasov evolution operator is ordered as[5]

$$\frac{d_{gc}}{dt} \equiv \epsilon_\tau \frac{\partial}{\partial \tau} + \left(v_\parallel \frac{B^\theta}{B} + \epsilon_B \dot{\theta}_B \right) \frac{\partial}{\partial \theta} + \epsilon_B \dot{\psi}_B \frac{\partial}{\partial \psi}, \quad (37)$$

where we used $\partial/\partial\phi \equiv 0$ (by definition of axisymmetry), explicit time dependence is over long time scales (i.e., $\partial/\partial t \equiv \epsilon_\tau \partial/\partial \tau$ is ordered small), and the guiding-center dynamics on the poloidal plane is represented by Eq. (11).

For each generic set of guiding-center invariants $(\bar{\psi}, \mathcal{E}, \mu)$, the guiding-center Fokker–Planck kinetic equation is expressed as

$$\epsilon_\tau \frac{\partial F}{\partial \tau} + \dot{\theta} \left(\frac{\partial}{\partial \theta} + \epsilon_B \frac{\partial \delta\psi}{\partial \theta} \frac{\partial}{\partial \psi} \right) F = \epsilon_\nu \mathcal{C}_{gc}[F], \quad (38)$$

where the magnetic-flux drift motion

$$\dot{\psi} \equiv \frac{\partial \delta\psi}{\partial \theta} \dot{\theta} \quad (39)$$

is expressed in terms of the bounce radius (10) when it is projected onto the guiding-center orbit $\mathcal{O}$ associated with the guiding-center invariants $(\bar{\psi}, \mathcal{E}, \mu)$. While the topology of these orbits can be rather complicated,[14,15,20–22] our discussion will remain as general as possible concerning the nature of the axisymmetric magnetic geometry. Whether a guiding-center orbit $\mathcal{O}$ corresponds to a trapped-particle orbit or a passing-particle orbit, however, the motion is periodic in the poloidal plane (ψ, θ) and a generic orbital period in the poloidal plane,

$$\tau_\mathcal{O} \equiv \oint_\mathcal{O} \frac{d\theta}{\dot{\theta}}, \quad (40)$$

is defined as a closed-loop integral along the drift-orbit curve $\mathcal{O}$ parametrized by the poloidal angle θ (at constant invariants $\bar{\psi}$, $\mathcal{E}$, and μ). Note that the definition (40) of the orbital period can also be given in terms of $d\psi/\dot{\psi}$,[14] which, when evaluated along the guiding-center orbit $\mathcal{O}$, yields $(d\psi/\dot{\psi})_\mathcal{O} \equiv d\theta (\partial \delta\psi/\partial \theta)/\dot{\psi} = d\theta/\dot{\theta}$ upon using Eq. (39).

IV. ORBIT-AVERAGED GUIDING-CENTER FOKKER–PLANCK EQUATION IN LOW-COLLISIONALITY REGIME

A. Orbit-averaging operator in axisymmetric magnetic geometry and low collisionality regime

In the neoclassical transport regime $\epsilon_\nu = \nu \tau_\mathcal{O} \ll 1$, the guiding-center Fokker–Planck kinetic Eq. (38) reduces to leading order to $\dot{\theta} \partial F/\partial \theta = 0$, which states that

$F \equiv \bar{F}(\bar{\psi}, \sigma, \mathcal{E}, \mu; \tau)$ is independent of the poloidal angle θ. The physical picture is that guiding centers orbits undergo many poloidal cycles before being perturbed by collisions.

We now introduce the *orbit*-averaging operation

$$\langle \cdots \rangle_\mathcal{O} \equiv \frac{1}{\tau_\mathcal{O}} \oint_\mathcal{O} (\cdots) \frac{d\theta}{\dot{\theta}}, \quad (41)$$

where the orbital period $\tau_\mathcal{O}$ is defined in Eq. (40). By orbit-averaging the guiding-center Fokker–Planck kinetic Eq. (38) in the low-collisionality approximation, we finally obtain the orbit-averaged guiding-center Fokker–Planck kinetic equation

$$\epsilon_\tau \frac{\partial \bar{F}}{\partial \tau} = \epsilon_\nu \langle \mathcal{C}_{gc}[\bar{F}] \rangle_\mathcal{O}, \quad (42)$$

which describes the collisional time evolution of the orbit-averaged guiding-center distribution

$$\langle F \rangle_\mathcal{O} = \bar{F}(\bar{\psi}, \sigma, \mathcal{E}, \mu; \tau), \quad (43)$$

and the guiding-center Fokker–Planck operator is given by Eq. (21).

Lastly, we note that the relation between the magnetic flux ψ and its orbit-averaged value $\langle \psi \rangle_\mathcal{O}$ (the "drift-center" position) is expressed as[18]

$$\psi \equiv \langle \psi \rangle_\mathcal{O} + (\delta\psi - \langle \delta\psi \rangle_\mathcal{O}), \quad (44)$$

where the drift-surface label is $\bar{\psi} \equiv \langle \psi \rangle_\mathcal{O} - \langle \delta\psi \rangle_\mathcal{O}$. For trapped and passing particles, we therefore obtain the following relations for the deviations from a magnetic-flux surface:

$$\begin{pmatrix} \psi - \langle \psi \rangle_b \\ \psi - \langle \psi \rangle_t \end{pmatrix} \equiv \begin{pmatrix} \delta\psi \\ \delta\psi - \langle \delta\psi \rangle_t \end{pmatrix}, \quad (45)$$

where we used the fact that $\langle \delta\psi \rangle_b \equiv 0$ for trapped particles, while $\langle \delta\psi \rangle_t \neq 0$ for passing particles (since $\delta\psi$ does not change sign along a passing-particle orbit). These relations show that, while the deviation $\delta\psi$ for a trapped particle may be large, the deviation $(\delta\psi - \langle \delta\psi \rangle_t)$ for a passing particle is in general small.

B. Approximate orbit-averaging operator

In this section, an explicit expression for the orbit-averaged guiding-center Fokker–Planck operator that appears on the right side of Eq. (42) is derived. In general, the orbit-averaging operation (41) must be computed along numerically calculated orbits. However, useful analytical expressions of the orbit-averaged guiding-center Fokker–Planck operator can be obtained when the orbit deviation $\delta\psi/|\nabla\psi|$ is small as compared to the local minor radius r. The difference between the exact incremental time element $d\theta/\dot{\theta}$ and the approximate time element $ds/v_\parallel$ is of order ϵ_B^2 and can be neglected when compared to the corrections of order ϵ_B in the Fokker–Planck operator. The orbital period (40) thus becomes

$$\tau_{\mathcal{O}}(\bar{\psi},\sigma,\mathcal{E},\mu) \simeq \oint_{\mathcal{O}} \frac{ds}{v_{\parallel}} = \frac{1}{v}\oint_{\mathcal{O}} \frac{d\theta}{\xi(\tilde{\psi},\sigma;\mathcal{E},\mu)} \frac{B(\tilde{\psi},\theta)}{B^{\theta}(\tilde{\psi},\theta)}.$$

(46)

The orbit topology is explicitly taken into account in Eq. (46) through the dependence of the integrand $B/(\xi B^{\theta})$ on the bounce-radius $\delta\psi(\theta,\sigma;\bar{\psi},\mathcal{E},\mu) \equiv \tilde{\psi}-\bar{\psi}$ defined in Eq. (10), where $\bar{\psi}=\psi_b$ (for trapped particles) or ψ_t (for passing particles). Note that the orbital period (46) is exact for stagnation orbits (where $\dot{\theta}=0$ yields $\dot{\theta}_B = -v_{\parallel}B^{\theta}/B$).

In the definition of the orbital period (46), it is convenient to extract the magnetic-geometric factor

$$\Lambda(\bar{\psi},p,\xi_0) \equiv \int_{-\pi}^{\pi} d\theta \frac{B(\tilde{\psi},\theta)}{B^{\theta}(\tilde{\psi},\theta)},$$

(47)

which defines the length of an orbit on the drift-surface labeled by $\bar{\psi}$. In the standard axisymmetric tokamak ordering, we find $\Lambda \simeq 2\pi q(\bar{\psi})R_0$, where R_0 denotes the major radius of the magnetic axis. Next, we define the normalized orbital period

$$\lambda(\bar{\psi},p,\xi_0) \equiv \frac{1}{\Lambda}\oint_{\mathcal{O}} d\theta \frac{\xi_0 B}{\xi B^{\theta}} = \left(\frac{v|\xi_0|}{\Lambda}\right)\tau_{\mathcal{O}},$$

(48)

which satisfies the condition $\lambda(\psi_t,p,|\xi_0|=1)\equiv 1$ for completely passing particles (i.e., $|\xi|=1$ and $\mu=0$ so that $v_{\parallel}=v$ is a constant of motion). Note that the orbital period (46) [or normalized orbital period (48)] becomes infinite on the boundary that separates trapped-particle orbits and passing-particle orbits (i.e., pinch orbits). Hence, the low-collisionality approximation ($\partial F/\partial\theta=0$) does not technically hold very close to the trapped-passing boundary, where the bounce and transit periods become much larger than the characteristic collisional time scale ν^{-1}. However, the fraction of particles with $\lambda \gg 1$ is very small since the corresponding singularity is integrable. Indeed, by introducing ξ_{0b} such that $1-\Psi(1-\xi_{0b}^2)=0$, it can be shown that

$$\int_0^1 \lambda(\bar{\psi},p,\xi_0)d\xi_0 = \frac{1}{\Lambda}\int_{-\pi}^{\pi}\frac{B}{B^{\theta}}d\theta\int_{\xi_{0b}}^1 \frac{\xi_0 d\xi_0}{\sqrt{1-\Psi(1-\xi_0^2)}}$$

$$= \frac{1}{\Lambda}\int_{-\pi}^{\pi}\frac{B_0}{B^{\theta}}d\theta,$$

which is a geometrical factor of order unity.

Using the Jacobian (15), the normalized bounce period (48) is now expressed as

$$\lambda(\bar{\psi},p,\xi_0) = \frac{B_0}{\Lambda p^2}\oint_{\mathcal{O}} \mathcal{J}d\theta \equiv 2\pi\mathcal{J}_{\mathcal{O}}\frac{B_0}{\Lambda p^2},$$

(49)

so that the orbital period (46) becomes

$$\tau_{\mathcal{O}} = \frac{2\pi\mathcal{J}_{\mathcal{O}}B_0}{vp^2|\xi_0|}.$$

(50)

The orbit-averaging operation (41) becomes

$$\langle\cdots\rangle_{\mathcal{O}}(\bar{\psi},p,\xi_0) = \frac{1}{\tau_{\mathcal{O}}}\oint_{\mathcal{O}}(\cdots)\frac{ds}{v_{\parallel}}$$

$$= \frac{1}{\tau_{\mathcal{O}}}\oint_{\mathcal{O}}(\cdots)\frac{Bd\theta}{v\xi B^{\theta}}$$

$$= \frac{1}{\mathcal{J}_{\mathcal{O}}}\oint_{\mathcal{O}}(\cdots)\mathcal{J}\frac{d\theta}{2\pi}.$$

(51)

For trapped-particle orbits, the orbit-average (51) yields the explicit formula for bounce-averaging operation

$$\langle\cdots\rangle_b(\psi_b,p,\xi_0) \equiv \frac{1}{\tau_b}\sum_{\sigma}\int_{\theta_b^-}^{\theta_b^+}(\cdots)\frac{d\theta}{v|\xi|}\frac{B}{B^{\theta}}$$

$$= \frac{1}{\mathcal{J}_{\mathcal{O}}}\left[\frac{1}{2}\sum_{\sigma}\right]\int_{\theta_b^-}^{\theta_b^+}(\cdots)\mathcal{J}\frac{d\theta}{2\pi}.$$

(52)

Here, the summation is over the sign σ of ξ and, therefore, the symbol $(1/2)\Sigma_{\sigma}$ takes the average (for trapped particles) between values of the integrand for $\xi>0$ and values of the integrand for $\xi<0$. For passing-particle orbits, on the other hand, the orbit-average (51) yields the explicit formula for transit-averaging operation

$$\langle\cdots\rangle_t(\psi_t,p,\xi_0) \equiv \frac{1}{\mathcal{J}_{\mathcal{O}}}\int_0^{2\pi}(\cdots)\mathcal{J}\frac{d\theta}{2\pi}.$$

(53)

C. Orbit-averaged guiding-center Fokker–Planck operator

In the expression for the guiding-center Fokker–Planck operator (21), the guiding-center function $\bar{F}$ is independent of the angle coordinates (θ,ϕ,ζ_g), in the limit of weak collisionality and under the assumptions of axisymmetry and gyroangle invariance. Hence, only derivatives of the guiding-center distribution $\bar{F}$ with respect to the guiding-center invariants I^a will remain in Eq. (21). The coefficients $(\mathcal{K}_{gc}^a,\mathcal{D}_{gc}^{ab})$ and the Jacobian $\mathcal{J}$ in the guiding-center Fokker–Planck operator (21), on the other hand, still depend on the poloidal angle θ. We therefore need to orbit average the guiding-center Fokker–Planck operator (21) according to the orbit-averaging procedure (51).

The orbit-averaged guiding-center Fokker–Planck operator is expressed as

$$\langle\mathcal{C}_{gc}[\bar{F}]\rangle_{\mathcal{O}} \equiv -\frac{1}{\mathcal{J}_{\mathcal{O}}}\frac{\partial}{\partial I^a}\left[\mathcal{J}_{\mathcal{O}}\left(\langle\mathcal{K}_{gc}^a\rangle_{\mathcal{O}}\bar{F} - \langle\mathcal{D}_{gc}^{ab}\rangle_{\mathcal{O}}\frac{\partial\bar{F}}{\partial I^b}\right)\right],$$

(54)

where the averaged Jacobian $\mathcal{J}_{\mathcal{O}}$ now becomes the Jacobian for the orbit-averaged guiding-center Fokker–Planck operator (54). Orbit-averaged guiding-center Fokker–Planck operator that treat friction and diffusion in three-dimensional space have been derived previously.[16] While these previous operators retained only the lowest-order terms in magnetic-field nonuniformity, the orbit-averaged guiding-center Fokker–Planck operator (54) retains first-order corrections

as well, which allows a realistic magnetic geometry to be considered.

The orbit-averaged guiding-center Fokker–Planck operator is widely used in the physics of magnetized plasmas (earth magnetosphere,[4] thermonuclear fusion,[17,23] etc.) For practical applications (such as studying fast particle dynamics generated by rf waves or a constant electric field[24]), analytical expressions often correspond to an oversimplified description of the problem of interest, and therefore a more realistic approach requires full numerical calculations.

For computational purposes, the orbit-averaged guiding-center Fokker–Planck operator (54) is recast in a flux-conservative form

$$\langle \mathcal{C}_{gc}[\bar{F}]\rangle_{\mathcal{O}} \equiv -\frac{1}{\mathcal{J}_{\mathcal{O}}}\frac{\partial}{\partial\bar{\psi}}(\mathcal{J}_{\mathcal{O}}|\nabla\bar{\psi}|_0 S_{\mathsf{L}}^{\bar{\psi}}) - \frac{1}{\mathcal{J}_{\mathcal{O}}}\frac{\partial}{\partial p}(\mathcal{J}_{\mathcal{O}} S_{\mathsf{L}}^p)$$
$$+ \frac{1}{\mathcal{J}_{\mathcal{O}}p}\frac{\partial}{\partial\xi_0}(\mathcal{J}_{\mathcal{O}}\sqrt{1-\xi_0^2}\, S_{\mathsf{L}}^{\xi_0}), \tag{55}$$

such that the usual two-grid discretization technique may be applied for the finite-difference method.[25] Particle conservation is therefore naturally satisfied numerically up to second order in the truncation error. In Eq. (55), $S_{\mathsf{L}}^{\bar{\psi}}$ describes particle flux across magnetic flux surfaces, while S_{L}^p and $S_{\mathsf{L}}^{\xi_0}$ account for momentum and pitch-angle dynamics, respectively. From Eq. (54), we have

$$\begin{pmatrix} S_{\mathsf{L}}^{\bar{\psi}} \\ S_{\mathsf{L}}^p \\ S_{\mathsf{L}}^{\xi_0} \end{pmatrix} \equiv \begin{pmatrix} K_{\mathsf{L}}^{\bar{\psi}} \\ K_{\mathsf{L}}^p \\ K_{\mathsf{L}}^{\xi_0} \end{pmatrix}\bar{F} - \begin{pmatrix} D_{\mathsf{L}}^{\bar{\psi}\bar{\psi}} & D_{\mathsf{L}}^{\bar{\psi}p} & D_{\mathsf{L}}^{\bar{\psi}\xi_0} \\ D_{\mathsf{L}}^{p\bar{\psi}} & D_{\mathsf{L}}^{pp} & D_{\mathsf{L}}^{p\xi_0} \\ D_{\mathsf{L}}^{\xi_0\bar{\psi}} & D_{\mathsf{L}}^{\xi_0 p} & D_{\mathsf{L}}^{\xi_0\xi_0} \end{pmatrix}$$
$$\times \begin{pmatrix} |\nabla\bar{\psi}|_0\,\partial/\partial\bar{\psi} \\ \partial/\partial p \\ -p^{-1}\sqrt{1-\xi_0^2}\,\partial/\partial\xi_0 \end{pmatrix}\bar{F}, \tag{56}$$

where the friction coefficients K_{L}^a and symmetric diffusion coefficients $D_{\mathsf{L}}^{ab} = D_{\mathsf{L}}^{ba}$ may be expressed in terms of guiding-center friction and diffusion components (22) and (23) according to the general relations

$$K_{\mathsf{L}}^{\bar{\psi}} = \frac{1}{|\nabla\bar{\psi}|_0}\langle\mathcal{K}_{gc}^{\bar{\psi}}\rangle_{\mathcal{O}}, \tag{57}$$

$$K_{\mathsf{L}}^p = \langle\mathcal{K}_{gc}^p\rangle_{\mathcal{O}}, \tag{58}$$

$$K_{\mathsf{L}}^{\xi_0} = -\frac{p}{\sqrt{1-\xi_0^2}}\langle\mathcal{K}_{gc}^{\xi_0}\rangle_{\mathcal{O}}, \tag{59}$$

and

$$D_{\mathsf{L}}^{\bar{\psi}\bar{\psi}} = \frac{1}{|\nabla\bar{\psi}|_0^2}\langle\mathcal{D}_{gc}^{\bar{\psi}\bar{\psi}}\rangle_{\mathcal{O}}, \tag{60}$$

$$D_{\mathsf{L}}^{\bar{\psi}p} = \frac{1}{|\nabla\bar{\psi}|_0}\langle\mathcal{D}_{gc}^{\bar{\psi}p}\rangle_{\mathcal{O}}, \tag{61}$$

$$D_{\mathsf{L}}^{\bar{\psi}\xi_0} = -\frac{p}{\sqrt{1-\xi_0^2}}\frac{1}{|\nabla\bar{\psi}|_0}\langle\mathcal{D}_{gc}^{\bar{\psi}\xi_0}\rangle_{\mathcal{O}}, \tag{62}$$

$$D_{\mathsf{L}}^{pp} = \langle\mathcal{D}_{gc}^{pp}\rangle_{\mathcal{O}}, \tag{63}$$

$$D_{\mathsf{L}}^{p\xi_0} = -\frac{p}{\sqrt{1-\xi_0^2}}\langle\mathcal{D}_{gc}^{p\xi_0}\rangle_{\mathcal{O}}, \tag{64}$$

$$D_{\mathsf{L}}^{\xi_0\xi_0} = \frac{p^2}{1-\xi_0^2}\langle\mathcal{D}_{gc}^{\xi_0\xi_0}\rangle_{\mathcal{O}}. \tag{65}$$

The numerical implementation of the orbit-averaged guiding-center Fokker–Planck operator (55) is described in a forthcoming paper[25] using a novel three-dimensional scheme with fully implicit time evolution. It is incorporated in the code LUKE,[26] which was initially developed for fast electron physics, and will therefore extend its range of applicability to multispecies physics and transport processes.

V. BOUNCE-CENTER FOKKER–PLANCK OPERATOR

The next step in the derivation of a reduced Fokker–Planck operator is to proceed with the construction of a *bounce-center* Fokker–Planck operator

$$\mathcal{C}_{bc}[\hat{F}] \equiv \langle\mathbf{T}_{bc}^{-1}\mathcal{C}_{gc}[\mathbf{T}_{bc}\hat{F}]\rangle_b, \tag{66}$$

where $\mathbf{T}_{bc}^{-1}$ and $\mathbf{T}_{bc}$ are the push-forward and pull-back operators associated with the bounce-center phase-space transformation[6] (and references therein), $\hat{F} \equiv \mathbf{T}_{bc}^{-1}F$ denotes the bounce-center distribution (which is independent of the bounce angle ζ_b), and $\langle\cdots\rangle_b$ denotes averaging with respect to ζ_b. In the bounce-center Fokker–Planck operator (66), finite-orbit effects will explicitly be taken into account.

To see how these finite-orbit effects might arise in the bounce-center Fokker–Planck operator (66), we first express the bounce-center push-forward and pull-back operators as $\mathbf{T}_{bc}^{\pm} \equiv \exp(\pm\epsilon\mathbf{G}_1\cdot\mathbf{d})$, where $\mathbf{G}_1\cdot\mathbf{d} \equiv G_1^a\partial_a$ is defined in terms of the components G_1^a of the first-order generating vector field for the bounce-center transformation and $\epsilon \equiv \epsilon_d = \tau_b/\tau_d \ll 1$ denotes its ordering parameter. Next, we expand the transformed operator,

$$\hat{\mathcal{J}}\mathbf{T}_{bc}^{-1}\mathcal{C}_{gc}[\mathbf{T}_{bc}\hat{F}] = \hat{\mathcal{J}}(e^{-\epsilon\mathbf{G}_1\cdot\mathbf{d}}\mathcal{C}_{gc}[e^{\epsilon\mathbf{G}_1\cdot\mathbf{d}}\hat{F}]), \tag{67}$$

in powers of ϵ, where the bounce-center Jacobian $\hat{\mathcal{J}}$ is defined in terms of the guiding-center Jacobian $\mathcal{J}$ as $\hat{\mathcal{J}} \equiv \mathcal{J} - \epsilon\partial_a(G_1^a\mathcal{J}) + \cdots$. By keeping terms up to second order in ϵ, and rearranging terms, we obtain

$$(\hat{\mathcal{J}} + \epsilon\partial_a(G_1^a\hat{\mathcal{J}}) + \cdots)\mathcal{C}_{gc}[\hat{F} + \epsilon G_1^b\hat{F} + \cdots]$$
$$- \epsilon\partial_a(\hat{\mathcal{J}}G_1^a\mathcal{C}_{gc}[\hat{F} + \epsilon G_1^b\partial_b\hat{F} + \cdots]) + \cdots, \tag{68}$$

where the first term is simply the original guiding-center Fokker–Planck operator multiplied by the guiding-center Jacobian $\mathcal{J} \equiv \hat{\mathcal{J}} + \epsilon\partial_a(G_1^a\hat{\mathcal{J}}) + \cdots$ (i.e., the first term is an exact divergence), while the second term is also an exact divergence in bounce-center phase space. The total bounce-center

Fokker–Planck operator is therefore guaranteed to retain its phase-space divergence form.

In future work, it will be our purpose to show that finite-orbit effects will appear in the bounce-center Fokker–Planck operator (66) in the form of the second-order term

$$\frac{\epsilon^2}{\hat{\mathcal{J}}}\frac{\partial}{\partial\hat{\psi}}\left\langle\hat{\mathcal{J}}G_1^\psi\mathcal{C}_{gc}\left[G_1^\psi\frac{\partial\hat{F}}{\partial\hat{\psi}}\right]\right\rangle_b,$$

where $\hat{\psi}$ denotes the bounce-center position and G_1^ψ denotes the corresponding component of the bounce-angle-dependent bounce-radius. In the present work, $\langle\psi\rangle_\mathcal{O}\equiv\bar{\psi}$ plays the role of the bounce-center position $\hat{\psi}$ for a trapped-particle orbit ($\delta\psi$ plays the role of G_1^ψ), while the nonlocal bounce-action J_b (canonically conjugate to the bounce angle ζ_b) is represented by the minimum-B pitch-angle coordinate ξ_0.

VI. SUMMARY

The present work presents the derivation of a general guiding-center Fokker–Planck equation in axisymmetric magnetic geometry (38) that allows first-order corrections in magnetic-field nonuniformity to be retained. The Fokker–Planck operator (21) is obtained from a guiding-center transformation using the set of invariants $(\bar{\psi},p,\xi_0)$. Next, the orbit-averaging procedure is introduced and an explicit Fokker–Planck operator (54) is derived in the low-collisionality limit. This operator is also expressed in a conservative form (55) that is best suited for numerical application. In the continuation of this paper (based on work briefly outlined in Sec. V), future theoretical work will consider finite-orbit effects in a different way by deriving a general bounce-center Fokker–Planck operator using Lie-transform methods.

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