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Title: Small deformation viscoplastic dynamic sphere problem

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Small deformation viscoplastic dynamic sphere problem

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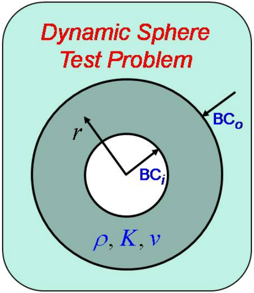
LA-UR-12-xxxxx

Research objectives

- Obtain analytic solution to dynamic sphere test problem
- Develop software to evaluate solution
- Demonstrate ‘convergence’ of computed solution

Schematic of dynamic sphere problem

- Analytic solution includes:
  - Dirichlet BC: Displacement
  - Neumann BC: Strain
  - Robin BC: Stress
- Time-varying BCs applied at r_i, r_o
- This talk focuses only on Dirichlet (displacement) BCs



Outline

- Describe Bodner-Partom model of viscoplasticity.
- Describe Bodner-Partom dynamic sphere problem.
- Derive series solution for displacement.
- Describe numerical implementation of analytic solution.
- Demonstrate self-convergence of solution.

Conservation law equations for isotropic viscoplastic material

- Formulated in material coordinates, so mass balance satisfied
- Constant temperature, so no energy equation
- Variables:
  - $u(x, t)$: displacement
  - $\epsilon(x, t) = \frac{1}{2}(\nabla u + \nabla u^T)$: total strain tensor
  - $\epsilon^p(x, t)$: plastic strain tensor

$$\rho \ddot{u} = \nabla \cdot \sigma + b$$

$$\sigma = C(\epsilon - \epsilon^p)$$

$$u(x, 0) = u^{(0)}(x), \quad \dot{u}(x, 0) = u^{(1)}(x)$$

$$u|_{\partial\Omega} = u_b(x, t)$$

Bodner-Partom viscoplastic constitutive model

The constitutive model variables are

- $\epsilon^p(x,t)$: plastic strain tensor
- $Z(x,t)$: isotropic hardening variable

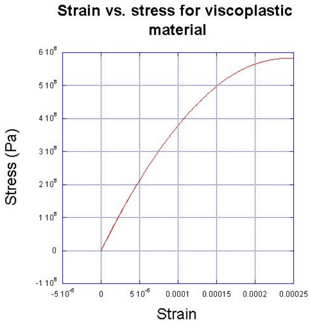
and the equations of the model are

$$\dot{\epsilon}^p = G(\frac{|S|}{Z})\frac{S}{|S|}$$

$$\dot{Z} = \gamma(Z)G(\frac{|S|}{Z})|S|$$

$$\epsilon^p(x,0) = \epsilon^{p,(0)}(x), \quad Z(x,0) = Z^{(0)}(x)$$

where $S(x,t) = \sigma - \frac{1}{3}\mathrm{tr}\sigma I$ is the stress deviator.



Radially symmetric dynamic sphere IBVP

Let $\lambda = -C\epsilon^p$ be plastic stress tensor. In spherical coordinates,

$$c_r^{-2}\ddot{u} = u'' + \frac{2}{r}u' - \frac{2}{r^2}u + c_r^{-2}f_r(\lambda)$$
$$\lambda_{rr} = -C_{rrrr}\epsilon_{rr}^p - 2C_{rr\theta\theta}\epsilon_{\theta\theta}^p$$
$$\lambda_{\theta\theta} = -C_{rr\theta\theta}\epsilon_{rr}^p - (C_{rrrr} + C_{rr\theta\theta})\epsilon_{\theta\theta}^p$$
$$u(r,0) = d_r^0(r) \quad \dot{u}(r,0) = v_r^0(r) \quad \epsilon_{rr}^p(r,0) = 0 \quad \epsilon_{\theta\theta}^p(r,0) = 0$$
$$u(r_i,t) = \mathrm{BC}_i(t) \quad u(r_o,t) = \mathrm{BC}_o(t)$$

where

$$f_r(\lambda) = b_r + \frac{1}{\rho}[\lambda_{rr}' + \frac{2}{r}(\lambda_{rr} - \lambda_{\theta\theta})].$$

Analytic solution for u

- Assume λ_{rr}, $\lambda_{\theta\theta}$ are known.
- u has the form $u(r,t) = r^{-1/2}[\overline{w}(r,t) + \tilde{w}(r,t)]$.
- $\overline{w}(r,t) = \gamma_0(t) + \gamma_1(t)r$ contains BCs.
- $\tilde{w}(r,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(r)$ contains plasticity.
- $a_n(t)$ is given by the equation

$$
\begin{aligned}
a_n(t) &= c_{3,n}\cos c_r\sqrt{\lambda_n}t + c_{4,n}\sin c_r\sqrt{\lambda_n}t \\
&\quad + \frac{1}{c_r\sqrt{\lambda_n}}\int_0^t F_n(s)\sin c_r\sqrt{\lambda_n}(t-s)ds, \\
F_n(t) &= c_r^2\frac{\int_{r_i}^{r_o}\tilde{f}(r,t)\psi_n(r)dr}{\int_{r_i}^{r_o} r\psi_n^2(r)dr}
\end{aligned}
$$

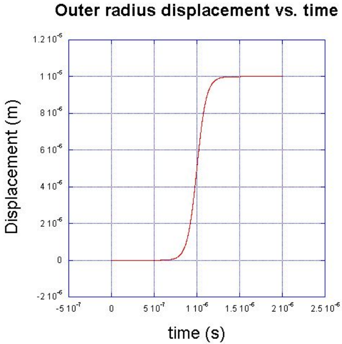
where $\tilde{f} = L(\tilde{w}) + r^{3/2}c_r^{-2}f_r(\lambda)$.

Numerical implementation

- Choose number of grid points *nr* through shell, number of time steps *nt*, and number of eigenmodes *nl*.
- Numerical quadrature in time and space, including hereditary integral.
- Take $\tilde{w}(r,t) = \sum_{n=1}^{nl} a_n(t)\psi_n(r)$ at each time step and grid point.
- Use material model to partition total strain (from u) into elastic (ϵ) and plastic (ϵ^p) components.
- At each time step, iterate to find ϵ^p, λ, and u.
- Numerical error arises from quadrature, series truncation, and discretization in computing ϵ^p.

Self-convergence for test problem

- Consider convergence of analytic solution in nr, nt, nl.
- Compute to a final time of $T = 2$ microseconds.
- At inner radius r_i, impose void displacement BCs: $u(r_i, t) = 0$.
- At outer radius r_o, impose time-varying smooth jump displacement BCs.



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Self-convergence: Convergence in space

Take $nt = 8000$, $nl = 300$.

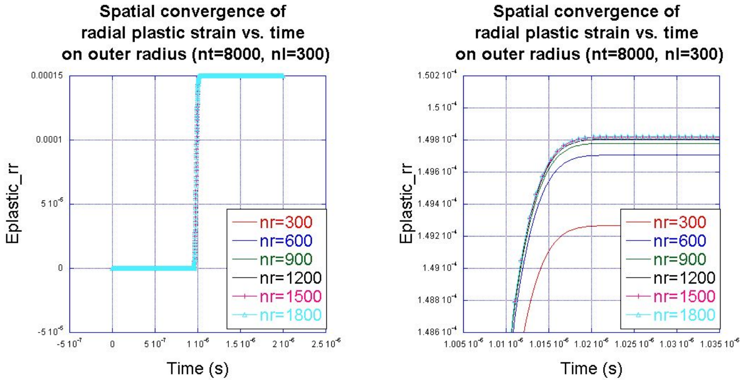


Figure: Left: *nr* convergence plot of ϵ^p_{rr} vs. time at r_o. Right: Zoomed in plot of ϵ^p_{rr}.

Self-convergence: Convergence in time

Take $nr = 1200$, $nl = 300$.

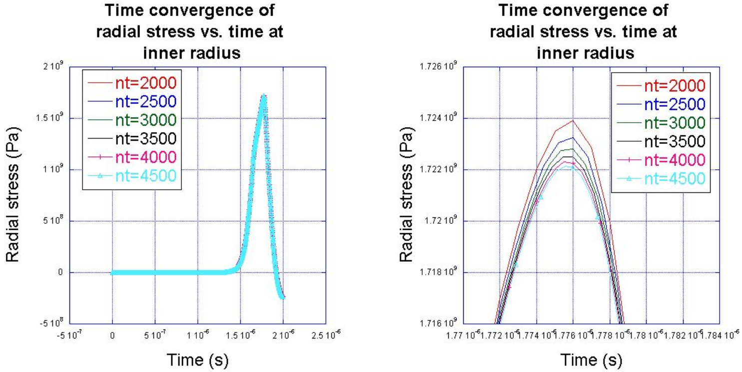
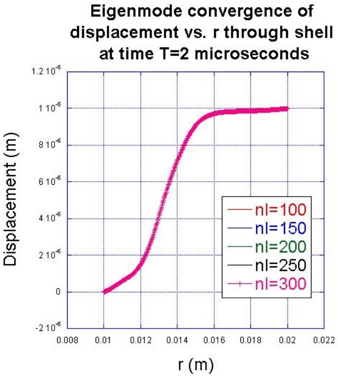


Figure: Left: *nt* convergence plot of σ_{rr} vs. time at r_i. Right: Zoomed in plot of σ_{rr}.

Self-convergence: Convergence in eigenmode

Compute through shell at final time $T = 2$ microseconds. Take $nr = 1200$, $nt = 3500$.



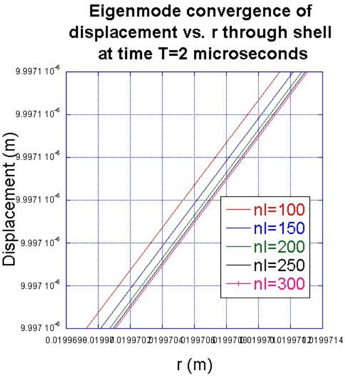


Figure: Left: *nl* convergence plot of u vs. r at time $T = 2$ microseconds. Right: Zoomed in plot of u.

Self-convergence: Quantitative analysis I

- We compute $L^1(0, T)$ norm of percent errors of plastic strain relative to reference solution computed on extremely fine mesh to test spatial convergence.
- The order of convergence is 2.25.

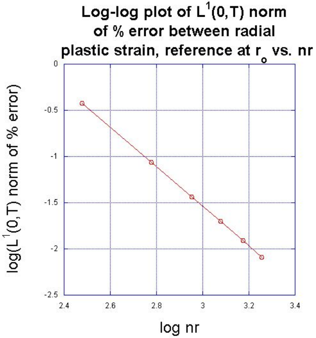


Figure: Log-log plot of $L^1(0, T)$ norm of percent error of ϵ^p_{rr} at r_o vs. nr.

LA-UR-12-xxxxx

Self-convergence: Quantitative analysis II

- We compute $L^1(0, T)$ norm of percent errors of stress relative to reference solution computed on extremely fine mesh to test time convergence.
- The order of convergence is 2.

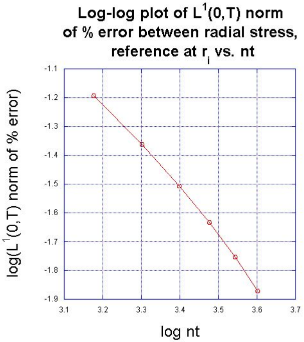


Figure: Log-log plot of $L^1(0, T)$ norm of percent error of σ_{rr} at r_i vs. nt.

Self-convergence: Quantitative analysis III

- We compute $L^1(r_i, r_o)$ norm of percent errors of displacement relative to reference solution computed on extremely fine mesh to test eigenmode convergence.
- The order of convergence is 3.2.

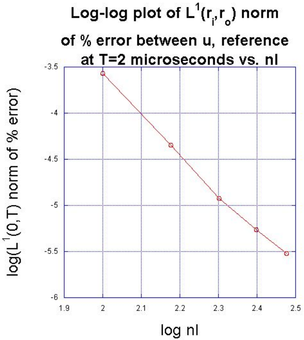


Figure: Log-log plot of $L^1(r_i, r_o)$ norm of percent error of u through shell at $T = 2$ microseconds vs. nl.

LA-UR-12-xxxxx

Summary

- We describe the Bodner-Partom constitutive model of plastic flow for a solid under small deformation.
- We derive an analytic solution for displacement in the form of an infinite series.
- We demonstrate convergence of a truncated solution under spatial, time, and eigenmode refinement.

Future work

- Compare analytic solution to LANL physics code.
- Derive and study solution for other boundary conditions (Neumann, Robin).
- Derive and study solution for finite deformations.