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Title: Finite Time Equations for High-Strain Rate Plasticity

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Finite Time Equations for High-Strain Rate Plasticity
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ABSTRACT

The inelastic response of solids to strong loading is usually simulated with plasticity models that are solved by a method of radial return. Such implementations in which stress is integrated forward in time are termed *hypoelastic*. Radial return methods are simple to implement and computationally efficient. However, for problems with high strain-rates, these solutions may be very inaccurate and noisy, and further may not converge under increasing mesh resolution. Here we describe an alternate solution method based on multiple time scale (perturbation) theory that addresses these issues.

First, we derive an analytic solution for the fast time scales (e.g., the computational time step) in a simple plasticity model. Replacing the radial return solution with this analytic formula produces an accurate and clean solution while preserving simplicity and computational efficiency.

However, the hypoelastic framework itself implies restrictions on the accuracy and physical realizability of the solution. In our second result, we show how to generate the slow time solution to the equations. That is, we find a functional relationship between stress, strain and internal variables. This resulting *hyperelastic* framework has further advantages over hypoelasticity both as regards the physical model and computational implementation.

Finite Time Equations for High-Strain Rate Plasticity

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Take Home Points

- **Numerical simulations of solid dynamics can involve many additional difficulties in comparison with fluids.**

  Inelastic solid dynamics is an inherently multiscale problem.

- **There are *better* ways to solve standard models based on multiscale perturbation methods.**

- **There are better formulations of the models – hyperelastic vs. hypoelastic.**

Finite Volume Methods

- are a popular and effective technique for CF(/S)D
- solve for averaged quantities over length and time x

$$\bar{\sigma}_{jk} \equiv \frac{1}{\Delta t \Delta x} \int_{t-\Delta t/2}^{t+\Delta t/2} \int_{x-\Delta x/2}^{x+\Delta x/2} \sigma_{jk}(x', t') \, dx' \, dt'$$

- Averaged variables of Navier-Stokes satisfy different equations because of nonlinearity of advection, etc.
- We term these *equations of finite scale*

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Hypoelastic Equations

Momentum

$$\rho \frac{du_j}{dt} = \frac{\partial \sigma_{jk}}{\partial x_k}$$

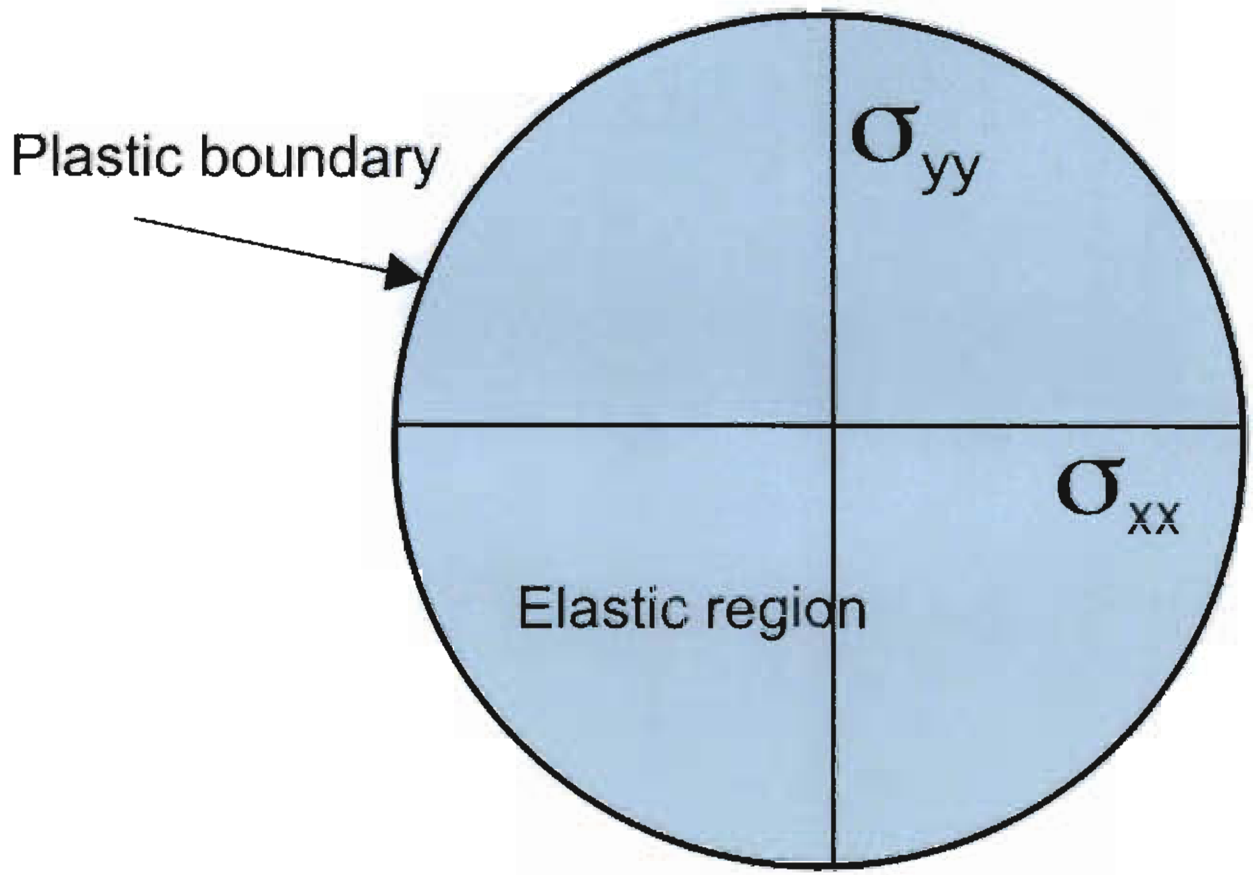
Strain rate

$$\dot{\epsilon}_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)$$

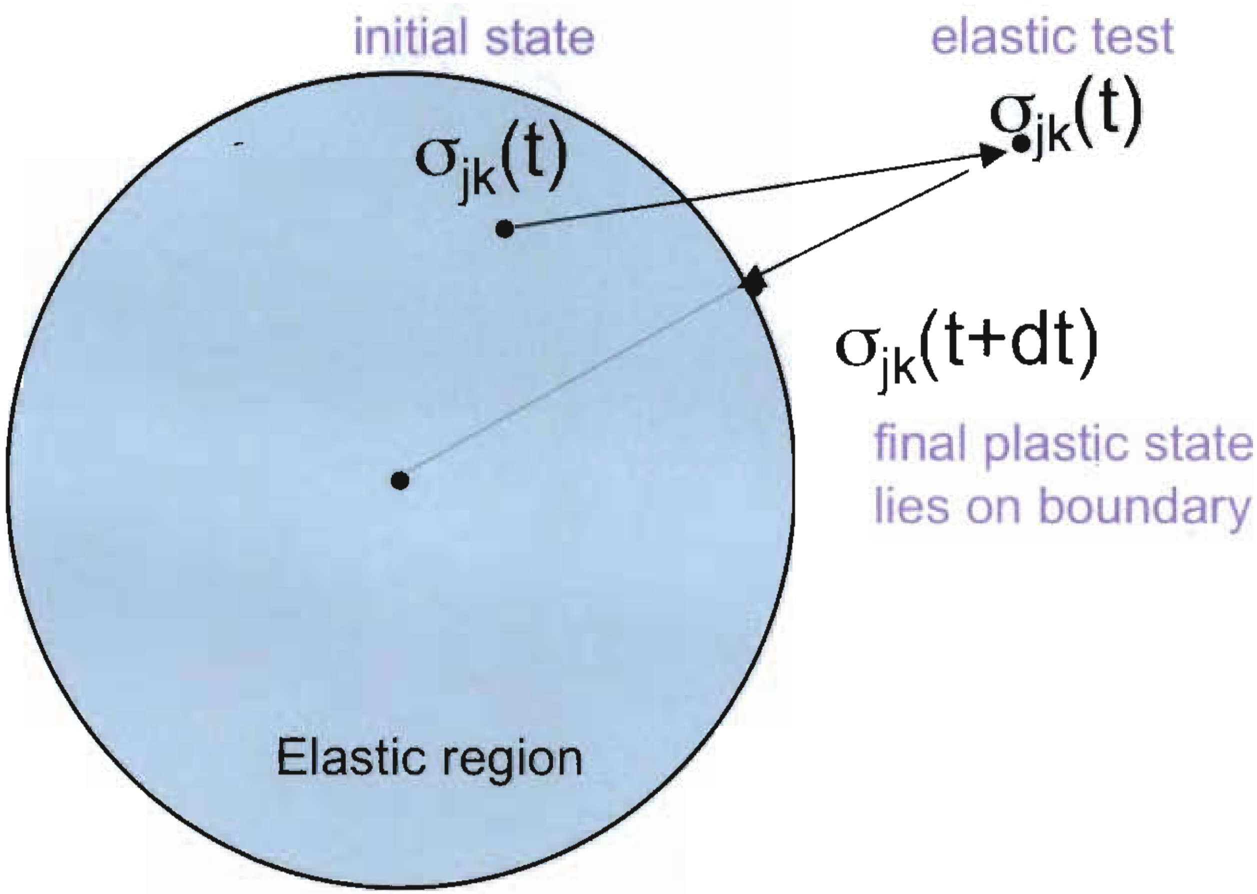
Constitutive (elastic plastic)

$$\frac{d\sigma_{jk}}{dt} = F(\sigma_{jk}, \dot{\epsilon}_{jk}, \alpha_i)$$

Accessible Regions of Stress Space



Radial Return (Wilkins)



Implied Constitutive Law

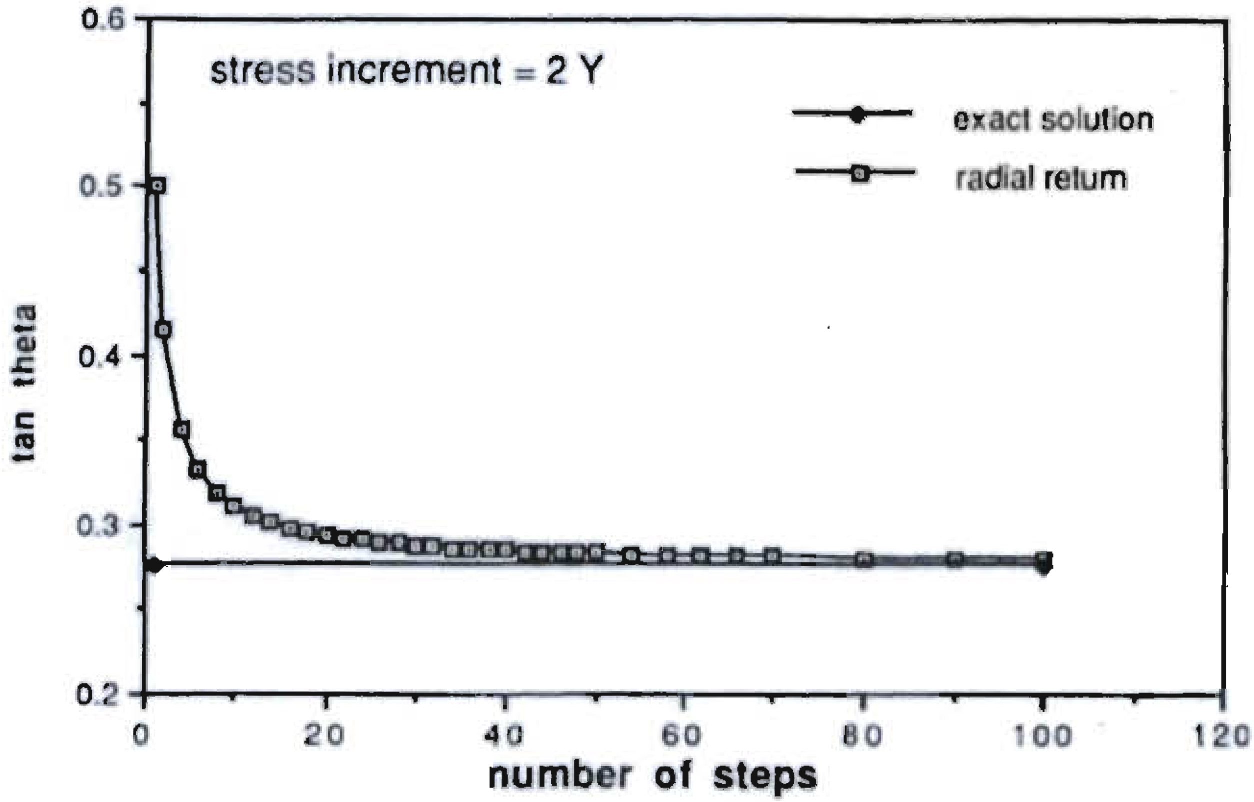
For perfect plasticity with constant yield stress Y

$$\frac{d\sigma_{jk}}{dt} = 2G\left(\dot{\epsilon}_{jk} - \frac{(\sigma_{mn}\dot{\epsilon}_{mn})}{2Y^2}\sigma_{jk}\right)$$

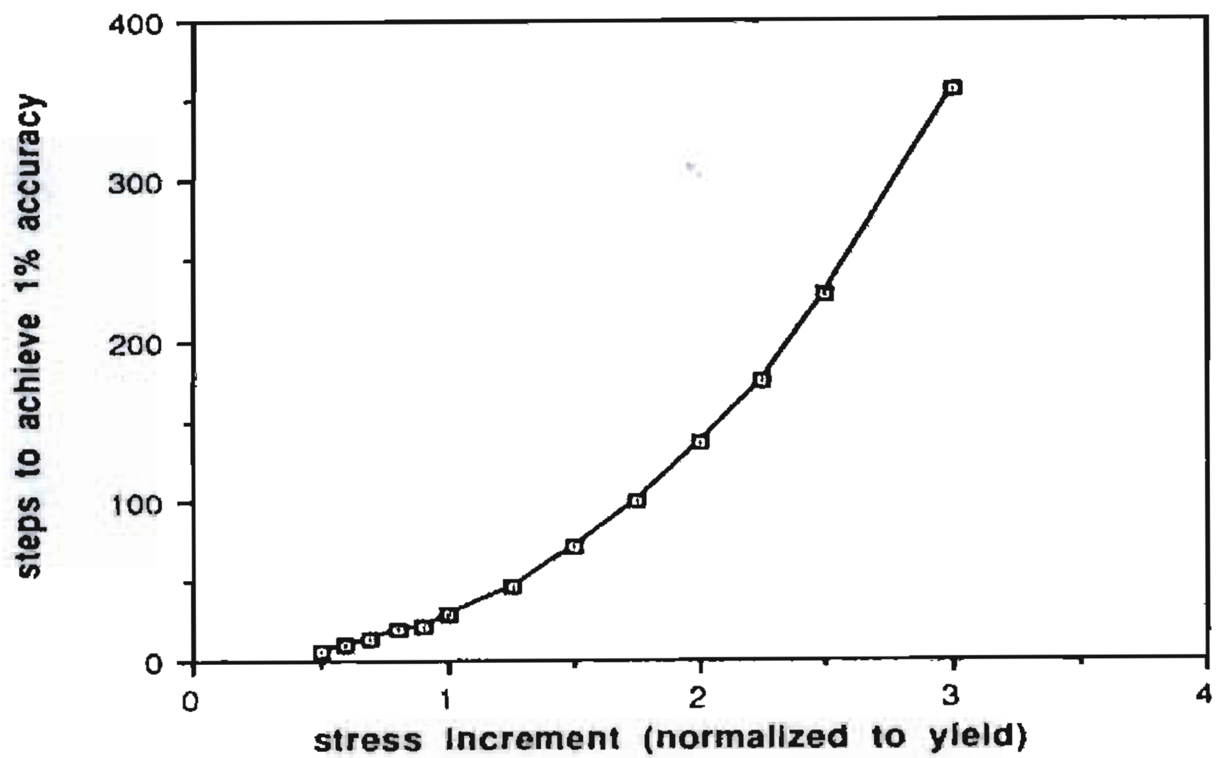
It can be shown that radial return is an explicit approximation to this equation

$$\frac{\sigma_{jk}^{n+1} - \sigma_{jk}^{n}}{\Delta t} = 2G\left(\dot{\epsilon}_{jk} - \frac{[\sigma_{lm}^{n}\dot{\epsilon}_{lm}]}{2Y^2}\sigma_{jk}^{n}\right) + \mathcal{O}(\Delta t)$$

For a large strain increment, error approaches 100% in one step



Number of subcycles to achieve 1% accuracy plotted against size of the stress increment.



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Elastic vs. Plastic Stress Increments

Are large elastic "test" increments possible – e.g., when is

$$2G\,||\dot{\epsilon}_{jk}||\,\Delta t >> 2Y$$

The time step is limited by the CFL condition $\boxed{\dfrac{u\,\Delta t}{\Delta x} < 1}$

Now $\boxed{||\dot{\epsilon}|| \approx \dfrac{\partial u}{\partial x}}$ and in most solids $\boxed{G \approx 100Y}$

So large increments are possible if $\boxed{\dfrac{\partial u}{\partial x} \approx \dfrac{u}{\Delta x}}$

$\Longrightarrow$ in regions of steep gradients, e.g., near shocks!

Multiple Time Scales

All the previous discussion is designed to justify the existence of two time scales, $\boxed{\Delta t}$ and $\boxed{\tau \approx \frac{Y}{G}\Delta t << \Delta t}$

So, to apply perturbation theory in the constitutive law

$$\boxed{\frac{d\sigma_{jk}}{dt} = 2G\left(\dot{\epsilon}_{jk} - \frac{(\sigma_{mn}\dot{\epsilon}_{mn})}{2Y^2}\sigma_{jk}\right)}$$

we will assume that $\dot{\epsilon}_{mn}$ is constant over a time interval Δt .

This leads to a very integrable ODE.

Short Time Scale Solution (1)

Defining the scalar function $\mathcal{W} \equiv \sigma_{jk}\dot{\epsilon}_{jk}$ (rate of work)

$$\frac{d\mathcal{W}}{dt} = 2G\left(I^2 - \frac{\mathcal{W}^2}{2Y^2}\right)$$

which has solution

$$\boxed{\mathcal{W} = \frac{Y^2}{G}\frac{d}{dt}\left[\ln\ F\right]}$$

Here, $F(t) \equiv A\exp(\alpha t) + \exp(-\alpha t)$ **and**

$$I = \sqrt{(\dot{\epsilon}_{jk})^2} \quad ; \quad \alpha = \sqrt{2}\,I\frac{G}{Y} \quad ; \quad A = \frac{\sqrt{2}\,IY + \mathcal{W}_0}{\sqrt{2}\,IY - \mathcal{W}_0}$$

are all constants.

Short Time Scale Solution (2)

Then

$$\frac{d\sigma_{jk}}{dt} = 2\,G\,\dot{\epsilon}_{jk} - \sigma_{jk}\,\frac{d}{dt}\,[\ln F(t)]$$

which has solution

$$\sigma_{jk}(t+\Delta t) = \sigma_{jk}(t)\left[\frac{\chi(A+1)}{A\chi^2+1}\right] + \dot{\epsilon}_{jk}\,\frac{\sqrt{2}Y}{I}\left[\frac{(A\chi^2-1)-\chi(A-1))}{A\chi^2+1}\right]$$

- **Here,** $\chi = \exp(\alpha\Delta t)$
- **Note that** $\alpha\Delta t = \sqrt{2}\,\frac{G\,I}{Y}\Delta t >> 1$

First Summary

1. The perturbation solution is accurate on time scales of Δt. In particular, it always stays *exactly* on the yield surface.

2. The procedure can be generalized to more complicated yield surfaces.

3. The method has been implemented and tested in large scale codes.

4. it is stable, convergent and more accurate as well as computationally efficient.

But, one can do more . . . we can find the <u>slow time</u> solution accurate over total problem time.

Slow Time Scale Solution (1)

In the fast time solution, the term $\chi = \exp(\alpha\Delta t)$ appears.

Note that as we let $\Delta t \rightarrow 0$, our fast time solution becomes even more accurate.

Our process now will become one of

1. **refining the increments – that is, letting $\Delta t \rightarrow 0$**
2. **expanding the fast time solution to first order in Δt**
3. **back substitute the solutions generating sums**
4. **converting the sums to integrals.**

Some Notation

We will write δt for the time increment to distinguish it (conceptually) from the computational time Δt.

We will write $\sigma^n = \sigma(n\delta t)$, $\chi_n = \exp(\alpha_n \delta t)$, $\alpha_n = \sqrt{2}\, I_n \frac{G}{Y}$

Then our fast time solution generically can be written

$$\sigma_{jk}^{n+1} = \sigma_{jk}^{n} \left[\frac{\chi_n(A_n+1)}{A_n\chi^2+1} \right] + \dot{\epsilon}_{jk}^{n} \frac{\sqrt{2}Y}{I} \left[\frac{(A_n\chi^2-1)-\chi(A_n-1))}{A_n\chi^2+1} \right]$$

Now expand $\chi_n \approx 1 + \alpha_n \delta t$. Setting $\mathcal{W}^n = \sigma_{jk}^{n} \dot{\epsilon}_{jk}^{n}$

$$\sigma_{jk}^{n+1} = \sigma_{jk}^{n} \left(1 - \delta t \frac{G\mathcal{W}^n}{Y^2} \right) + 2\,\delta t G\, \dot{\epsilon}_{jk}^{n}$$

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Back Substituting

$$\sigma_{jk}^{n+1} = \sigma_{jk}^{n}\left(1 - \delta t \frac{G\mathcal{W}^{n}}{Y^{2}}\right) + 2\delta t G \dot{\epsilon}_{jk}^{n}$$

We also have

$$\sigma_{jk}^{n} = \sigma_{jk}^{n-1}\left(1 - \delta t \frac{G\mathcal{W}^{n-1}}{Y^{2}}\right) + 2\delta t G \dot{\epsilon}_{jk}^{n-1}$$

and so forth. Back substituting, we end up with to $\mathcal{O}(\Delta t)$

$$\sigma_{jk}^{n} = \sigma_{jk}^{1}\left(1 - \frac{G}{Y^{2}} \sum_{m=1}^{n} \mathcal{W}^{m} \delta t\right) + 2G \sum_{m=1}^{n} \dot{\epsilon}_{jk}^{m} \delta t$$

Slow Time Solution

Now letting $\delta t \to 0$ while simultaneously fixing $n\delta t = t$ is a problem time that could be much greater than the computational time step

$$\boxed{\sigma_{jk}(t) = \sigma_{jk}(0)\left(1 - \frac{G}{Y^2}W^p\right) + 2G\epsilon_{jk}}$$

where $W^p = \int_0^t \mathcal{W}(t')\,dt'$ is total plastic work and

where $\epsilon_{jk} = \int_0^t \dot{\epsilon}_{jk}(t')\,dt'$ is total strain.

Plastic work and total strain are internal variables.

Hyperelastic Equations

Momentum

$$\rho \frac{du_j}{dt} = \frac{\partial \sigma_{jk}}{\partial x_k}$$

Strain

$$\epsilon_{jk} = \frac{1}{2} \int_0^t \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) dt'$$

Plastic work

$$W^p = \int_0^t \mathcal{W}(t')\, dt' = \int_0^t \sigma_{jk} \dot{\epsilon}^p_{jk}\, dt'$$

Stress

$$\sigma_{jk}(t) = \sigma_{jk}(0) \left(1 - \frac{G}{Y^2} W^p \right) + 2G\epsilon_{jk}$$

Summary for Hyperelastic Equations

The hyperelastic equations use *strain* as the dependent variable, not stress. Some advantages:

1. Replacing the constitutive (ODE) law with a functional EOS is numerically more stable (Hicks, 1978).

2. Strain and plastic work have conservation laws – better for Eulerian and ALE frameworks.

3. Hyperelasticity is well-posed for all loading (Naghdi, 1990).

4. Hyperelasticity framework allows for combining other inelastic processes with plasticity – e.g., brittle fracture, damage, etc.

Conclusions

1. High strain rate plasticity is a multiple time scale process.

2. Perturbation theory can improve the accuracy, stability and efficiency of standard numerical models.

3. Perturbation theory can also improve the mathematical and physical framework of plasticity theory.