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Title: The jump-off velocity of an impulsively loaded spherical shell

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The jump-off velocity of an impulsively loaded spherical shell

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1 Introduction

We consider a constant temperature spherical shell of isotropic, homogeneous, linearly elastic material with density ρ and Lamé coefficients λ and μ. The inner and outer radii of the shell are r_i and r_o, respectively. We assume that the inside of the shell is a void. On the outside of the shell, we apply a uniform, time-varying pressure $p(t)$. We also assume that the shell is initially at rest. The equation of motion and boundary and initial conditions for this problem are

$$c^{-2}\ddot{u} = u'' + \frac{2}{r}u' - \frac{2}{r^2}u; \quad r_i < r < r_o,\ t > 0 \tag{1}$$

$$\alpha_i u'(r_i, t) + \beta_i u(r_i, t) = 0; \quad t > 0 \tag{2}$$

$$\alpha_o u'(r_o, t) + \beta_o u(r_o, t) = -p(t); \quad t > 0 \tag{3}$$

$$u(r, 0) = 0, \quad \dot{u}(r, 0) = 0; \quad r_i \le r \le r_o \tag{4}$$

where $u(r,t)$ is the displacement, $u' = \frac{\partial u}{\partial r}$ is the radial strain, $\dot{u} = \frac{\partial u}{\partial t}$ is the velocity, $\alpha_i = \alpha_o = \lambda + 2\mu$, $\beta_i = \frac{2}{r_i}\lambda$, $\beta_o = \frac{2}{r_o}\lambda$, and

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

is the speed of sound through the material [1, 2, 3].

We want to compute the jump-off time and velocity of the pressure wave, which are the first time after $t = 0$ at which the pressure wave from the outer surface reaches the inner surface. This analysis computes the jump-off velocity and time for both compressible and incompressible materials. This differs substantially from [3], where only incompressible materials are considered. We will consider the behavior of an impulsively loaded, exponentially decaying pressure wave $p(t) = P_0 e^{-\alpha t}$, where $\alpha \ge 0$. We notice that a constant pressure wave $P(t) = P_0$ is a special case ($\alpha = 0$) of a decaying pressure wave. Both of these boundary conditions are considered in [3].

As described in [1] using the method of characteristics, the displacement and velocity have the forms

$$u(r,t) = \frac{1}{r}[\psi_1'(r - ct) + \psi_2'(r + ct)] - \frac{1}{r^2}[\psi_1(r - ct) + \psi_2(r + ct)], \tag{5}$$

$$\dot{u}(r,t) = \frac{c}{r}[\psi_2''(r + ct) - \psi_1''(r - ct)] - \frac{c}{r^2}[\psi_2'(r + ct) - \psi_1'(r - ct)], \tag{6}$$

where the functions $\psi_1(\xi)$ and $\psi_2(\eta)$ for $\xi = r - ct$ and $\eta = r + ct$ are determined from (2), (3), and (4). Figure 1 gives a picture of the strips used to compute the functions ψ_1 and ψ_2 and shows how a pressure wave front originating on the outer radius at time $t = 0$ propagates through the shell. In strip [0] of the figure, ψ_1 is computed from (4), so ψ_1 and all of its derivatives are identically 0 for all ξ in the strip. Likewise, in strip {0}, ψ_2 and all of its derivatives are identically 0.

2 Solution for compressible material

In strip $\{N\}$ for $N = 1, 2, \ldots$, ψ_2 is computed from (3). When the material is compressible (i.e. when the Poisson ratio $\nu \neq 1/2$), $\psi_2(\eta)$ in strip $\{N\}$ is given by

$$\psi_{2\{N\}}(\eta) = E_{\{N\}1} e^{a_{\{1\}}\eta} \cos b_{\{1\}}\eta + E_{\{N\}2} e^{a_{\{1\}}\eta} \sin b_{\{1\}}\eta - \frac{1}{b_{\{1\}}} \int_{\eta^*_{\{N\}}}^{\eta} e^{a_{\{1\}}(\eta-\gamma)} [\frac{r_o}{\alpha_o} p(\frac{\gamma - r_o}{c}) \tag{7}$$
$$+\psi''_{1[N-1]}(2r_o - \gamma) + \frac{1}{r_o}(\frac{\beta_o r_o}{\alpha_o} - 2)\psi'_{1[N-1]}(2r_o - \gamma) + \frac{1}{r_o^2}(2 - \frac{\beta_o r_o}{\alpha_o})\psi_{1[N-1]}(2r_o - \gamma)]$$
$$\sin b_{\{1\}}(\eta - \gamma) d\gamma,$$

where $\psi_{1[N-1]}$ is $\psi_1(\xi)$ in strip $[N-1]$,

$$a_{\{1\}} = \frac{1 - 2\nu}{r_o(1 - \nu)}, \quad b_{\{1\}} = \frac{\sqrt{1 - 2\nu}}{r_o(1 - \nu)},$$

$\eta^*_{\{N\}} = r_o + (N-1)R$ is the lower boundary of strip $\{N\}$, $R = r_o - r_i$ is the thickness of the shell, and $E_{\{N\}1}$, $E_{\{N\}2}$ are constants chosen so that

$$\psi_{2\{N\}}(\eta^*_{\{N\}}) = \psi_{2\{N-1\}}(\eta^*_{\{N\}}),$$
$$\psi'_{2\{N\}}(\eta^*_{\{N\}}) = \psi'_{2\{N-1\}}(\eta^*_{\{N\}}).$$

Thus both ψ_2 and ψ'_2 are continuous across strip boundaries. The constants $E_{\{N\}1}$, $E_{\{N\}2}$ are given by the vector equation

$$\begin{bmatrix} E_{\{N\}1} \\ E_{\{N\}2} \end{bmatrix} = \frac{e^{-a_{\{1\}}\eta^*_{\{N\}}}}{b_{\{1\}}} \begin{bmatrix} a_{\{1\}} \sin b_{\{1\}}\eta^*_{\{N\}} + b_{\{1\}} \cos b_{\{1\}}\eta^*_{\{N\}} & -\sin b_{\{1\}}\eta^*_{\{N\}} \\ -a_{\{1\}} \cos b_{\{1\}}\eta^*_{\{N\}} + b_{\{1\}} \sin b_{\{1\}}\eta^*_{\{N\}} & \cos b_{\{1\}}\eta^*_{\{N\}} \end{bmatrix} \begin{bmatrix} \psi_{2\{N-1\}}(\eta^*_{\{N\}}) \\ \psi'_{2\{N-1\}}(\eta^*_{\{N\}}) \end{bmatrix}.$$

In strip $[N]$ for $n = 1, 2, \ldots$, ψ_1 is computed from (2). When the material is compressible, $\psi_1(\xi)$ in strip $[N]$ is given by

$$\psi_{1[N]}(\xi) = E_{[N]1} e^{a_{[1]}\xi} \cos b_{[1]}\xi + E_{[N]2} e^{a_{[1]}\xi} \sin b_{[1]}\xi - \frac{1}{b_{[1]}} \int_{\xi^*_{[N]}}^{\xi} e^{a_{[1]}(\xi-\gamma)} [\psi''_{2\{N-1\}}(2r_i - \gamma) \tag{8}$$
$$+\frac{1}{r_i}(\frac{\beta_i r_i}{\alpha_i} - 2)\psi'_{2\{N-1\}}(2r_i - \gamma) + \frac{1}{r_i^2}(2 - \frac{\beta_i r_i}{\alpha_i})\psi_{2\{N-1\}}(2r_i - \gamma)] \sin b_{[1]}(\xi - \gamma) d\gamma,$$

where $\psi_{2\{N-1\}}$ is $\psi_2(\eta)$ in strip $\{N-1\}$,

$$a_{[1]} = \frac{1 - 2\nu}{r_i(1 - \nu)}, \quad b_{[1]} = \frac{\sqrt{1 - 2\nu}}{r_i(1 - \nu)},$$

$\xi^*_{[N]} = r_i - (N-1)R$ is the lower boundary of strip $[N]$, and $E_{[N]1}$, $E_{[N]2}$ are constants chosen so that

$$\psi_{1[N]}(\xi^*_{[N]}) = \psi_{1[N-1]}(\xi^*_{[N]}),$$
$$\psi'_{1[N]}(\xi^*_{[N]}) = \psi'_{1[N-1]}(\xi^*_{[N]}).$$

Thus both ψ_1 and ψ'_1 are continuous across strip boundaries. The constants $E_{[N]1}$, $E_{[N]2}$ are given by the vector equation

$$\begin{bmatrix} E_{[N]1} \\ E_{[N]2} \end{bmatrix} = \frac{e^{-a_{[1]}\xi^*_{[N]}}}{b_{[1]}} \begin{bmatrix} a_{[1]} \sin b_{[1]}\xi^*_{[N]} + b_{[1]} \cos b_{[1]}\xi^*_{[N]} & -\sin b_{[1]}\xi^*_{[N]} \\ -a_{[1]} \cos b_{[1]}\xi^*_{[N]} + b_{[1]} \sin b_{[1]}\xi^*_{[N]} & \cos b_{[1]}\xi^*_{[N]} \end{bmatrix} \begin{bmatrix} \psi_{1[N-1]}(\xi^*_{[N]}) \\ \psi'_{1[N-1]}(\xi^*_{[N]}) \end{bmatrix}.$$

We see from Figure 1 that the pressure wave originates at $r = r_o$, $t = 0$ and initially travels along the characteristic line $r + ct = r_o$, so it first reaches r_i at time $t = R/c$, so this is the jump-off time. We recall that the speed of sound through a compressible linearly elastic isotropic material is

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

After jump-off, the wave travels back into the shell along the characteristic line $r - ct = r_i - R$. In order to obtain the jump-off velocity, we notice from (6) that the velocity $\dot{u}$ depends on the second derivatives of ψ_1 and ψ_2. Since the pressure wave travels along a strip boundary, $\dot{u}$ along the wave may be discontinuous because ψ_1'' and ψ_2'' may be discontinuous.

For simplicity we consider a special type of pressure wave, an exponentially decaying impulse:

$$p(t) = \begin{cases} 0; & t < 0 \\ P_0 e^{-\alpha t}; & t \geq 0 \end{cases} \tag{9}$$

with $\alpha \geq 0$. Notice that when $P_0 \neq 0$, the outer boundary pressure $p(t)$ is not consistent with the initial conditions (i.e. $-p(0) \neq \alpha_o u'(r_o, 0) + \beta_o u(r_o, 0)$), so there is a discontinuity that propagates along the characteristics. We compute the velocity as t approaches the jump-off time R/c both for $\eta \in \{1\}$, $\xi \in [1]$ and for $\eta \in \{1\}$, $\xi \in [2]$, which correspond to the velocities obtain by the pressure wave traveling along the characteristic line $\eta = r_o$ and $\xi = r_i - R$, respectively. These velocities correspond to what [3] calls the particle velocity and the material surface velocity, respectively. Along $\eta = r_o$, the jump-off velocity is

$$\dot{u}_{\{1\}[1]}(r_i, \frac{R}{c}) = \lim_{t \to R/c, \eta \in \{1\}, \xi \in [1]} \dot{u}(r_i, t) = \frac{c}{r_i}[\psi''_{2\{1\}}(r_o) - \psi''_{1[1]}(r_i - R)] - \frac{c}{r_i^2}[\psi'_{2\{1\}}(r_o) - \psi'_{1[1]}(r_i - R)].$$

Plugging these arguments into the derivatives of ψ_1 and ψ_2 in the appropriate strips, we obtain the jump-off velocity

$$\dot{u}_{\{1\}[1]}(r_i, \frac{R}{c}) = -\frac{r_o P_0}{r_i \rho c}$$

after using the fact that $\alpha_o = \rho c^2$. This velocity is called the particle velocity in [3]. Along $\xi = r_i - R$, the jump-off velocity is

$$\dot{u}_{\{1\}[2]}(r_i, \frac{R}{c}) = \lim_{t \to R/c, \eta \in \{1\}, \xi \in [2]} \dot{u}(r_i, t) = \frac{c}{r_i}[\psi''_{2\{1\}}(r_o) - \psi''_{1[2]}(r_i - R)] - \frac{c}{r_i^2}[\psi'_{2\{1\}}(r_o) - \psi'_{1[2]}(r_i - R)].$$

Plugging these arguments into the derivatives of ψ_1 and ψ_2 in the appropriate strips, we obtain the jump-off velocity

$$\dot{u}_{\{1\}[2]}(r_i, \frac{R}{c}) = -\frac{2 r_o P_0}{r_i \rho c}.$$

This velocity is called the material surface velocity in [3], and it is consistent with the inner surface velocity computed in [1, pg. 22] for general stress boundary conditions.

We conclude this section by making a few observations about the jump-off velocities computed here. First, we notice that the decay parameter α does not appear anywhere in the jump-off velocities. Therefore, these velocities apply to the case where a constant pressure $p(t) = P_0$ is applied to the outer surface for all time. Second, we notice that the material surface velocity $\dot{u}_{\{1\}[2]}(r_i, \frac{R}{c})$ is twice the particle velocity $\dot{u}_{\{1\}[1]}(r_i, \frac{R}{c})$. This difference is explained by the velocity doubling rule described in [4, pg. 716-719].

3 Solution for incompressible material

For our purposes, we define an incompressible material to be a linearly elastic solid with Poisson ratio $\nu = 1/2$ and finite bulk modulus K. The speed of sound through such a material is

$$c = \sqrt{\frac{K}{\rho}}.$$

This is not physically realistic, but it is sometimes used as a simplifying assumption [1, 2, 3]. For such a spherical shell, the equation of motion (1) and the initial conditions (4) are the same as in the compressible case, but the boundary conditions take a slightly different form. In this case, $\beta_i = \frac{2}{r_i}\alpha_i$, $\beta_o = \frac{2}{r_o}\alpha_o$, and $\alpha_i = \alpha_o = \rho c^2$.

$$\rho c^2[u'(r_i,t) + \frac{2}{r_i}u(r_i,t)] = 0, \quad t > 0 \tag{10}$$

$$\rho c^2[u'(r_o,t) + \frac{2}{r_o}u(r_o,t)] = -p(t), \quad t > 0 \tag{11}$$

The velocity $\dot{u}$ has the same form (6) as in the compressible case, and the functions ψ_1, ψ_2 are computed by the same method of characteristics. However, the equations for ψ_1 and ψ_2 in each strip in Figure 1, and hence their solutions, take a different form. Taking $p(t)$ to have the form given in (9), the equation for ψ_2 in strip $\{1\}$ is

$$\psi''_{2\{1\}}(\eta) = -\frac{r_o}{\rho c^2}P_0 e^{-\alpha(\eta - r_o)/c}$$

and its solution is

$$\psi_{2\{1\}}(\eta) = -\int_{r_o}^{\eta}\int_{r_o}^{\beta}\frac{r_o}{\rho c^2}P_0 e^{-\alpha(\gamma - r_o)/c}d\gamma d\beta$$

after accounting for continuity of ψ_2 and its first derivative across the $\eta = r_o$ strip boundary. In strip [1],

$$\psi_{1[1]}(\xi) = 0$$

for all ξ inside the strip. ψ_1 in strip [2] is

$$\psi_{1[2]}(\xi) = \int_{r_i-R}^{\xi}\int_{r_i-R}^{\beta}\frac{r_o}{\rho c^2}P_0 e^{-\alpha(r_i - R - \gamma)/c}d\gamma d\beta.$$

Finally, computing the jump-off velocities along the characteristic lines $\eta = r_o$ and $\xi = r_i - R$, we find that the particle velocity is

$$\dot{u}_{\{1\}[1]}(r_i, \frac{R}{c}) = -\frac{r_o P_0}{r_i \rho c}$$

and the material surface velocity is

$$\dot{u}_{\{1\}[2]}(r_i, \frac{R}{c}) = -\frac{2 r_o P_0}{r_i \rho c}.$$

Thus the jump-off velocities are the same as in the compressible case. Also, as in the compressible case, the decay parameter α does not appear in the final result, so these jump-off velocities apply equally well to a constant loading stress $p(t) = P_0$. Finally, we note that these jump-off velocities are equal to the particle and inner surface material jump-off velocities computed in [3].

References

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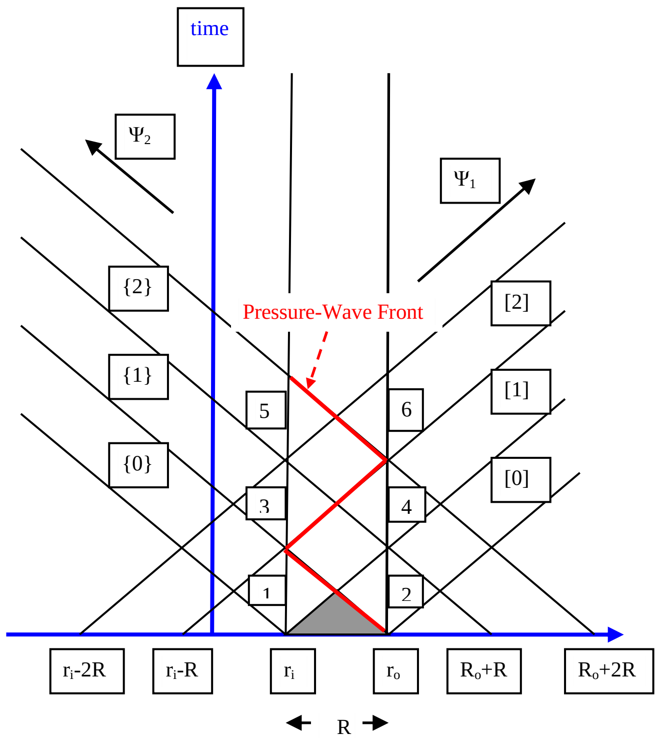

Figure 1: Pressure wave front.