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UPLIFT AND ROCKING OF A DEFORMABLE BODY SUBJECT TO BASE EXCITATION*

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ABSTRACT

The rolling and sliding motions of a rigid body subject to gravity and supported by a plane surface are treated in elementary texts on dynamics. Rocking of a riding body supported by a horizontal surface which experiences oscillatory accelerations due to an earthquake has been discussed by Housner[1]. If the body is deformable there is a potential for the dynamics of the body deformations to couple with the rocking mode; in particular, resonances in the deformation response can develop sufficient reaction moment at the base to cause base uplift which would not occur if the body were rigid. The paper presents a model suitable for studying this phenomena including the magnitude of the uplift, impacts occurring during stable rocking motions, and overturning.

The equations governing the plane motion of a deformable body with rocking boundary conditions supported by a horizontal flat surface subject to vertical and horizontal accelerations are derived. These equations depend on dynamic parameters of the body which are defined in terms of integrals of assumed modes of deformation. The number of assumed modes is arbitrary. Motions which involve uplift but not overturning are termed rocking motions and are characterized by impacts with the supporting plane. Integration of these equations requires care in dealing with high frequency rocking motions may occur.

NOMENCLATURE

F	force acting on base of the body (N)
I	impulse acting on base (N-s)
I, J, K	unit vectors in an inertial frame
M	total mass of the body (kg)
R	vector from inertial frame to an arbitrary point (m)
T	kinetic energy (kg-m^2/s^2)
V	potential energy (N-m); volume (m^3)
X, Y	inertial coordinates of supporting plane (m)
f	transformation from inertial to body coordinates
g	acceleration of gravity (m/s^2)
h	height of c.g. (m)
h	linear momentum (kg-m/s)
i, j, k	unit vectors fixed in the body
n	number of assumed modes of deformation
t	time (s)
x, y, z	body coordinates of an arbitrary point (m)
α	tipping parameter
ζ	dimensionless radius of gyration
θ	tipping angle
$\boldsymbol{\rho}$	position vector of an arbitrary point in body coordinates
τ	dimensionless time
$\mathbf{H}_0$	moment of momentum about c.g. (kg-m^2/s)
b_1, b_2	base distance from c.g. to left and right tipping axes (m)
k_{ij}	stiffness matrix (N/m)
q_i	generalized coordinate
$\mathbf{r}_0$	vector from inertial frame to tipping axis (m)
$\mathbf{r}_1$	vector from tipping axis to c.g. (m)
α_1, α_2	left and right tipping parameters
β_i	shear deformation parameters
γ_{ij}	coriolis parameters
δ_i	dilation parameters
κ_{ij}	dimensionless stiffness matrix
μ_{ij}	dimensionless mass matrix
$\boldsymbol{\phi}_i$	assumed mode deformation vector function
ψ_i	dimensionless participation vector

INTRODUCTION

The rolling and sliding motions of a rigid body subject to gravity and supported by a plane surface are treated in elementary texts on dynamics. Rocking of a rigid body supported by a horizontal surface which experiences oscillatory accelerations due to an earthquake has been discussed by Housner[1]. If the body is deformable there is a potential for the dynamics of the body deformations to couple with the rocking mode; in particular, resonances in the deformation response can develop sufficient reaction moment at the base to cause base uplift which would not occur if the body were rigid. The purpose of this work is to develop a model suitable for studying this phenomena including the magnitude of the uplift, impacts occurring during stable rocking motions, and overturning.

GEOMETRY

Consider a deformable body subject to gravity and supported on a horizontal surface which accelerates in the vertical and one horizontal direction. Restrict the deformation and rotation of the body to the plane of the supporting surface accelerations. The body can tip (rotate) about two points, called tipping axes, located at the "left" (point 1) and "right" (point 2) extremities of it's "base". These two points are assumed to be the same material points regardless of the deformation of the body. Two coordinate systems are used to describe the rotation and deformation of the body: an inertial frame with unit vectors **I, J** and a body-fixed frame with unit vectors **i** and **j**. **I** is parallel to the support plane, horizontal, and directed from point 1 to point 2; **J** is in the direction of the gravitational acceleration, up. The body fixed vector, **i**, is directed from point 1 to point 2 whatever the deformation or rotation and **i** and **j** are parallel to **I** and **J** when the body is not tipped. The rotation of the body is described by the tipping angle θ measured counter-clockwise from **I** to **i**; when $\theta > 0$ the body is tipped about the left tipping axis, point 1, and when $\theta < 0$ the body is tipped about the right tipping axis, point 2. The boundary conditions imposed at points 1 and 2 are zero applied moment, zero displacement in the $\pm$**I** direction, zero displacement in the -**J** direction and no tensile force in the **J** direction. Positive displacements are allowed in the +**J** direction, however, we will be interested in tipping motions where only one point has a non-zero displacement at any given time. These boundary conditions are termed "rocking" boundary conditions.

The parameter α is defined for the purpose of locating the tipping axis of the body relative to the center of mass of the body. α is termed the "tipping parameter" because it measures the stability of the body against tipping. Three physical dimensions are used in this definition. They are:

- h the height of the center of gravity (c.g.) of the undeformed and untipped body above the horizontal support plane,
- b_1 the distance from the projection of the c.g. on the support plane to the left tipping axis, point 1,
- b_2 the distance from the projection of the c.g. on the support plane to the right tipping axis, point 2.

Then α is given by:

$$\alpha(\theta) = -\alpha_1 = -b_1/h \qquad \text{for } \theta > 0$$

$$\alpha(\theta) = \alpha_2 = b_2/h \qquad \text{for } \theta < 0$$

The vector from the inertial reference frame to the tipping

$$\mathbf{r}_0 = (X + h\alpha)\mathbf{I} + Y\mathbf{J} = \mathbf{f}(X + h\alpha, Y, \theta)$$

axis is where X and Y are the horizontal and vertical displacements of the support plane from an inertial frame and the function $\mathbf{f}(a,b,\theta) \equiv (a\cos\theta + b\sin\theta)\mathbf{i} + (b\cos\theta - a\sin\theta)\mathbf{j}$. $\mathbf{f}(a,b,\theta)$ is the transformation of the vector $a\mathbf{I} + b\mathbf{J}$ to the body coordinates. Note that the function **f** is linear in its first two arguments; in particular

$$\mathbf{f}(X + h\alpha, Y, \theta) = h\mathbf{f}\left(\frac{X}{h} + \alpha, \frac{Y}{h}, \theta\right); \quad \mathbf{f}(0, g, \theta) + \mathbf{f}(\ddot{X}, \ddot{Y}, \theta) = g\,\mathbf{f}\left(\frac{\ddot{X}}{g}, \frac{\ddot{Y}}{g} + 1, \theta\right)$$

When $\alpha = -\alpha_1$, $\mathbf{r}_0$ points to the left tipping axis and when $\alpha = \alpha_2$, $\mathbf{r}_0$ points to the right tipping axis.

The position of the center of mass of the undeformed body relative to the tipping axis is given by the vector $\mathbf{r}_1 = -\alpha h\mathbf{i} + h\mathbf{j}$. The position of an arbitrary point in the undeformed body relative to the center of mass is $\boldsymbol{\rho}_0 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The location of an arbitrary point in the deformed body relative to the center of mass of the undeformed body is given by the vector

$$\rho(x,y,z,t) = \rho_0(x,y,z) + \sum_{i=1}^{n} \phi_i(x,y,z) q_i(t) = \rho_0 + \phi_i q_i$$

where the vector functions $\phi_i(x,y,z)$ are assumed-mode deformation functions and the scalar functions $q_i(t)$ are dimensionless generalized coordinates.

Finally, the position of an arbitrary point in the deformed body relative to the inertial frame is:

$$\mathbf{R} = \mathbf{r}_0 + \mathbf{r}_1 + \rho = h\left[\mathbf{f}\left(\frac{x}{h} + \alpha, \frac{y}{h}, \theta\right) - \alpha\mathbf{i} + \mathbf{j}\right] + \rho_0 + \theta_i(x,y,z) q_i(t)$$

KINEMATICS

Begin by introducing a dimensionless time variable $\tau = \sqrt{g/h}\, t$ where g is the acceleration of gravity and t is time in seconds. Use "dot" to denote derivatives with respect to real time, t, and "prime" to denote derivatives with respect to τ. Then:

$$\dot{\mathbf{r}}_1 = \frac{D\mathbf{r}_1}{Dt} = \sqrt{g/h}\,\frac{D\mathbf{r}_1}{D\tau} = \sqrt{g/h}\,\theta'\mathbf{k} \times (-h\alpha\mathbf{i} + h\mathbf{j}) = -\sqrt{gh}(\mathbf{i} + \alpha\mathbf{j})\theta'$$

$$\frac{D\mathbf{f}(X,Y,\theta)}{D\tau} = \mathbf{f}(X', Y', \theta) = \mathbf{f}(\sqrt{h/g}\dot{X}, \sqrt{h/g}\dot{Y}, \theta) = h\,\mathbf{f}(\dot{X}/\sqrt{gh}, \dot{Y}/\sqrt{gh}, \theta)\Big]$$

$$\frac{D\rho}{D\tau} = \theta'\mathbf{k} \times (\rho_0 + \phi_j q_j) + \phi_j q_j'$$

The velocity of an arbitrary point in the body relative to the inertial frame is:

$$\dot{\mathbf{R}} = \frac{D\mathbf{R}}{Dt} = \sqrt{g/h}\,\frac{D\mathbf{R}}{D\tau} = \sqrt{gh}\Big[\mathbf{f}(\dot{X}/\sqrt{gh}, \dot{Y}/\sqrt{gh}, \theta) - \Big(\mathbf{i} - \alpha\mathbf{j} - \frac{1}{h}\mathbf{k}\times(\rho_0 + \phi_j q_j)\Big)\theta' + \frac{1}{h}\phi_j(x,y,z)\,q_j'\Big]$$

and the inertial acceleration of an arbitrary point is:

$$\ddot{\mathbf{R}} = \frac{D^2\mathbf{R}}{Dt^2} = \frac{g}{h}\frac{D^2\mathbf{R}}{D\tau^2} = g\Big[\mathbf{f}(\frac{\ddot{X}}{g}, \frac{\ddot{Y}}{g}, \theta) + \Big[\alpha\mathbf{i} - \mathbf{j} + \mathbf{k}\times\Big(\mathbf{k}\times\frac{1}{h}(\rho_0 + \phi_j q_j)\Big)\Big]\theta'^2 + \Big[\mathbf{j} - \alpha\mathbf{i} + \mathbf{k}\times\frac{1}{h}(\rho_0 + \phi_j q_j)\Big]\theta'' + \frac{2}{h}\mathbf{k}\times\phi_j q_j'\theta' + \frac{1}{h}\phi_j q_j''\Big]$$

KINETICS

The linear momentum of the body is given by the volume integral over the body of the point inertial velocity times the elemental mass

$$\mathbf{h} = \int_V \frac{D\mathbf{R}}{Dt}\,dm$$

and the moment of momentum about the center of gravity is

$$\mathbf{H}_0 = \int_V \rho \times \frac{D\mathbf{R}}{Dt}\,dm.$$

The kinetic energy, T, is 1/2 the integral of the inertial velocity squared:

$$T = \frac{1}{2}\int_V \frac{D\mathbf{R}}{Dt}\cdot\frac{D\mathbf{R}}{Dt}\,dm = \frac{g}{2h}\int_V \frac{D\mathbf{R}}{D\tau}\cdot\frac{D\mathbf{R}}{D\tau}\,dm$$

The potential energy, V, for the system is the gravitational potential plus the strain energy of deformation, $V = V_g + V_{se}$. The gravitational potential is the negative of the work done by the gravity force in an arbitrary displacement of the body. Since the gravity force is $\mathbf{f}(0,-g,\theta)$, the gravity potential is

$$V_g = \int_V g\,\mathbf{f}(0,1,\theta)\cdot\mathbf{R}\,dm.$$

The strain energy for a linear deformation theory takes the form $V_{se} = \frac{1}{2}k_{ij}q_i q_j$ where the stiffness matrix, $k_{ij} = k_{ji}$, depends on a volume integral of the derivatives of the deformation functions $\phi_i(x,y,z)$.

The following parameters describe the dynamic properties of the body. With the exception of the total mass, M, they are given in a dimensionless form.

$$M = \int_V dm$$

(Total mass of the body)

$$\zeta^2 = \frac{1}{Mh^2}\int_V (\mathbf{k}\times\rho_0)\cdot(\mathbf{k}\times\rho_0)\,dm = \frac{I_{zz}}{Mh^2}$$

(Radius of gyration squared of the undeformed body)

$$\mu_{ij} = \frac{1}{Mh^2}\int_V \phi_i\cdot\phi_j\,dm$$

(Deformation mass matrix)

$$\kappa_{ij} = \frac{1}{Mgh}k_{ij}$$

(Deformation stiffness matrix)

$$\gamma_{ij} = \frac{1}{Mh^2}\int_V \phi_i\times\phi_j\,dm\cdot\mathbf{k}$$

(Gyroscopic or "Coriolis" terms)

$$\psi_i = \psi_{xi}\mathbf{i} + \psi_{yi}\mathbf{j} = \frac{1}{Mh}\int_V \phi_i(x,y,z)\,dm$$

(Base excitation participation factors)

$$\delta_i = \frac{1}{Mh^2}\int_V \rho_0\cdot\phi_i\,dm$$

(Measure of modal dilation of the body)

$$\beta_i = \frac{1}{Mh^2}\mathbf{k}\cdot\int_V \rho_0\times\phi_i\,dm$$

(Measure of modal shear of the body)

Using these defined parameters, the kinetic energy may be written as

$$T = Mgh\Big\{\frac{1}{2}\Big[\frac{\dot{X}^2}{gh} + \frac{\dot{Y}^2}{gh} + \mu_{ij}q_i'q_j' + (1 + \zeta^2 + \alpha^2 + 2\delta_j q_j + \mu_{ij}q_i q_j - 2\alpha\psi_{xj}q_j + 2\psi_{yj}q_j)\theta'^2\Big] + (\beta_j + \gamma_{ij}q_i - \alpha\psi_{yj} - \psi_{xj})\,q_j'\theta' + \mathbf{f}(\frac{\dot{X}}{\sqrt{gh}}, \frac{\dot{Y}}{\sqrt{gh}}, \theta)\cdot\psi_j q_j' + \mathbf{k}\cdot\Big[(\mathbf{j} - \alpha\mathbf{i} + \psi_j q_j)\times\mathbf{f}(\frac{\dot{X}}{\sqrt{gh}}, \frac{\dot{Y}}{\sqrt{gh}}, \theta)\Big]\theta'\Big\},$$

the potential energy takes the form

$$V = Mgh\Big\{\frac{Y}{h} + \mathbf{f}(0,1,\theta)(-\alpha\mathbf{i} + \mathbf{j} + \psi_j q_j) + \frac{1}{2}\kappa_{ij}q_i q_j\Big\};$$

and the linear momentum takes the form

$$\mathbf{h} = \sqrt{gh}M\Big[\mathbf{f}(\frac{\dot{X}}{\sqrt{gh}}, \frac{\dot{Y}}{\sqrt{gh}}, \theta) + \psi_j q_j' + \mathbf{k}\times(-\alpha\mathbf{i} + \mathbf{j} + \psi_j q_j)\theta'\Big].$$

The equations of motion of the body can be derived from the Lagrangian formulation using the expressions for kinetic and potential energy. Using the usual summation convention for repeated subscripts the equations of motion take the form:

$$\mu_{ij}q_j'' + (\beta_i - \gamma_{ij}q_j - \alpha\psi_{yi} - \psi_{xi})\theta'' - 2\gamma_{ij}q_j'\theta' - (\delta_i + \mu_{ij}q_j - \alpha\psi_{xi} + \psi_{yi})\theta'^2 + \kappa_{ij}q_j = \psi_i\cdot\mathbf{f}(\frac{\ddot{X}}{g}, \frac{\ddot{Y}}{g} + 1, \theta) \quad (1)$$

$$\left(1+\zeta^2+\alpha^2+\mu_{ij}q_iq_j+2(\delta_j-\alpha\psi_{xj}+\psi_{yj})q_j\right)\theta'' \quad (2)$$
$$+(\beta_j+\gamma_{ij}q_i-\alpha\psi_{yj}-\psi_{xj})q_j''$$
$$+2(\mu_{ij}q+\delta_j-\alpha\psi_{xj}+\psi_{yj})q_j'\theta'$$
$$k\cdot\left[(-\alpha\mathbf{i}+\mathbf{j}+\psi_jq)\times\mathbf{f}(\frac{\ddot{X}}{g},\frac{\ddot{Y}}{g}+1,\theta)\right]$$

where $\alpha=-\alpha_1$ for $\theta>0$; $\alpha=\alpha_2$ for $\theta<0$ and

$$\mathbf{f}(\frac{\ddot{X}}{g},\frac{\ddot{Y}}{g}+1,\theta)=\left[\frac{\ddot{X}}{g}\cos\theta+(\frac{\ddot{Y}}{g}+1)\sin\theta\right]\mathbf{i}+\left[(\frac{\ddot{Y}}{g}+1)\cos\theta-\frac{\ddot{X}}{g}\sin\theta\right]\mathbf{j}$$

The stable equilibrium state when $\ddot{X}=\ddot{Y}=0$ is

$$\theta=\theta'=q_i'=0,\quad q_i=-[k_{ij}]^{-1}\psi_{yj}$$

which represents the deformation of the body under gravity loading.

It is usual to introduce "material damping" into equation (1) by adding a term $\eta_{ij}q'_j$ where η_{ij} is proportional to a linear combination of the mass matrix μ_{ij} and the stiffness matrix κ_{ij}.

The resultant of the reaction force between the floor and the body is:

$$\mathbf{F}=\int_V\ddot{\mathbf{R}}\,dm=Mg\Big[\mathbf{f}(\frac{\ddot{X}}{g},\frac{\ddot{Y}}{g}+1,\theta)+[(\alpha-\psi_{xj}q_j)\theta'^2$$
$$-2\psi_{yj}q_j'\theta'-(1+\psi_{yj}q_j)\theta''+\psi_{xj}q_j'']\mathbf{i}+[-(1+\psi_{yj}q_j)\theta'^2 \quad (3)$$
$$+2\psi_{xj}q_j'\theta'-(\alpha-\psi_{xj}q_j)\theta''+\psi_{yj}q_j'']\mathbf{j}\Big]$$

This force acts at the tipping axis when the body is tipped. When the body is not tipped the angle theta is zero and the location of the resultant reaction force is such as to have zero moment about the center of gravity of the deformed body. Define the parameter alpha for $\theta=0$ so that the vector $h(\alpha\mathbf{i}+\mathbf{j}-\psi_jq_j)$ is the location of the resultant reaction force on the base of the body relative to the deformed c.g. Then the condition that the moment of **F** about the c.g. of the deformed body be zero is $h(\alpha\mathbf{i}+\mathbf{j}-\psi_jq_j)\times\mathbf{F}=0$ which may be solved for α:

$$\alpha=\frac{(\frac{\ddot{Y}}{g}+1)\psi x_jq_j-\frac{\ddot{X}}{g}(\psi_{yj}q_j+1)+(\beta_j+\gamma_{ij}q_i-\psi_{xj})q_j''}{1+\frac{\ddot{Y}}{g}+\psi_{yj}q_j''} \quad (4)$$

UPLIFT

Consider a body initially at rest on the surface subjected to moderate base excitation. The early motion is governed by equation (1) with $\theta=0$. The reaction force is given by (3) located at (4). If the base acceleration is sufficiently small, α will never reach the extremities of the interval $[-\alpha_1,\alpha_2]$ and the body will not lift off its base. If the excitation is more severe α will eventually reach the extremities of the interval $[-\alpha_1,\alpha_2]$ and uplift will occur. During uplift α remains fixed at the extreme value, and equation (2) governs the evolution of θ. If the excitation is not too severe, the uplift angle will reach a maximum, the angular velocity will change sign, θ will return to 0 and the body will impact the support plane. Such a stable tipping motion is termed "rocking". If, however, the excitation is quite severe θ will grow sufficiently large that the center of gravity of the body is outside the base support and gravity will become destabilizing. In this case θ will begin to grow rapidly and the body is said to overturn.

OVERTURNING

An exact condition for overturning would depend explicitly on both the base excitation and the deformation q(t), that is, the body may tip arbitrarily far and still return to an upright position provided the future accelerations are sufficient to right it. However, an unstable static equilibrium condition can be given which approximates the limit of stable rocking provided the future base acceleration and the deformation accelerations are small. This equilibrium is given by the solution to the equations:

$$\kappa_{ij}q_j=-\left(\psi_{xi}\sin(\theta)+\psi_{yi}\cos(\theta)\right)$$
$$\tan(\theta)=\frac{\psi_{xj}q_j-\alpha}{\psi_{yj}q_j+1}$$

If the static deformations are small, the unstable equilibrium is approximately

$$\theta=-\tan^{-1}(\alpha). \quad (5)$$

When integrating the equations of rocking motion after uplift it is prudent to check condition (5) to avoid following an unstable trajectory.

IMPACTS

The second condition which must be checked while integrating the rocking equations is the impact condition, $\theta=0$. When impact occurs a discontinuity takes place in the angular velocity, θ', and the generalized velocities, q'_i. Consider first an impact with the body angular velocity positive prior to the impact, i.e. $\theta'^- >0$. For such an impact the impulsive reaction occurs at point 1 and is given by the change in linear momentum:

$$\hat{\mathbf{I}}_1=\Delta h=\sqrt{gh}\,M\left\{\begin{array}{l}[\psi_{xi}(q_i'^+-q_i'^-)-(1+\psi_{yi}q_i)(\theta'^+-\theta'^-)]\mathbf{i}\\+[\psi_{yi}(q_i'^+-q_i'^-)+\psi_{xi}q(\theta'^+-\theta'^-)-\alpha_2\theta'^+-\alpha_1\theta'^-]\mathbf{j}\end{array}\right\} \quad (6)$$

At impact, i.e. $\theta=0$, the moment of momentum about the center of gravity is:

$$H_0(\alpha,q',\theta')=\int_V\rho\times\frac{DR}{Dt}dm$$
$$=Mh\sqrt{gh}\Big\{(\psi_jq_i\times\mathbf{f}(\frac{\dot{X}}{\sqrt{gh}},\frac{\dot{Y}}{\sqrt{gh}},0)$$
$$+[(\beta_j+\gamma_{ij}q_i)q_j'+(\zeta^2+2\delta_jq_j+\mu_{ij}q_iq_j-\alpha\psi_{xj}q_j$$
$$+\psi_{yj}q_j)\theta']\mathbf{k}\Big\}$$

The change in moment of momentum about point 1 is:

$$\Delta H_1 = H_0(-\alpha_1, q'^{+}, \theta'^{+}) - H_0(\alpha_2, q'^{-}, \theta'^{-}) + h(\alpha_1 \mathbf{i} + \mathbf{j}) \times \hat{I}_1$$

The condition that linear momentum at each point be preserved within the constraints of the assumed deformations leads to an expression for the change in the generalized velocities in terms of the body angular velocity before and after the impact:

$$\Delta \int_V \phi_i \dot{R}\, dm = 0 \quad \text{for } i=1 \ldots n \text{ implies}$$

$$\Delta q_i' = [\mu_{ij}]^{-1}[(\psi_{xi} - \beta_i + \gamma_{ij} q_j)\Delta\theta' \;\; \psi_{yi}(\alpha_1 \theta'^{+} + \alpha_2 \theta'^{-})] \tag{7}$$

The condition that moment of momentum about the point of impact be preserved gives the new angular velocity of the body in terms of the old angular velocity:

$$\theta'^{+} = \left[\frac{\begin{array}{c} 1 + \zeta^2 + 2\delta_j q_j + \mu_{ij} q_i q_j + 2\psi_{yj} q_j + (\alpha_1 - \alpha_2)\psi_{xj} q_j - \alpha_1 \alpha_2 \\ -(\psi_{xi} - \beta_i - \gamma_{ij} q_j + \alpha_1 \psi_{yi})\, [\mu_{ij}]^{-1}\, (\psi_{xj} - \beta_j + \gamma_{ij} q_i + \alpha_2 \psi_{yj}) \end{array}}{\begin{array}{c} 1 + \zeta^2 + 2\delta_j q_j + \mu_{ij} q_i q_j + 2\psi_{yj} q_j + 2\alpha_1 \psi_{xj} q_j + \alpha_1^2 \\ -(\psi_{xi} - \beta_i - \gamma_{ij} q_j + \alpha_1 \psi_{yi})\, [\mu_{ij}]^{-1}\, (\psi_{xj} - \beta_j + \gamma_{ij} q_i - \alpha_1 \psi_{yj}) \end{array}} \right] \theta'^{-} \tag{8}$$

For an impact with 0 impact occurs at point 2. The same expressions for the impulse, change in generalized velocities, and new angular velocities hold with the substitution:

$$\alpha_1 \rightarrow -\alpha_2 ; \qquad \alpha_2 \rightarrow -\alpha_1 \tag{9}$$

ENERGY LOSS

The impact conditions do not conserve energy for the system. The energy loss during impact is equal to the work done by the impulsive force minus the change in the kinetic energy of the body. The work done by the impulsive force is:

$$W_{\hat{I}} = \dot{r}_0 \cdot \hat{I} = \dot{X}\hat{I}_x + \dot{Y}\hat{I}_y$$

and the energy loss in an impact at point 1 with $\theta'^{-} > 0$ is:

$$E_I = Mgh \left[\begin{array}{c} (1 + \zeta^2 + 2\delta_j q_j + \mu_{ij} q_i q_j + 2\psi_{xj} q_j)(\theta'^{2+} \theta'^{2-}) \\ + \mu_{ij} q_i'^{+} q_j'^{+} \mu_{ij} q_i'^{-} q_j'^{-} + (\alpha_1^2 + 2\alpha_1 \psi_{xj} q_j)\theta'^{2+} (\alpha_2^2 - 2\alpha_2 \psi_{xj} q_j)\theta'^{2-} \\ + (\beta_j + \gamma_{ij} q_i - \psi_{xj})(q_j'^{+}\theta'^{+} - q_j'^{-}\theta'^{-}) + \alpha_1 \psi_{yj} q_j'^{+}\theta'^{+} + \alpha_2 \psi_{yj} q_j'^{-}\theta'^{-} \end{array} \right] \tag{10}$$

The same expression holds at point 2 when $\theta'^{-} < 0$ with the substitution (9). The energy lost during impact may be considered to go into higher modes of vibration not modeled in the assumed mode model and into plastic deformation of the body. The energy in higher modes will be dissipated by material damping and radiation. Thus the energy lost in an impact is an upper bound on the plastic deformation and may be used as a damage function in assessing the effect of rocking motion on the body.

CHATTERING

As energy is lost in each impact, the period between impacts can become arbitrarily small. The high frequency of this chattering motion and the associated overhead of impact calculations becomes an impediment to efficient integration of the equations. A practical solution is to stop the chattering motion by setting $\theta = 0, \theta' = 0$ when the frequency of impacts becomes large compared to the natural frequencies of the deformation.

SLIDING

This mathematical model does not allow for sliding relative to the support surface during impact, tipping, or non-uplift conditions. Sliding is presumed to be prevented by friction. The required friction coefficient can be determined by comparing the ratio of the horizontal to vertical component of the reaction force or impulse. For this purpose the reaction force (3) may be written in inertial coordinates as:

$$\begin{aligned} \mathbf{F} = Mg \Big\{ \Big[& \frac{\ddot{X}}{g} + [(\alpha - \psi_{xj} q_j)\theta'^2 - 2\psi_{yj} q_j' \theta' - (1 + \psi_{yj} q_j)\theta'' + \psi_{xj} q_j''] \cos(\theta) \\ & - [-(1 + \psi_{yj} q_j)\theta'^2 + 2\psi_{xj} q_j' \theta' - (\alpha - \psi_{xj} q_j)\theta'' + \psi_{yj} q_j''] \sin(\theta) \Big] \mathbf{I} \\ & + \Big[1 + \frac{\ddot{Y}}{g} + [(\alpha - \psi_{xj} q_j)\theta'^2 - 2\psi_{yj} q_j' \theta' - (1 + \psi_{yj} q_j)\theta'' + \psi_{xj} q_j''] \sin(\theta) \\ & + [-(1 + \psi_{yj} q_j)\theta'^2 + 2\psi_{xj} q_j' \theta' - (\alpha - \psi_{xj} q_j)\theta'' + \psi_{yj} q_j''] \cos(\theta) \Big] \mathbf{J} \Big\} \end{aligned}$$

Sliding will not occur provided the ratio of the **I** component to the **J** component is less than the coefficient of friction between the body and the surface. For impulses, equation (6) may be used directly since $\theta = 0$ so that $\mathbf{i} = \mathbf{I}$ and $\mathbf{j} = \mathbf{J}$.

NUMERICS

Under most circumstances the fundamental deformation mode will dominate the uplift moment; higher modes may contribute to the internal stress state of the body but normally they will require minimum reaction moment at the base. Thus if the normal modes in a pinned base support configuration are known, a single mode of deformation is sufficient for a tipping analysis of a deformable body. With this observation in mind a FORTRAN computer code was written to integrate the rocking equations for a deformable body with a single deformation mode. A Runge-Kutta integration scheme was used along with the logic to determine the time of impacts, check for overturning, eliminate chatter, and retain records of extreme values of uplift, impulse, and energy loss. The code runs efficiently on a 386 PC computer.

SUMMARY

The equations governing the plane motion of a deformable body with rocking boundary conditions supported by a horizontal flat surface subject to vertical and horizontal accelerations are given by (1) and (2). These equations depend on dynamic parameters of the body which are defined in terms of integrals of assumed modes of deformation. The number of assumed modes is arbitrary. The reaction force at the surface is given by (3) and is located using (4). Equation (4) also governs uplift. Overturning of the body is approximated by (5). Motions which involve uplift but not overturning are termed rocking motions and are characterized by impacts with the supporting plane. The impulses are given by (6), the corresponding changes in the generalized deformation velocities are given by (7) and the changes in the angular velocity are given by (8). These impulses result in the loss of energy for the body given by (10). Integration of these equations requires some care because high frequency rocking motions can occur. Two examples are considered using a computer simulation with a single deformation mode.

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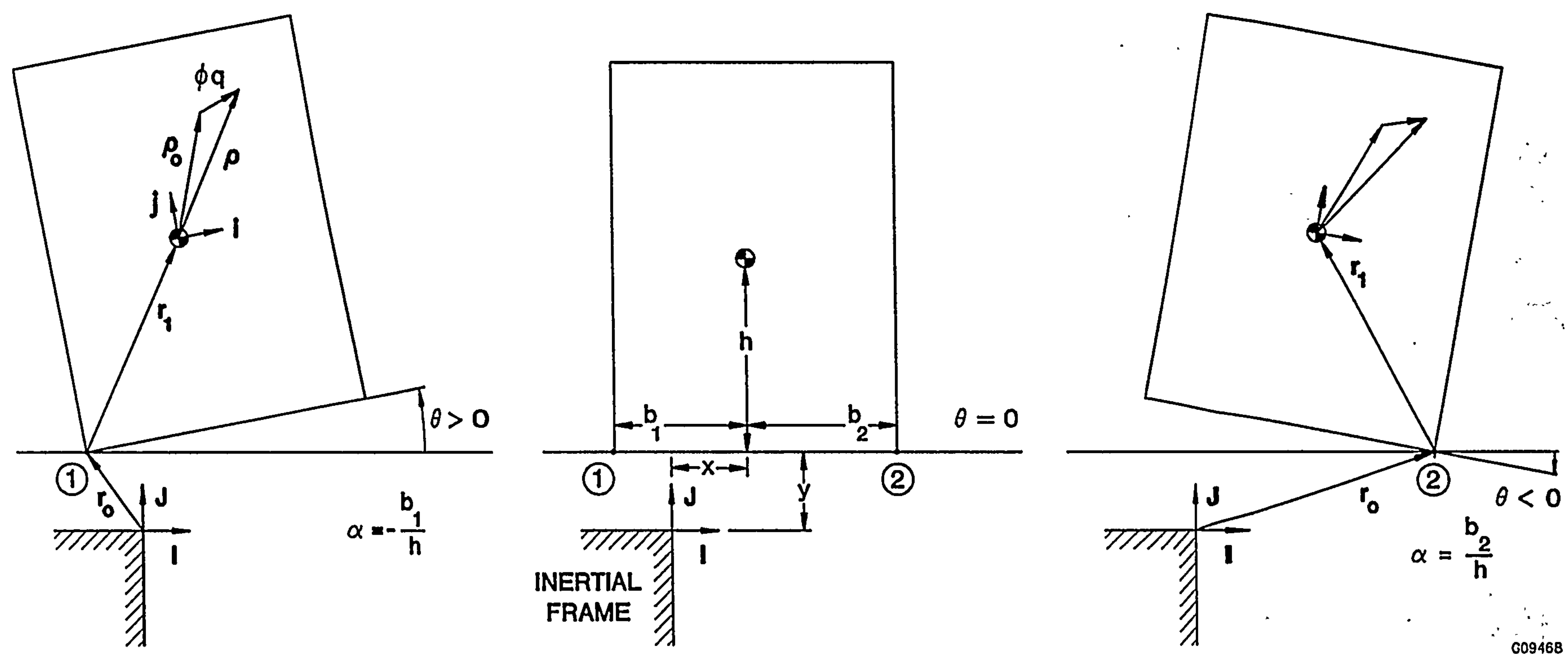
REFERENCE:

1. Housner, G. W., 1963, "The Behavior of Inverted Pendulum Structures during Earthquakes", *Bull. of the Seism. Soc. of Amer.*, Vol. 53, No. 1.



φq
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