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Title: Tri-cubic manufactured solutions ~~of the static diffusion equation~~

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Tri-cubic Manufactured Solutions

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We wish to verify software that solves the nonlinear diffusion equation

$$\begin{aligned}
&\nabla\cdot\left(K(u)\nabla u\right)=f(u,x),\quad x\in V\\
&u(x)=c_i,\quad x\in\partial V_i,\quad i=\{1,\ldots,m\}\\
&\nabla u\cdot n=d_j,\quad x\in\partial V_j,\quad j=\{m+1,\ldots,m+n\}\\
&\bigcup_i\partial V_i=\partial V,\quad \operatorname{int}\partial V_i\cap\operatorname{int}\partial V_j=\varnothing
\end{aligned}.$$

where K is a complicated analytic or tabular function representing a realistic material property and c_i are constants. To do so we construct an arbitrary function $u^*(x)$ that satisfies the above boundary conditions. Then we define a source term

$$f^*(x)=\nabla\cdot\left(K(u^*)\nabla u^*\right).$$

We then set our solver on the original equations with $f=f^*$. It should reproduce the solution u^*. The hard part is getting nontrivial functions that satisfy the boundary conditions (BC's), as they must be constant. After a couple of days with Mathematica, a symbolic computation method for deriving tri-quadratic and tri-cubic (in 3D) polynomials that satisfy a number of types of BC's on the unit cube and a cylindrical shell and rod has been developed. Some examples follow. To date, we have solutions for the following types of problems:

Geometry	Order	Dirichlet	Neumann
Unit Cube	Quadratic		X,Y,Z={0,1}
Unit Cube	Quadratic	X=0	X=1, Y,Z={0,1}
Unit Cube	Quadratic	X={0,1}	Y,Z={0,1}
Unit Cube	Cubic		X,Y,Z={0,1}
Unit Cube	Cubic	X=0	Y,Z={0,1}
Unit Cube	Cubic	X={0,1}	X,Y,Z={0,1}
Cylindrical ¼ shell	Quadratic		R={1/2,1}, $\theta=\{0,\pi/2\}$,Z={0,1}
Cylindrical ¼ shell	Cubic		R={1/2,1}, $\theta=\{0,\pi/2\}$,Z={0,1}
Cylindrical ¼ shell	Cubic Periodic		R={1/2,1}, $\theta=\{0,\pi/2\}$,Z={0,1}
Cylindrical shell	Quadratic		R={1/2,1},Z={0,1}
Cylindrical shell	Cubic		R={1/2,1}, ,Z={0,1}
Cylindrical shell	Cubic Periodic		R={1/2,1}, ,Z={0,1}

Cylindrical rod	Cubic Periodic	Z={0,1}	R=1

In the examples below it is seen that each function has a number of parameters. The c_i are the input Dirichlet or Neumann BC's. The a_i are free parameters that can be chosen at will. Some of the solutions have many of these, in one case up to 96.

It is curious that relatively few solutions involving Dirichlet conditions could be found. None involving r or θ for example.

A code generator has been built in Mathematic to write c++ code for these functions as well as their first and second derivatives. Using the resulting code, it is possible to generate manufactured solutions as outlined above.

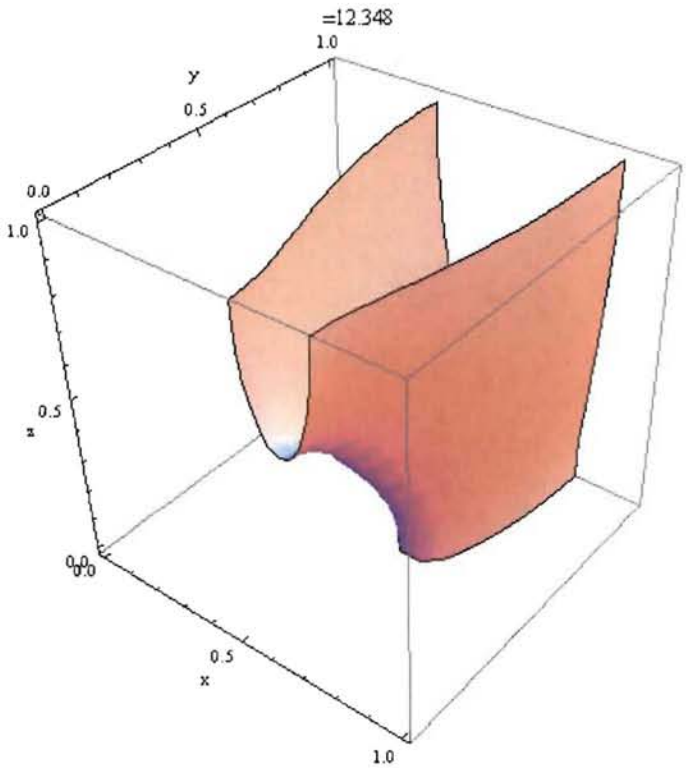
CUBIC, DOUBLE-DIRICHLET ON X

$$u\big|_{x=0} = c_0,\ u\big|_{x=1} = c_1,\ \frac{\partial u}{\partial y}\bigg|_{y=0} = 0,\ \frac{\partial u}{\partial y}\bigg|_{y=1} = 0,\ \frac{\partial u}{\partial z}\bigg|_{z=0} = 0,\ \frac{\partial u}{\partial z}\bigg|_{z=1} = 0$$

$$\mathrm{u} = \frac{1}{9}(x((1-x)(4a_3xy^3z^3 + 4a_7xy^3z^3 - 6a_3xy^3z^2 - 6a_7xy^3z^2 - 6a_2xy^3 - 6a_6xy^3 - 6a_3xy^2z^3 - 6a_7xy^2z^3 + 9a_3xy^2z^2 + 9a_7xy^2z^2 + 9a_2xy^2 + 9a_6xy^2 - 6a_5xz^3 - 3a_1(x+1)z^2(2z-3) + 9a_5xz^2 + 9a_0(x+1) + 9a_4x + 4a_3y^3z^3 - 6a_3y^3z^2 - 6a_2y^3 - 6a_3y^2z^3 + 9a_3y^2z^2 + 9a_2y^2) + 9c_1x^2) - 9c_0(x^3-1))$$

$\{a_0 \to 11.6634, a_1 \to 9.47814, a_2 \to 18.2946, a_3 \to 28.0376, a_4 \to 25.5023, a_5 \to 19.5, a_6 \to 25.2602, a_7 \to 18.4571\}$

$\{c_0 \to 1.84406, c_1 \to 4.96337\}$



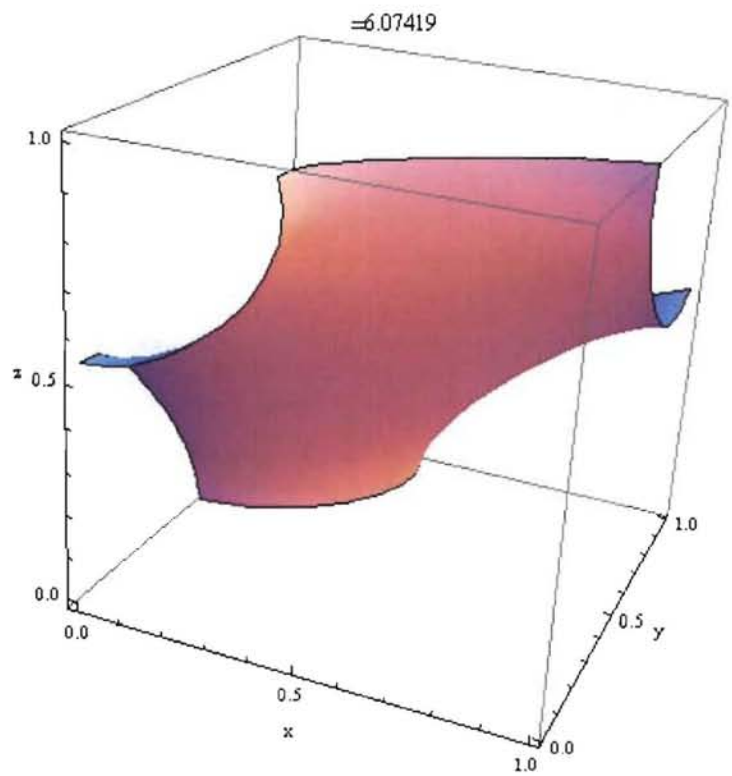
CUBIC, BRICK, 6 NEUMANN CONDITIONS

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = c_0,\ \left.\frac{\partial u}{\partial x}\right|_{x=1} = c_1,\ \left.\frac{\partial u}{\partial y}\right|_{y=0} = c_2,\ \left.\frac{\partial u}{\partial y}\right|_{y=1} = c_3,\ \left.\frac{\partial u}{\partial z}\right|_{z=0} = c_4,\ \left.\frac{\partial u}{\partial z}\right|_{z=1} = c_5$$

$$\begin{aligned} u = {} & x^3\left(-\frac{2a_4}{3}+\frac{c_0}{3}+\frac{c_1}{3}\right)+y^3\left(-\frac{2a_2}{3}+\frac{c_2}{3}+\frac{c_3}{3}\right)+z^3\left(-\frac{2a_1}{3}+\frac{c_4}{3}+\frac{c_5}{3}\right)-\frac{8}{27}a_7x^3y^3z^3 \\ & +\frac{4}{9}a_7x^3y^3z^2+\frac{4}{9}a_6x^3y^3+\frac{4}{9}a_7x^3y^2z^3-\frac{2}{3}a_7x^3y^2z^2-\frac{2}{3}a_6x^3y^2+\frac{4}{9}a_5x^3z^3 \\ & -\frac{2}{3}a_5x^3z^2+\frac{4}{9}a_7x^2y^3z^3-\frac{2}{3}a_7x^2y^3z^2-\frac{2}{3}a_6x^2y^3-\frac{2}{3}a_7x^2y^2z^3+a_7x^2y^2z^2 \\ & +a_6x^2y^2-\frac{2}{3}a_5x^2z^3+a_5x^2z^2+a_4x^2+\frac{4}{9}a_3y^3z^3-\frac{2}{3}a_3y^3z^2-\frac{2}{3}a_3y^2z^3+a_3y^2z^2 \\ & +a_2y^2+a_1z^2+a_0-c_0x-c_2y-c_4z \end{aligned}$$

$\{a_0\to 10.0192, a_1\to -17.4407, a_2\to -28.825, a_3\to -29.7872, a_4\to 14.1264, a_5\to 2.58884, a_6\to 24.1892, a_7\to -0.639174\}$

$\{c_0\to 0.0156487, c_1\to 3.13805, c_2\to -1.52654, c_3\to 7.06751, c_4\to 1.68383, c_5\to 3.08845\}$



CUBIC, CYLINDRICAL QUARTER-SHELL, ALL NEUMANN

$$\left.\frac{\partial u}{\partial r}\right|_{r=1/2} = c_0,\ \left.\frac{\partial u}{\partial r}\right|_{r=1} = c_1,\ \left.\frac{1}{r}\frac{\partial u}{\partial \theta}\right|_{\theta=0} = c_2,\ \left.\frac{1}{r}\frac{\partial u}{\partial \theta}\right|_{\theta=\pi/2} = c_3,\ \left.\frac{\partial u}{\partial z}\right|_{z=0} = c_4,\ \left.\frac{\partial u}{\partial z}\right|_{z=1} = c_5$$

$$\begin{aligned} \mathrm{u} = \frac{1}{54\pi}(&32a_7r^3z^3\theta^3 - 24\pi a_7r^3z^3\theta^2 - 24\pi a_5r^3z^3 - 48a_7r^3z^2\theta^3 + 36\pi a_7r^3z^2\theta^2 + 36\pi a_5r^3z^2 \\ &- 48a_6r^3\theta^3 + 36\pi a_6r^3\theta^2 + 36\pi a_4r^3 - 72a_7r^2z^3\theta^3 + 54\pi a_7r^2z^3\theta^2 + 54\pi a_5r^2z^3 \\ &+ 108a_7r^2z^2\theta^3 - 81\pi a_7r^2z^2\theta^2 - 81\pi a_5r^2z^2 + 108a_6r^2\theta^3 - 81\pi a_6r^2\theta^2 \\ &- 81\pi a_4r^2 + 48a_7rz^3\theta^3 - 36\pi a_7rz^3\theta^2 - 36\pi a_5rz^3 - 72a_7rz^2\theta^3 + 54\pi a_7rz^2\theta^2 \\ &+ 54\pi a_5rz^2 - 72a_6r\theta^3 + 54\pi a_6r\theta^2 + 54\pi a_4r + 48a_3z^3\theta^3 - 36\pi a_3z^3\theta^2 \\ &- 36\pi a_1z^3 - 72a_3z^2\theta^3 + 54\pi a_3z^2\theta^2 + 54\pi a_1z^2 - 72a_2\theta^3 + 54\pi a_2\theta^2 + 54\pi a_0 \\ &+ 36\pi c_0(2r-3)r^2 + 9\pi c_1(4r-3)r^2 + 18\pi c_2z^3 + 18\pi c_3z^3 - 54\pi c_2z) \end{aligned}$$

$\{a_0\to 15.1921, a_1\to -25.9486, a_2\to 22.1572, a_3\to -27.0624, a_4\to 23.3677, a_5\to 10.5747, a_6\to 26.6375, a_7\to -13.3463\}$

$\{c_0\to 3.83392, c_1\to 1.06843, c_2\to -4.31864, c_3\to 3.78729, c_4\to -7.69857, c_5\to -2.97175\}$