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Title: An Exact Calculation of Electron-Ion Energy Splitting in a Hot Plasma

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An Exact Calculation of Electron-Ion Energy Splitting in a Hot Plasma

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Abstract

In this brief report, I summarize the rather involved recent work of Brown, Preston, and Singleton (BPS). In Refs. [2] and [3], BPS calculate the energy partition into ions and electrons as a charged particle traverses a non-equilibrium two-temperature plasma. These results are exact to leading and next-to-leading order in the plasma coupling g, and are therefore extremely accurate in a weakly coupled plasma. The new BPS calculations are compared with the more standard work of Fraley *et al.* [12]. The results differ substantially at higher temperature when $T_{\rm I} \neq T_e$.

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I. INTRODUCTION AND CONTEXT

For the last several years, in collaboration with Lowell Brown and Dean Preston, I have been studying Coulomb energy exchange processes in hot multi-component but weakly coupled plasmas [1–8]. In Ref. [1], we first calculated several physical processes relevant to thermonuclear burn, namely, the stopping power of a non-thermal charged particle as it traverses a plasma, and the electron-ion temperature equilibration in a non-equilibrium two-temperature plasma. Our results are essentially exact in a weakly coupled plasma, thereby eliminating the uncertainty associated with the notorious *Coulomb Logarithm*. In our most recent work [2, 3], supported by PEM and outlined in this document, we have now calculated another relevant process: the exact form of the electron-ion energy splitting of a non-thermal charged particle in a non-equilibrium two-temperature plasma. Our new calculation is again accurate to leading and next-to-leading order in the plasma coupling parameter, and is therefore almost exact in a weakly coupled system. The formalism use in these calculations starts from first principles, exploiting an advanced regularization technique from Quantum Field Theory (called *dimensional regularization*). This regularization scheme accurately captures both the short and long range contributions from the Coulomb interaction, which allows one to find the exact form of the Coulomb Log for relevant Coulomb energy exchange processes, such as the charged particle stopping power and electron-ion temperature relaxation. Finding these Coulomb logarithms is equivalent to calculating the physical process to leading and next-to-leading order in the plasma coupling constant. The first order term has been known since Landau [9] and Spitzer [10]. These authors calculated Coulomb relaxation times, and their work was extended to stopping power by Corman *et al.* [11]. The second order term, which uniquely captures the Coulomb logarithm, was not rigorously calculated until our work in Ref. [1]. In our new calculation [2, 3], summarized in this report, we find the exact form of the electron-ion energy splitting of a non-thermal charged particle moving through the plasma, again, to leading and next-to-leading order in the plasma coupling parameter.

Coulomb energy exchange processes have historically been prohibitively difficult to calculate because of the long-range nature of the Coulomb interaction, and all such calculations, until Ref. [1], were only accurate to a logarithmic term called the *Coulomb Logarithm*. The energy splitting problem was addressed quite some time ago in 1974 by Fraley *et al.* [12]. These authors proposed a simple form for the electron-ion splitting factors based on a phenomenological model of the stopping power,

$$f_{\rm I} \;=\; \frac{T_e}{T_e + T_0} \tag{1.1}$$

$$f_e \;=\; \frac{T_0}{T_e + T_0} \quad \text{with} \quad T_0 = 32\,\text{keV} \;. \tag{1.2}$$

Other common variants of this model use different values for the 50-50 splitting temperature T_0. Also note that the Fraley *et al.* model is independent of the ion temperature $T_{\rm I}$ and only depends upon the electron temperature T_e, a result that cannot be correct in a two-temperature plasma, *i.e.* when $T_e \neq T_{\rm I}$. The BPS calculations have rectified this situation, and we have found the exact form of the coefficients under the logarithm for the electron-ion energy splitting process. In our most recent work [2, 3], we have now calculated the splitting factors $f_{\rm I}$ and f_e to the same level of accuracy as the other processes we have studied. For the simple case in which $T_e = T_{\rm I} = T$, our results take the form

$$f_{\rm I} \;=\; \int_0^{E_0} \frac{dE}{E_0}\, \frac{\mathcal{A}_{\rm I}(E)}{\mathcal{A}_e(E) + \mathcal{A}_{\rm I}(E)} \left[{\rm erf}\left(\sqrt{E/T}\right) + \sqrt{\frac{4E}{\pi T}}\, e^{-E/T} \right] \tag{1.3}$$

$$f_e \;=\; \int_0^{E_0} \frac{dE}{E_0}\, \frac{\mathcal{A}_e(E)}{\mathcal{A}_e(E) + \mathcal{A}_{\rm I}(\mathcal{E})} \left[{\rm erf}\left(\sqrt{E/T}\right) + \sqrt{\frac{4E}{\pi T}}\, e^{-E/T} \right] . \tag{1.4}$$

Here, E_0 is the threshold energy of the charged particle ($E_0 = 3.5\,{\rm MeV}$ for the DT α particle), and $\mathcal{A}_e$ and $\mathcal{A}_{\rm I}$ are the longitudinal components of the BPS Fokker-Planck equation introduced in Ref. [1]. I will write down the forms of the $\mathcal{A}$-coefficients in a later section. I should note, as advertised, that we have also calculated the splitting factors for the more interesting two-temperature plasma with $T_e \neq T_{\rm I}$. The two-temperature expressions are far more complicated than Eqs. (1.3) and (1.4),[1] and they will be discussed in a future section.

II. THE FRALEY, LINNEBUR, MASON, AND MORSE MODEL

I will first outline the work of Fraley *et al.* (FLMM) of Ref. [12]. These authors are concerned with 50-50 mixtures of deuterium and tritium (DT) and the corresponding 3.5 MeV α particle resulting from DT fusion. They start with a phenomenological model of the stopping power for this system, with the ion and electron contributions given by

$$\frac{dU_{\rm I}}{dx} \;=\; -0.047\,\frac{\rho}{\rho_0}\,\frac{1}{U} \left[1 + 0.075\,\ln\left\{ T_e^{1/2} \left(\frac{\rho_0}{\rho}\right)^{1/2} U^{1/2} \right\} \right] \tag{2.1}$$

$$\frac{dU_e}{dx} \;=\; -23.2\,\frac{\rho}{\rho_0}\,\frac{U^{1/2}}{T_e^{3/2}} \left[1 + 0.17\,\ln\left\{ T_e \left(\frac{\rho_0}{\rho}\right)^{1/2} \right\} \right] , \tag{2.2}$$

where the dimensionless energy U and the density ρ_0 are

[1] We have, however, found a simple parametrization of the splitting factor $f_{\rm I}$, which is useful for numerical work [3].

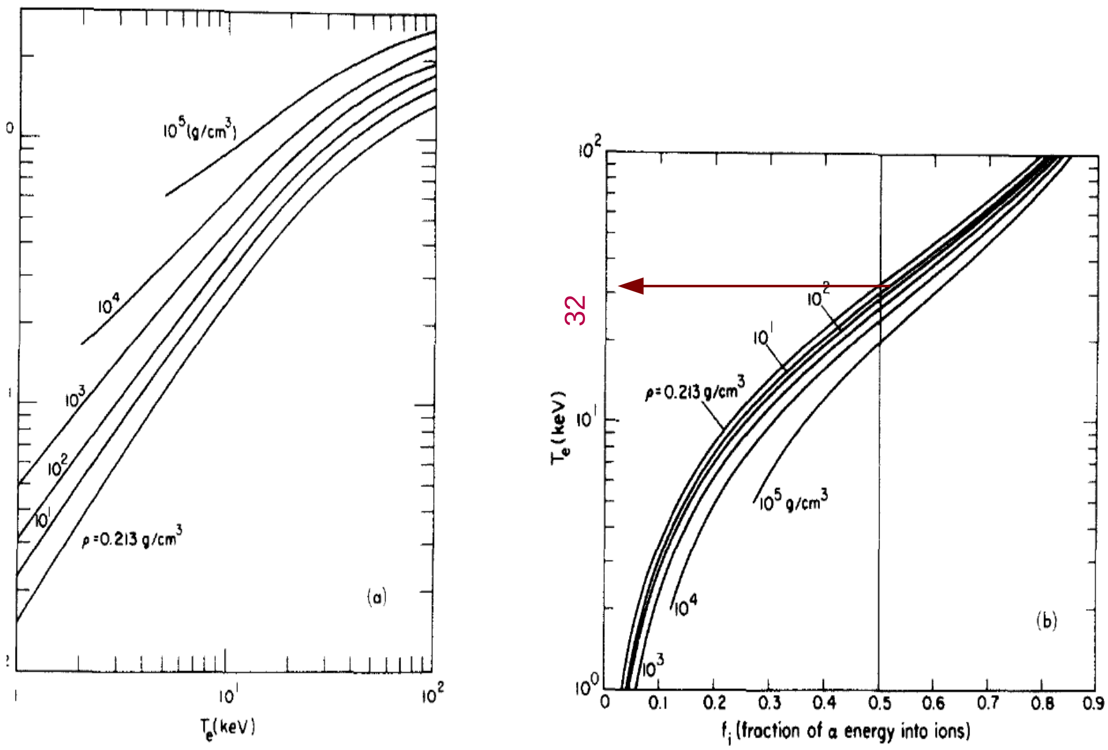

FIG. 1. **(a) α particle range λ_α vs T_e, (b) α energy absorption.**

FIG. 1: Figure 1 from Fraley *et al.* [12] has been reproduced here. Figure (a) is the α particle range for various plasma densities ρ versus the electron temperature T_e. Figure (b) plots the ion energy absorption factor $f_{\rm I}$ for various plasma densities over a range of electron temperatures. These results are best fit using the 50-50 crossover temperature $T_0 = 32\,{\rm keV}$.

$$U = E/3.5\,{\rm MeV} \tag{2.3}$$

$$\rho_0 = 0.213\,{\rm g/cm}^3 \ . \tag{2.4}$$

Note that the density $\rho = 0.213\,{\rm g/cm}^3$ corresponding to an electron number density of $n_e = 5 \times 10^{22}\,{\rm cm}^{-3}$. The total stopping power is obtained by adding the electron and ions contributions,

$$\frac{dU}{dx} = \frac{dU_e}{dx} + \frac{dU_{\rm I}}{dx} \ . \tag{2.5}$$

It should be noted that such a clean pair-wise split between electrons and ions only occurs to second order in the plasma coupling, *i.e.* only for weakly coupled plasmas. The energy of higher order terms involves the Coulomb potential, and will therefore not split in such a pair-wise fashion.

Recall that the stopping power measures the energy deposition of the charged particle per unit distance, and by integrating appropriate ratios of Eqs. (2.1) and (2.1), Fraley *et al.*

calculated the electron-ion energy splitting of the 3.5 MeV α particle in a 50-50 molar DT plasma. This is illustrated in Fig. 1. The ion splitting factor was then fit to the following form,

$$f_{\rm I} = \frac{1}{1 + T_0/T_e} \ . \tag{2.6}$$

The crossover temperature T_0, where the electron and ion fraction is 50-50, can be determined from Fig. 1b, and Fraley *et al.* find $T_0 = 32\,\text{keV}$. Variants on this model use different values for T_0.

III. THE PROBLEM WITH INTEGRATING THE STOPPING POWER

When a fast charged particle with initial energy E_0 traverses a plasma, it looses energy at a rate dE/dx per unit of distance, and it comes into thermal equilibrium after depositing its energy into the electrons and ions that make up the plasma. The traditional method for computing the energy partition between electrons and ions involves integrating the stopping power, but, as we shall see, there are conceptual problems with this approach.

Recall that the total stopping power dE/dx is a function of the projectile energy E, and as the particle slows down, this can be inverted to obtain the distance traveled as a function of energy, $x = x(E)$. One may then express the the stopping power as a function of x to determine the path of the charged particle, and the resulting integration along this trajectory gives the energy contribution to ions and electrons. This procedure can be written schematically as

$$E_{\rm I} = \int_0^{E_0} dE \, \frac{dE_{\rm I}/dx}{dE/dx} \tag{3.1}$$

and

$$E_e = \int_0^{E_0} dE \, \frac{dE_e/dx}{dE/dx} \, , \tag{3.2}$$

where $dE_{\rm I}/dx$ and dE_e/dx are the stopping power contributions from the ions and electrons, and the total stopping power is

$$\frac{dE}{dx} = \frac{dE_{\rm I}}{dx} + \frac{dE_e}{dx} \ . \tag{3.3}$$

Within the context of this approach, we can integrate the stopping power to obtain the seemingly reasonable result

$$E_{\rm I} + E_e = E_0 \ . \tag{3.4}$$

However, this is only an approximate description because the fast charged particle does not simply come to rest within the plasma, but instead it becomes thermalized at the temperature T. One should not extend the integrals in Eqs. (3.1) and (3.2) down to zero, but rather,

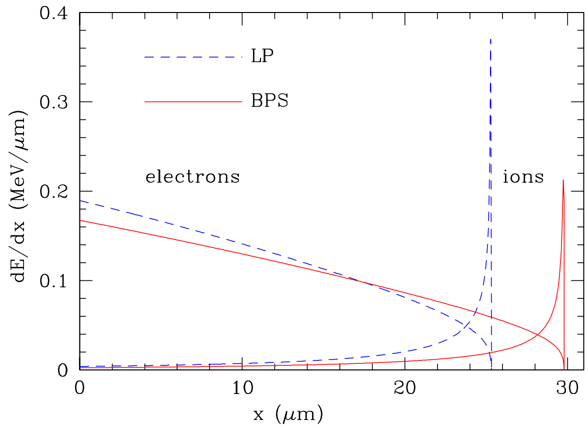

FIG. 2: LIP vs BPS stopping power in a weakly coupled plasma: the α particle $dE(x)/dx$ (in MeV/μm) *vs.* x (in μm) split into separate ion (peaked curves) and electron components (softly decreasing curves). The solid line is the BPS result, and the dashed line is the result of Li and Petrasso (LP). The area under each curve gives the corresponding energy partition into electrons and ions for this work and that of Li and Petrasso. For our results, the total energy deposited into electrons is $E_e^{\rm BPS} = 3.16$ MeV and into ions is $E_{\rm I}^{\rm BPS} = 0.38$ MeV, while LP gives $E_e^{\rm LIP} = 3.09$ MeV and $E_{\rm I}^{\rm LIP} = 0.45$ MeV. Note that these energies sum to the initial α particle energy of $E_0 = 3.54\,$MeV, in agreement with Eq. (3.4).

one should use a lower limit $E_{\rm min}$ on the order of the thermal plasma energy, $E_{\rm min} \sim T$. Consequently the systematic error in the above calculation of $E_{\rm I}$ and E_e is of order T/E_0, and as we shall see, the correct electron-ion energy partition relation reads (for a simple one-temperature plasma)

$$E_{\rm I} + E_e + \frac{3}{2} T = E_0 \ . \tag{3.5}$$

As we shall see, the correct expression for the energy splitting $f_{\rm I}$ arises from a Fokker-Planck equation derived in Ref. [1], which takes into account the thermalization process as the charged particle slows down and becomes part of the plasma background. Since energy is conserved in this formalism, Eq. (3.5) will arise naturally during the calculation. I should emphasize that the BPS formalism provides a means of regulating the kinetic equations at short and long distances in a consistent manner while treating quantum mechanical effects exactly.

Another problem with using the stopping power to determine the energy splitting is that most authors use an inaccurate form of dE/dx. For example, Fig. 2 illustrates the difference between the stopping powers of Li and Petrasso [13] and the exact form calculated by BPS [1]. To illustrate the importance of the stopping power on thermonuclear burn and ignition, let us examine a simple test problem introduced in Ref. [12] (I call this the *Fraley Sphere*). Consider a sphere of DT plasma with a given initial mass and radius, a uniform initial density, and a specified initial temperature. We choose a parameter regime in which

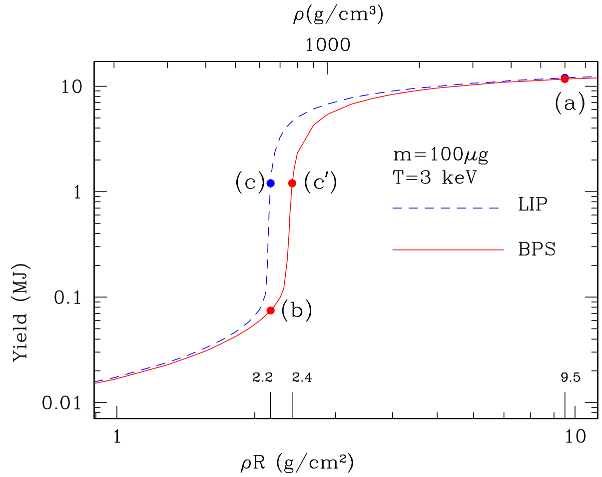

FIG. 3: The yield in MJ as a function of initial ρR in g/cm^2 for a DT microsphere of Ref. [12]. The initial uniform temperature is taken to be $T_0 = 3\,\text{keV}$ and the initial mass of the sphere is $m_0 = 100\,\mu\text{g}$. The upper axis gives the initial density ρ in g/cm^3. The dashed curve is the yield profile for the stopping power of Li and Petrasso (LIP), while the solid curve corresponds to that of Brown, Preston, and Singleton (BPS). The points labeled by (a) and (b) correspond to full yield 1% maximum yield, respectively. The points labeled by (c) and (c′) correspond to 10% maximum yield, the first for LIP and the second for BPS. The BPS stopping power has a longer α particle range than LIP, and it delivers less α particle energy to the ions. This has the effect of increasing the ignition threshold for BPS relative to LIP.

DT fusion is taking place, and we let the sphere evolve in time until it disassembles, thereby turning off the fusion yield. For various stopping power models, it is instructive to plot the total yield as a function of the initial density of the Fraley sphere. Figure 3 illustrates this exercise for the LP and BPS stopping powers, using an initial DT mass of $m_0 = 100\,\mu\text{g}$ with an initial temperature of $T_0 = 3\,\text{keV}$. As we see, the ignition curve varies significantly between the two forms of the stopping power. Other Coulomb energy exchange processes, such as electron-ion temperature equilibration, can produce similar effects in various physical quantities.

IV. THE NEW BPS ENERGY SPLITTING FORMALISM

1. The BPS Fokker-Planck Equation

Besides the stopping power and the temperature equilibration rate, BPS also derived a Fokker-Planck equation accurate to leading and next-to-leading order in the plasma coupling [1]. Denoting the phase space density for the dilute collection of charged particles by $f(\mathbf{r}, \mathbf{p}, t)$, this equation reads

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right] f(\mathbf{r}, \mathbf{p}, t) = \sum\nolimits_b \frac{\partial}{\partial p^k} C_b^{k\ell}(\mathbf{p}) \left[\beta\, v^\ell + \frac{\partial}{\partial p^\ell}\right] f(\mathbf{r}, \mathbf{p}, t)\,, \tag{4.1}$$

where the sum runs over the plasma components b, $\beta = 1/T$, the vector $\mathbf{v} = \mathbf{p}/m$ is the velocity of a particle with momentum $\mathbf{p}$, and the summation convention is used for repeated vector indices. The symmetric tensor $C_b^{k\ell}$ has longitudinal and transverse components,

$$C_b^{k\ell}(\mathbf{p}) = \mathcal{A}_b(E)\, \frac{\hat{v}^k \hat{v}^\ell}{\beta v} + \mathcal{B}_b(E)\, \frac{1}{2}\, \left(\delta^{k\ell} - \hat{v}^k \hat{v}^\ell\right)\,, \tag{4.2}$$

where $v = |\mathbf{v}|$ and $\hat{\mathbf{v}} = \mathbf{v}/v$, and $E = \frac{1}{2}\, m v^2$. We denote the sum of the ion components of the $\mathcal{A}$-coefficients by $\mathcal{A}_{\rm I} = \sum_i \mathcal{A}_i$ and the electron component by $\mathcal{A}_e$, with $\mathcal{A} = \mathcal{A}_{\rm I} + \mathcal{A}_e$. Expressions for the $\mathcal{A}_b$ can be found in BPS [1]. With our conventions, the number and kinetic energy densities of the charged particles are given by

$$n(\mathbf{r}, t) \;=\; \int \frac{d^3p}{(2\pi\hbar)^3}\, f(\mathbf{r}, \mathbf{p}, t) \tag{4.3}$$

and

$$\mathcal{E}(\mathbf{r}, t) \;=\; \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{\mathbf{p}^2}{2m}\, f(\mathbf{r}, \mathbf{p}, t)\,. \tag{4.4}$$

The energy exchange dE/dx is determined only by the $\mathcal{A}$-coefficients, while the momentum exchange is given by both $\mathcal{A}$- and $\mathcal{B}$-coefficients. By calculating the energy and momentum exchange to leading and next-to-leading order using the method of dimensional continuation, BPS determined the kernel C_b of the Fokker-Planck equation (4.1) [1].

We can derive a relation between the stopping power and the $\mathcal{A}$-coefficients in the following manner. For a single particle at $\mathbf{r}_p$ moving with velocity $\mathbf{v}_p$, the distribution function takes the form $f(\mathbf{r}, \mathbf{p}, t) = (2\pi\hbar)^3 \delta(\mathbf{r} - \mathbf{r}_p)\delta(\mathbf{p} - \mathbf{p}_p)$, the Fokker-Planck equation gives the particle's rate of energy loss as

$$v_p\, \frac{dE}{dx} = \frac{dE}{dt} = \sum\nolimits_b \left[\beta v_p^\ell - \frac{\partial}{\partial p_p^\ell}\right] v_p^k\, C_b^{k\ell}(\mathbf{p}_p)\,. \tag{4.5}$$

Upon substituting the decomposition (4.2) for the scattering tensor, the contribution from species b appears as

$$\frac{dE_b}{dx} = \left[1 - \frac{1}{\beta m v}\, \frac{\partial}{\partial v^\ell}\, \hat{v}^\ell\right] \mathcal{A}_b(E)\,. \tag{4.6}$$

As v gets large ($\beta m v^2 \gg 1$) note that $dE_b/dx \to \mathcal{A}_b$.

As we shall see, to use these results to obtain an unambiguous formulation of the fractions of the total energy deposited into the ions and electrons, we first need to compute the asymptotic distribution into which an initial swarm of projectile particles relaxes in the presence of a background plasma of differing electron and ion temperatures. In spherical symmetry the the BPS Fokker-Planck equation takes the form

$$\left\{\frac{\partial}{\partial t} - \frac{2}{mv}\frac{\partial}{\partial E}\, E\left[\mathcal{A}(E) + \langle T\mathcal{A}(E)\rangle\,\frac{\partial}{\partial E}\right]\right\} f(E,t) = 0 \;, \tag{4.7}$$

where the total $\mathcal{A}$-coefficient and the temperature-weighted coefficient are respectively defined by

$$\mathcal{A}(E) \;=\; \mathcal{A}_{\rm I}(E) + \mathcal{A}_e(E)\,, \tag{4.8}$$

$$\langle T\mathcal{A}(E)\rangle \;=\; T_{\rm I}\,\mathcal{A}_{\rm I}(E) + T_e\,\mathcal{A}_e(E)\,. \tag{4.9}$$

The quasi-static distribution will be spherically symmetric and a function of E (or equivalently of p), which we express in terms of a function $S(E)$ as

$$f_\infty(E) = \mathcal{N}\, e^{-S(E)}\;, \tag{4.10}$$

where we choose $\mathcal{N}$ to normalize the distribution to unity,

$$1 = \int \frac{d^3p}{(2\pi\hbar)^3}\, f_\infty(E) = \mathcal{N}\int \frac{d^3p}{(2\pi\hbar)^3}\, e^{-S(E)}\,. \tag{4.11}$$

This choice of normalization gives the asymptotic solution

$$f_{\rm asym} = n(t) f_\infty(E)\;. \tag{4.12}$$

The function $S(E)$ is determined by [2]

$$\mathcal{A}(E) - \langle T\mathcal{A}(E)\rangle\,\frac{dS(E)}{dE} = 0\;, \tag{4.13}$$

and the solution can be obtained by a simple integration

$$S(E;T_e,T_{\rm I}) \;=\; \int_0^E dE'\,\frac{\mathcal{A}(E')}{\langle T\mathcal{A}(E')\rangle}\,. \tag{4.14}$$

For the special case of a common plasma temperature $T = T_{\rm I} = T_e$, we see that this solution reduces to the Maxwell-Boltzmann distribution

$$S(E;T,T) \;=\; \frac{E}{T}\,, \tag{4.15}$$

and normalization

$$\mathcal{N} = \left(\frac{2\pi\hbar^2}{mT}\right)^{3/2}\;. \tag{4.16}$$

With this background in hand, we turn now to the energy splitting problem. Rather than tracking an individual charged particle slowing down in the plasma, it is much simpler to examine a homogeneous and isotropic source of charged particles produced at a single energy E_0. For example, the fusion process in a homogeneous DT plasma produces α particles uniformly in space and isotropically in angle with an initial energy of $E_0 = 3.54\,\text{MeV}$. The background plasma parameters, such as its density and temperatures, change very little over distances that are large in comparison with the stopping distance of the charged projectile particles, and the plasma parameters also change very little during the stopping time. Thus the plasma is treated as homogeneous and static, and the distribution function will therefore depend only upon the energy and time, $f = f(E,t)$. We therefore consider the non-homogeneous Fokker-Planck equation with a source,

$$\left\{\frac{\partial}{\partial t} - \frac{2}{mv}\frac{\partial}{\partial E}\, E\left[\mathcal{A}(E) + \langle T\mathcal{A}(E)\rangle\,\frac{\partial}{\partial E}\right]\right\} f(E,t) = \delta\left(E - E_0\right)\, s(t)\,. \tag{4.17}$$

We can write the non-homogeneous asymptotic late time solution more suggestively as

$$f(E,t) = n(t) f_\infty(E) + \bar{f}(E)\,, \tag{4.18}$$

with $f_\infty(E) = \mathcal{N}\, e^{-S(E)}$ as in the previous section. We emphasize that expression (4.18) is the asymptotic late-time solution to the inhomogeneous Fokker-Planck equation. We proceed now to motivate the structure of the solution (4.18) and the nature of the time independent function $\bar{f}(E)$. The particles will eventually thermalize to a Maxwell-Boltzmann distribution, and the first term of Eq. (4.18) merely represents the buildup and subsequent thermalization of the particles produced by the source. The normalization factor $\mathcal{N}$ is chosen so that $n(t)$ is the number density of the produced particles once they have thermalized into the Maxwell-Boltzmann distribution $\propto e^{-\beta E}$. The time-independent piece $\bar{f}(E)$ describes the steady state of nonthermal particles losing energy to the plasma, *i.e.* particles cascading down "energy bins" from the initial energy E_0 to the final thermal energy. The situation described here can be pictured as the flow of water over a rocky waterfall that slows the motion of the water as it descends. The initial rate of flow of the river corresponds to the rate of produces particles; the height of the waterfall to the initial energy E_0. The energy dissipated in the fall corresponds to the energy lost to the ions and electrons and is determined by $\bar{f}(E)$. The final flow into a horizontal lake corresponds to the build up of the particles into their final thermal equilibrium state. This is illustrated in Fig. 4.

The number density $n(t)$ is given by (4.3), and the inhomogeneous Fokker-Planck equation (4.17) allows us to express the rate of change as

$$\dot{n}(t) = \int \frac{d^3p}{(2\pi\hbar)^3}\,\delta\left(E - E_0\right)\, s(t) = \frac{s(t)}{2\pi^2\hbar^3}\,\sqrt{2m^3E_0}\,, \tag{4.19}$$

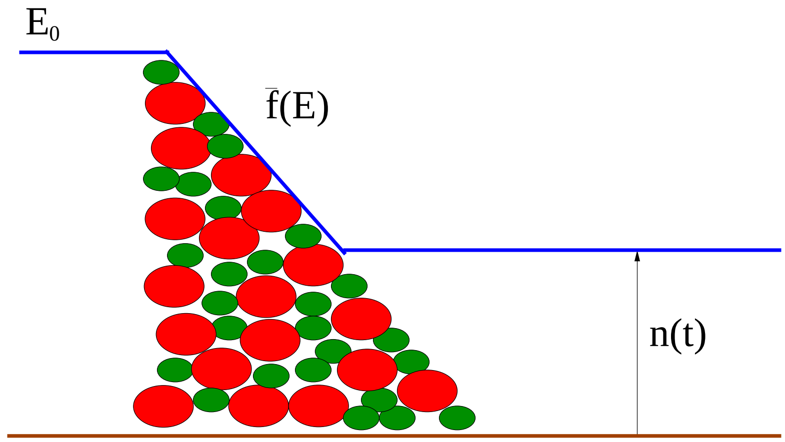

FIG. 4: The waterfall analogy. The small green rocks represent the plasma electrons while the larger red rocks are the plasma ions, with the flowing 'water' in blue representing the evolution of the produced charged particles (α particles for DT fusion). As 'water' falls down the electron-ion slope at a constant rate determined by $\bar{f}(E)$, energy is deposited into electrons and ions. At the bottom of the fall is a lake into which the excess 'water' drains, representing the final thermalized particles, with height $n(t)$ and Maxwell-Boltzmann distribution $\propto e^{-\beta E}$.

When the projectile particle source $s(t)$ is turned on and then attains a constant fixed value s_0, the number density $n(t)$ eventually increases linearly in time,

$$\begin{aligned} n(t) &= \int_{-\infty}^{t} dt'\, \dot{n}(t') = \frac{\sqrt{2m^3 E_0}}{2\pi^2\hbar^3} \int_{-\infty}^{t} dt'\, s(t') \\ &= \dot{n}_\infty\, t + \text{constant}\,, \end{aligned} \tag{4.20}$$

where

$$\dot{n}_\infty = \frac{s_0}{2\pi^2\hbar^3}\sqrt{2m^3 E_0}\,. \tag{4.21}$$

The energy density $\mathcal{E}(t)$ is given by (4.4), and the inhomogeneous Fokker-Planck equation (4.17) allows us to express the rate of energy change of the charged particles as,

$$\dot{\mathcal{E}}(t) = E_0\,\dot{n}(t) - \int \frac{d^3p}{(2\pi\hbar)^3}\, v \left\{ \Big[\mathcal{A}_{\rm I}(E) + \mathcal{A}_e(E)\Big] + \Big[T_{\rm I}\,\mathcal{A}_{\rm I}(E) + T_e\,\mathcal{A}_e(E)\Big]\frac{\partial}{\partial E} \right\} f(E,t)\,. \tag{4.22}$$

Since the first term in (4.18) corresponds to the thermal background and produces a zero contribution, expression (4.22) allows us to identify the energy exchange contributions to plasma electrons and ions,

$$f_{{\rm I},\,e} = \frac{1}{\dot{n}_\infty\, E_0} \int \frac{d^3p}{(2\pi\hbar)^3}\, v\, \mathcal{A}_{{\rm I},\,e}(E) \left[1 + T_{{\rm I},\,e}\,\frac{\partial}{\partial E}\right] \bar{f}(E)\,. \tag{4.23}$$

Reference [2] finds the solution $\bar{f}(E)$, whose derivation I will skip for the purposes of this summary. Instead, I will give the results of the integrals, which, for equal temperatures,

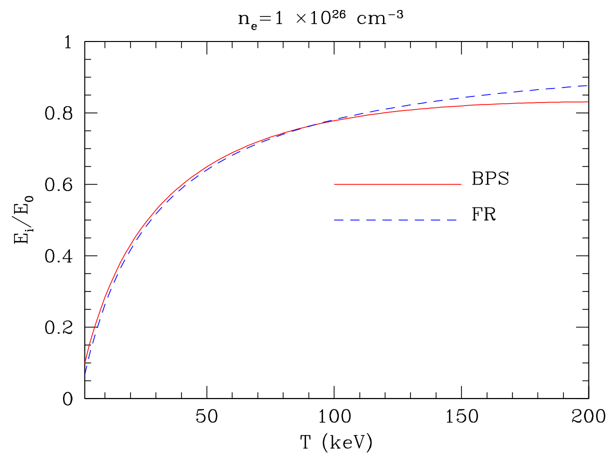

FIG. 5: The fractional energy loss into ions as a function of the plasma temperature for an α particle in an equimolar DT plasma with initial energy $E_0 = 3.54\,\mathrm{MeV}$. The electrons and ions have a common temperature T and the electron number density of the plasma is $n_e = 1 \times 10^{26}\,\mathrm{cm}^{-3}$, which corresponds to $T_0 = 28\,\mathrm{keV}$. The solid line is the analytic calculation of this work, and the dashed line is the fit provided by Fraley *et al.*

reduces to [6]

$$f_{\rm I} \;=\; \int_0^{E_0} \frac{dE}{E_0}\, \frac{\mathcal{A}_{\rm I}(E)}{\mathcal{A}(E)} \left[{\rm erf}(\sqrt{\beta E}) - \sqrt{\frac{4\beta E}{\pi}}\, e^{-\beta E} \right] \tag{4.24}$$

and

$$f_e \;=\; \int_0^{E_0} \frac{dE}{E_0}\, \frac{\mathcal{A}_e(E)}{\mathcal{A}(E)} \left[{\rm erf}(\sqrt{\beta E}) - \sqrt{\frac{4\beta E}{\pi}}\, e^{-\beta E} \right] \,, \tag{4.25}$$

where ${\rm erf}(x)$ is the error function. Since $dE_b/dx \to \mathcal{A}_b$ for large energies, and the error function approaches unity, at high energies E we see that Eqs. (4.24) and (4.25) approach the same form as the more intuitive but less accurate results (3.1) and (3.2). The primary differences occur for $E \sim T$. In the steady state, the energy density build up of final particles is their thermal energy per particle times the increase in number density, $\dot{\mathcal{E}} = \frac{3}{2}\, T\, \dot{n}_\infty$, and so the energy balance expression (4.22) now appears as

$$\frac{3}{2}\, T\, \dot{n}_\infty = \left[E_0 - E_{\rm I} - E_e \right] \dot{n}_\infty \,, \tag{4.26}$$

which gives expression (3.5): the original energy of a produced particle is lost to the ions and electrons with the remainder being the thermal energy of a free particle. Figure 5 gives the comparison with Fraley *et al.*. The case of differing electron and ion temperatures is much more complicated, and takes the form [2],

$$f_{\rm I} \;=\; +\int_0^\infty \frac{dE}{E_0}\,\frac{T_{\rm I}\,\mathcal{A}_{\rm I}(E)}{\langle T\mathcal{A}(E')\rangle}\,\left\{\theta\left(E_0-E\right)\;-\int_E^\infty dE'\sqrt{E'}\,\overline{\mathcal{N}}\,e^{-S(E')}\right\}+ \left[\frac{T_e-T_{\rm I}}{E_0}\right]\,G(T_{\rm I},T_e;E_0) \tag{4.27}$$

and

$$f_e \;=\; \int_0^\infty \frac{dE}{E_0}\,\frac{T_e\,\mathcal{A}_e(E)}{\langle T\mathcal{A}(E')\rangle}\,\left\{\theta\left(E_0-E\right)\;-\int_E^\infty dE'\sqrt{E'}\,\overline{\mathcal{N}}\,e^{-S(E')}\right\}+ \left[\frac{T_{\rm I}-T_e}{E_0}\right]\,G(T_{\rm I},T_e;E_0)\;, \tag{4.28}$$

with the normalization factor

$$\overline{\mathcal{N}} = \frac{m\,\sqrt{2m}}{2\pi^2\hbar^3}\,\mathcal{N}\,. \tag{4.29}$$

The work of Ref. [2] shows that the function G can be approximated, with an accuracy of a few percent, by

$$G(T_{\rm I},T_e;E_0) \;=\; \int_0^{E_0} dE\,E\,\frac{\mathcal{A}_{\rm I}(E)\mathcal{A}_e(E)}{\langle T\mathcal{A}(E)\rangle}e^{-S(E)}\int_0^E \frac{dE'}{E'}\,\frac{e^{+S(E')}}{\langle T\mathcal{A}(E')\rangle}\int_0^{E'} dE''\sqrt{E''}\,\overline{\mathcal{N}}\,e^{-S(E'')} +\frac{\mathcal{A}_{\rm I}(E_0)\,\mathcal{A}_e(E_0)}{\mathcal{A}^2(E_0)}\,, \tag{4.30}$$

The final energy integrals in Eqs. (4.27) and (4.28) run from $E=0$ to $E\to\infty$. In each case, the final integration region involves the exponentially small factor $\exp\{-S(E_0)\}\simeq\exp\{-E_0/\bar{T}\}$, where $\bar{T}$ is a typical plasma temperature. This is a very small factor, and hence this upper portion of the integration region may be safely neglected to write the results as

$$\begin{aligned} f_{\rm I} \;&=\; \int_0^{E_0} \frac{dE}{E_0}\,\frac{T_{\rm I}\,\mathcal{A}_{\rm I}(E)}{\langle T\mathcal{A}(E')\rangle}\,\left\{1-\int_E^\infty dE'\sqrt{E'}\,\overline{\mathcal{N}}\,e^{-S(E')}\right\}+\left[\frac{T_e-T_{\rm I}}{E_0}\right]\,G(T_{\rm I},T_e;E_0) \\ &=\; \int_0^{E_0} \frac{dE}{E_0}\,\frac{T_{\rm I}\,\mathcal{A}_{\rm I}(E)}{\langle T\mathcal{A}(E')\rangle}\,\int_0^E dE'\sqrt{E'}\,\overline{\mathcal{N}}\,e^{-S(E')}\;+\left[\frac{T_e-T_{\rm I}}{E_0}\right]\,G(T_{\rm I},T_e;E_0)\;, \end{aligned} \tag{4.31}$$

and

$$f_e \;=\; \int_0^{E_0} \frac{dE}{E_0}\,\frac{T_e\,\mathcal{A}_e(E)}{\langle T\mathcal{A}(E')\rangle}\,\int_0^E dE'\sqrt{E'}\,\overline{\mathcal{N}}\,e^{-S(E')}+\left[\frac{T_{\rm I}-T_e}{E_0}\right]\;G(T_{\rm I},T_e;E_0)\;, \tag{4.32}$$

Since the integration in the first line is over the finite interval $(0,E_0)$ and since it involves only nested integrals, rather than a three-dimensional integral with an arbitrary integrand

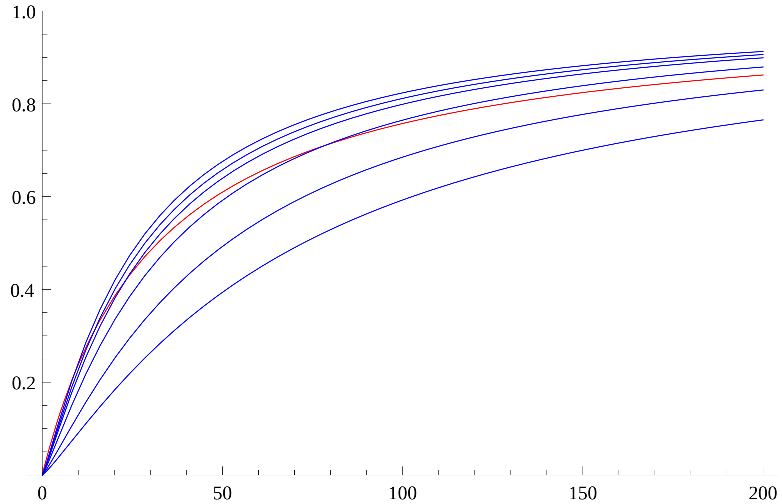

FIG. 6: The ion splitting fraction $f_{\rm I}$ for BPS [blue] and Fraley *et al.* [red]. The bottom axis is the electron temperature T_e in keV, and the left axis is the fraction $f_{\rm I}$. The blue BPS curves correspond to $T_{\rm I} = 10, 30, 50, 100, 200, 300\,{\rm keV}$ from the top to bottom.

that involves an general function of three variables, its numerical evaluation is not difficult. Reference [2, 3] exhibits an analytic fit for 50-50 molar DT gas with an α particle, that works for all densities between 10^{24}–$10^{25}\,{\rm cm}^{-3}$ and all temperatures in the tables, *i.e.* the following fit is 3% accurate on all three tables in the paper.

$$\frac{E_I}{E_0} = \frac{1}{2}\left\{1 + \tanh\left[1.34\left(\log_{10} T_e - 0.0015\, T_I - c(n_e)\right)\right]\right\} \tag{4.33}$$

$$c(n_e) = -0.005\log_{10}^2 n_e + 0.215\log_{10} n_e - 0.87\ . \tag{4.34}$$

APPENDIX A: THE BPS FORMALISM IN A NUTSHELL

Calculating Coulomb energy exchange processes in a hot plasma is notoriously difficult because of the subtleties of the Coulomb interaction, which produce logarithmic divergences at both long and short distance scales. This problem was first spelled out and solved to leading order by Landau and then Spitzer in the context of electron-ion temperature equilibration, and later by Corman *et al.* for the charged particle stopping power [9–11]. Since the divergences are only logarithmic, one introduces *ad hoc* short and long distance cutoffs $b_{\rm min}$ and $b_{\rm max}$, and the rate of energy loss of some process (such as temperature equilibration or stopping power) can be cast in the form

$$\frac{d\mathcal{E}}{dt} = K \int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b} = K \underbrace{\ln\left\{\frac{b_{\rm max}}{b_{\rm min}}\right\}}_{\text{Coulomb Logarithm}}. \tag{A1}$$

The prefactor K is easy to compute exactly. The logarithmic term, conventionally called the Coulomb logarithm, can only be approximated within the above scheme. The long distance scale $b_{\rm max}$ is set by the relevant Debye screening length, while the short distance scale $b_{\rm min}$ is determined by either the Landau length or the thermal de Broglie wave length (or some interpolation between them). As such, this method suffers a systematic uncertainty in the argument of the Coulomb logarithm.[2] In the language of perturbation theory, Eq. (A1) is accurate to leading order in the plasma coupling constant. This accuracy was extended to subleading order by BPS [1] which performed a first principles controlled calculation, including the exact terms under the logarithm and a rigorous treatment of the quantum to classical transition.

One can expand thermodynamic quantities as a perturbation series in integer powers (plus additional logs) of a dimensionless plasma coupling parameter g, and this approach was applied in BPS [1] to the stopping power and temperature equilibration problems. This coupling is generally defined by

$$g = \frac{e^2\kappa}{T} \ , \tag{A2}$$

where κ is a typical Debye wave number. This parameter is just the ratio of the potential energy of two electrons a Debye distance apart to the thermal kinetic energy of the plasma, and it is related to the usual plasma parameter by $g \propto \Gamma^{3/2}$. For this discussion, it is not critical whether one chooses the electron or the total Debye wave number, as we are primarily

[2] The constant under the logarithm sometimes varies by an order of magnitude from paper to paper within the literature, depending upon the choices of $b_{\rm min}$ and $b_{\rm max}$.

interested in constructing a small dimensionless quantity that will indicate how a systematic perturbative calculation should be organized. One may consult the Introduction of Ref. [1] for more details concerning this coupling constant. The main point is that quantities expand in integer powers of g, except for possible $\ln g$ terms, and the rate of energy exchange for the stopping power takes the form

$$\frac{d\mathcal{E}}{dt} = -\underbrace{A\, g^2 \ln g}_{\text{LO}} + \underbrace{B\, g^2}_{\text{NLO}} + \mathcal{O}(g^3) = A\, g^2 \ln \Lambda_{\text{coul}} + \mathcal{O}(g^3) \ . \tag{A3}$$

We have indicated the leading order (LO) and the next-to-leading order (NLO) terms in the g-expansion, and in a weakly coupled plasma these two contributions provide a good approximation to the exact result since the cubic order term will be quite small. Here $\ln \Lambda_{\text{coul}} = -\ln\{Cg\}$, with C defined by $B = -A \ln C$, and therefore knowing the next-to-leading order coefficient B is equivalent to knowing the exact coefficient C under the logarithm. To get a feel for the numbers, at the center of the sun $g = 0.04$, and the error term in Eq. (A3) is consequently small.

The problem with directly calculating A and B is that the kinetic equations diverge and must be regularized in the appropriate manner. Furthermore, to find the coefficient under the logarithm, this regularization procedure must preserve the delicate balance between the long and short distance physics. Indeed, the BPS calculation [1] includes both short distance physics and *dynamic* collective long distance physics, joined together exactly and unambiguously (and this is the rub), systematized by a power series expansion in the plasma coupling constant g. The coefficients are also calculated to all orders in the dimensionless quantum two-body scattering parameter η, thereby providing an exact interpolation between the extreme classical and quantum regimes.

The rigorous starting point is the BBGKY hierarchy (or its quantum generalization), which is finite and well defined and does not suffer from the aforementioned divergences. One must of course truncate this vast number of equations to something manageable, such as the Boltzmann or Lenard-Balescu equations, and it is this truncation process that renders the various three dimensional integrals divergent. However, as shown in Ref. [14], in ν spatial dimensions these divergences become simple poles of the form $1/(\nu-3)$. In spatial dimensions $\nu > 3$ the BBGKY hierarchy reduces to the Boltzmann equation (BE) to leading order in g (the BE is finite and does not have the usual long distance divergence for $\nu > 3$). Calculating the rate of energy loss using the ν-dimensional BE gives a result of the form

$$\frac{d\mathcal{E}^{>}}{dt} = H(\nu)\, \frac{g^2}{\nu - 3} + \mathcal{O}(\nu - 3) \qquad : \ \text{LO in } g \text{ when } \nu > 3 \ . \tag{A4}$$

The "greater-than" superscript is to remind us that the calculation has been performed in dimensions $\nu > 3$. In a similar manner, to leading order in g the BBGKY hierarchy reduces to the Lenard-Balescu equations (LBE) for $\nu < 3$ (the LBE is finite and does not suffer from short distance divergences when $\nu < 3$). A calculation of the energy rate with the LBE gives a form

$$\frac{d\mathcal{E}^<}{dt} \;=\; G(\nu)\,\frac{g^{\nu-1}}{3-\nu} + \mathcal{O}(3-\nu) \qquad : \text{ LO in } g \text{ when } \nu < 3\;. \tag{A5}$$

Note that both rates are of order g^2 in three dimensions, and they both suffer from a divergent simple pole. The coefficients $H(\nu)$ and $G(\nu)$ can be expanded in powers of $\epsilon = \nu - 3$, with

$$H(\nu) = -A + \epsilon\, H_1 + \mathcal{O}(\epsilon^2) \quad \text{and} \quad G(\nu) = -A + \epsilon\, G_1 + \mathcal{O}(\epsilon^2)\;. \tag{A6}$$

The leading terms must be equal, $H(3) = G(3) = -A$. This arises from the calculation itself and is not imposed by hand, and it makes the short- and long-distance poles cancel, thereby giving a finite result.

Since the rates $d\mathcal{E}^>/dt$ and $d\mathcal{E}^</dt$ were calculated in mutually exclusive dimensional regimes, one might think that they cannot be compared. However (and this is perhaps the most crucial step in the method, and certainly the most subtle), we can analytically continue the quantity $d\mathcal{E}^</dx$ to dimensional values $\nu > 3$, after which we can directly compare the rates (A4) and (A5) in a common dimension ν, and the limit $\nu \to 3$ may then be taken. Upon writing the g-dependence of Eq. (A5) as $g^{2+(\nu-3)}$, when $\nu > 3$ we see that the rate (A5) is indeed higher order in g than Eq. (A4) since $\epsilon = \nu - 3 > 0$:

$$\frac{d\mathcal{E}^<}{dt} \;=\; -G(\nu)\,\frac{g^{2+\epsilon}}{\nu-3} + \mathcal{O}(\nu-3) \quad : \text{ NLO in } g \text{ when } \nu > 3\;. \tag{A7}$$

The individual pole-terms in Eqs. (A4) and (A7) will cancel giving a finite result when the leading and next-to-leading order terms are added. Summing terms (A4) and (A7), using the relation $g^\epsilon = \exp\{\epsilon \ln g\} = 1 + \epsilon \ln g + \mathcal{O}(\epsilon^2)$, and taking the $\epsilon \to 0$ limit gives

$$\frac{d\mathcal{E}}{dt} = -A\, g^2 \ln g + B\, g^2 + \mathcal{O}(g^3)\;, \tag{A8}$$

with $B = H_1 - G_1$. This is in agreement with Eq. (A3). In this way, BPS has calculated the charged particle stopping power accurate to leading order and next-to-leading order in g. In the context of inertial confinement fusion and ignition, these arguments are presented in more detail in Ref. [7], while in the context of temperature equilibration one may consult Ref. [4].

APPENDIX B: THE A-COEFFICIENTS

The Fokker-Planck equation described in the text involves two scalar coefficient functions with only one of them, the $\mathcal{A}$ coefficient, entering into our problem of the partition of the energy loss of a fast charged particle into the ions and electrons in the plasma. The Fokker-Planck equation, and the coefficients $\mathcal{A}_{\rm I}$ and $\mathcal{A}_e$ coming from the ions and electrons that are needed for our problem, were discussed extensively in BPS [1]. There a method was employed to compute the $\mathcal{A}_b$ which enables the short-distance, point Coulomb scattering to be joined with the long-distance, collective force in an unambiguous fashion that has no double counting. This method was used to evaluate the $\mathcal{A}_b$ both to leading and to subleading order — roughly speaking — to order $n \ln n$ and order n, where n is the plasma number density (made dimensionless by the adduction of suitable parameters). For completeness, we present here the results of BPS.

The coefficient for the interaction of a projectile particle of energy E or velocity v_p, $(E = m_p\, v_p^2/2)$ with the species b of the background plasma may conveniently be written as

$$\mathcal{A}_b(v_p) = \mathcal{A}_b^{\rm C}(v_p) + \mathcal{A}_b^{\Delta Q}(v_p) \,, \tag{B1}$$

which is the same as Eq. (10.25) of BPS, with

$$\mathcal{A}_b^{\rm C}(v_p) = \mathcal{A}_{b,{\rm S}}^{\rm C}(v_p) + \mathcal{A}_{b,{\rm R}}^{<}(v_p) \,, \tag{B2}$$

which is the same as Eq. (9.6) of BPS. Here $\mathcal{A}_b^{\rm C}(v_p)$ has two terms. The first accounts for the hard Coulomb scattering in the classical limit, while the second accounts for the collective, long-distance effects, which are entirely classical. The term $\mathcal{A}_b^{\Delta Q}(v_p)$ is the quantum-mechanical correction to the scattering that vanishes in the limit in which Planck's constant vanishes, $\hbar \to 0$.

The first classical piece is given by

$$\begin{aligned} \mathcal{A}_{b,{\rm S}}^{\rm C}(v_p) \; &= \; \frac{e_p^2\, \kappa_b^2}{4\pi} \left(\frac{\beta_b m_b}{2\pi}\right)^{1/2} v_p \int_0^1 du\, u^{1/2}\, \exp\left\{-\frac{1}{2}\,\beta_b m_b v_p^2\, u\right\} \\ &\qquad \left[-\ln\left(\beta_b \frac{e_p e_b}{4\pi}\, K\, \frac{m_b}{m_{pb}}\, \frac{u}{1-u}\right) - 2\gamma + 2\right] \,, \end{aligned} \tag{B3}$$

which is contained in Eq. (9.5) of BPS. The wave number K.*** The function $F(v_p \cos\theta)$ is related to the classical dielectric function $\epsilon(k, kv_p \cos\theta)$ by

$$k^2\, \epsilon(k, kv_p \cos\theta) = k^2 + F(v_p \cos\theta) \,. \tag{B4}$$

Here, consistent with our leading orders evaluation, the dielectric function corresponds to the classical limit of the quantum ring sum. Hence the complex-valued function $F(v)$ is defined by

$$F(v) = -\int_{-\infty}^{\infty} du \, \frac{\rho_{\rm total}(u)}{v - u + i\eta} \,, \tag{B5}$$

with $\eta \to 0^+$. Note that, since $(x - i\eta)^{-1} - (x + i\eta)^{-1} = 2\pi i \, \delta(x)$, and since

$$\rho_b(-u) = -\rho_b(u) \,, \tag{B6}$$

we have

$$F(v) - F(-v) = 2\pi i \, \rho_{\rm total}(v) \,. \tag{B7}$$

Equations (B4) and (B5) are the formulae (7.7) and (7.8) of BPS.

The quantum correction is contained in Eq. (10.27) of BPS, and it reads

$$\begin{aligned}
\mathcal{A}_b^{\Delta Q}(v_p) \;=\; & -\frac{e_p^2 \, \kappa_b^2}{4\pi} \left(\frac{\beta_b m_b}{2\pi} \right)^{1/2} \frac{1}{2} \int_0^\infty dv_{pb} \Big\{ 2 \, {\rm Re} \, \psi \left(1 + i\eta_{pb} \right) - \ln \eta_{pb}^2 \Big\} \\
& \frac{1}{\beta_b m_b v_p v_{pb}} \left[\exp\left\{ -\frac{1}{2} \, \beta_b m_b \left(v_p - v_{pb} \right)^2 \right\} \left(1 - \frac{1}{\beta_b m_b v_p v_{pb}} \right) \right. \\
& \left. + \exp\left\{ -\frac{1}{2} \beta_b m_b \left(v_p + v_{pb} \right)^2 \right\} \left(1 + \frac{1}{\beta_b m_b v_p v_{pb}} \right) \right] .
\end{aligned} \tag{B8}$$

Here $\psi(z) = d \ln \Gamma(z)/dz$ and

$$\eta_{pb} = \frac{e_p e_b}{4\pi \hbar v_{pb}} \tag{B9}$$

is the dimensionless quantum coupling parameter. Note that we use the rationalized Gaussian units that were used by BPS in which the Coulomb potential energy between charges e_a and e_b a distance r_{ab} apart is given by $V = e_a e_b / (4\pi \, r_{ab})$.

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