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Title: **PHYSICAL MODELLING OF TRAFFIC WITH STOCHASTIC CELLULAR AUTOMATA**

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# Physical Modelling of Traffic with Stochastic Cellular Automata

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**95ATS094**

A new type of probabilistic cellular automaton for the physical description of single and multilane traffic is presented. In this model space, time and the velocity of the cars are represented by integer numbers (as usual in cellular automata) with local update rules for the velocity. The model is very efficient for both numerical simulations and analytical investigations. The numerical results from extensive simulations reproduce very well data taken from real traffic (e.g. fundamental diagrams). Several analytical results for the model are presented as well as new approximation schemes for stationary traffic. In addition the relation to continuum hydrodynamic theory (Lighthill-Whitham) and the follow-the-leader models is discussed. The model is part of an interdisciplinary research program in Northrhine-Westfalia ("NRW Forschungsverbund Verkehrssimulation") for the construction of a large scale microsimulation model for network traffic, supported by the government of NRW.

## 1 Introduction

The dynamical properties of flow processes is an interesting and very active field of research in physics. Realizations of this kind of systems range from water in a river or sand ("granular material") in a pipe over car traffic and pedestrian dynamics to the propagation of earthquakes. All these systems have in common that two phenomena are in competition with each other: on one hand a driving force (gravitation, acceleration desire, stress in earth's crust etc.) giving rise to the so-called *driven diffusion* and on the other hand energy loss through dissipation (heating of particles through collisions, decelaration of cars and heating of the environment).

On a macroscopic scale it is very difficult to distinguish between the various systems mentioned above although the microscopic processes are completely different. The phenomenological understanding therefore is possible on a macroscopic scale but the analysis of the resulting equations poses serious problems if one tries to understand the dynamical properties in detail. So it is possible to solve the hydrodynamical equations for traffic flow of the Lighthill-Whitham type [1, 2, 3] numerically but the discretization neither gives a corresponding microscopic model nor allows for the treatment of larger systems (with millions of cars) in a reasonable time and with an appropriate effort and more complex situations (e.g. network traffic).

In order to avoid these problems it is necessary to try to formulate a microscopic model directly. One step in this direction are the *Follow-the-Leader* type models where each car is treated separately but space and time and the interaction between the cars are treated in a continuous way [4] (for a newer approach in this direction see also [5]). An even more simple starting point is given by using the ideas of the so-called *Cellular Automata* (CA) models which have been used in physics for a long time in order to simulate complex dynamical phenomena. The main advantage is that in CA models one deals exclusively with discrete variables both for time and space (and consequently for the velocity of the cars) with *local* update rules for the internal parameters, i.e. the velocity. These models allow for large scale simulations on (parallel) computers with results comparable to data measured in real traffic (see 95ATS089, this book). On the other hand it is possible to derive several analytical results and to apply approximation schemes where this is not possible. We will show the results below.

## 2 Description of the Model

In the following we will first describe the simplest situation: Single-lane traffic on a ring of length $L$ with periodic boundary conditions ("Indianapolis situation"). The dynamics of the model which has been introduced in [7] is defined by a set of four rules. These rules fix the update of the velocity and the movement of each car and have to be applied for all car simultaneously ("parallel update"). Each lattice site can be occupied by a car or it is empty. The velocity of a car is stored as an internal parameter ("memory") $v = 0, 1, \ldots, v_{max}$ where $v_{max}$ is the maximum velocity possible in the system. At each discrete timestep $t \to t+1$ an arbitary arrangement of the $N$ cars is then updated according to the following rules:

1) **Acceleration:** If the velocity $v$ of a vehicle is lower than $v_{max}$ the speed is advanced by one $[v = v + 1]$.
2) **Slowing down (due to other cars):** If the distance $d$ to the next car ahead is not larger than $v$ ($d \leq v$) the speed is reduced to $d - 1$ $[v = d - 1]$.
3) **Randomization:** With probability $p$, the velocity of a vehicle (if greater than zero) is decreased by one $[v = v - 1]$.
4) **Car motion:** Each vehicle is advanced $v$ sites.

Rule 1 reflects the permanent wish of the car driver to accelerate and to approach his desired (maximum) velocity. In order to avoid car crashes one has to decelerate cars when their velocity is not less than the distance to the car ahead (which would, according to rule 4, lead to a crash in the next timestep). Without rule 3 the motion of the cars would be completely deterministic and the consequence would be a strong dependence on the initial condition of the system, a very unphysical property. Finally, rule 4 describes simply the act of the motion of the cars according to their (just determined) velocity.

The performance of the model on macroscopic scales (time and space) is fairly good in comparison with real data. The only point one has to worry about is the exact value of the maximum velocitiy $v_{max}$ and the deceleration probality $p$. We found out that $v_{max} = 5$ and $p = 0.5$ is a reasonable choice (for more details see 95ATS089, this issue). We want to discuss in this article more the theoretical aspects of the model and certain limiting cases as well as extensions.

## 3 Relation to traffic flow theory

The model can also be seen as a discrete particle hopping model [9]. For many particle hopping models the so-called fluid-dynamical limit is known, which is roughly speaking the limit for large temporal and spatial scales when one can smear out the particles. The following particle hopping models are important in this context:

1.) The *Asymmetric Exclusion Process.* This is probably the most-researched particle hopping process [10]. The update rule is very simple: Pick one particle at random and move it one site to the right if this site is empty. The random picking of particles introduces noise into the model ("random sequential" update) which will lead to different stationary (asymptotic) states than the (determinitic) parallel update (see below). The dynamics corresponds to the CA-model with $v_{max} = 1$ and therefore one needs no velocity memory.

The asymmetric exclusion process is, in the hydrodynamical (continuum) limit, identical to the Lighthill-Whitham-theory when one adds diffusion and noise, and specializes it to a quadratic flow-density relation (known as the Greenshields relation in traffic science; see, e.g., [11]):

$$\partial_t \rho + \partial_x q = D\, \partial_x^2 \rho + \eta \ , \qquad q = \rho(1-\rho) \ ,$$

where $\partial_t$, $\partial_x$ and $\partial_x^2$ denote first and second order partial derivatives with respect to time $t$ and space $x$. $\rho$ is the density, $q$ is the throughput, $D$ is the diffusion coefficient, $\eta$ is a noise term.

Being described by this theory, the asymmetric exclusion process displays the same kinematic waves as the Lighthill-Whitham-theory; the additional diffusion term on the right-hand side of the above equation leads wave dissipation with damping constant $D$. The noise $\eta$ introduces stochasticity due to the influence of external random forces.

2.) The *Deterministic Traffic Cellular Automaton.* This is the deterministic limit of the CA-model for traffic flow without rule 3) which is equivalent to setting $p = 0$. It is equivalent to the Lighthill-Whitham-theory with an "inverse V" flow-density relation, and is therefore fairly equivalent to, say, Daganzo's cell-transmission-model [12]. It produces laminar flow at low densities and start-stop-waves at high densities. But due to the lack of the important ingredient of external noise this model is only of restricted interest for real world applications.

These special cases of the CA traffic model so far do not explain the spontaneous phase separation into relatively free driving cars and rather dense regions observed in real traffic. Thus, one needs all three ingredients — 1.) the parallel update as opposed (in contrast to the asymmetric exclusion process), 2.) the randomness in the update rule (in contrast to the deterministic variant), and 3.) a maximum velocity larger than one — in order to obtain plausible traffic jam dynamics. See [19] for further details.

## 4 Exact results

The model as defined above does not allow for a complete analytical solution due to the non-linear ("hard core") interaction between the cars (in order to avoid crashes) and the descrete character of the model. Although this is the simplest choice possible it leads to complicated behaviour both in space and time. Especially a phase transition in the sense of statistical physics occurs from laminar flow to a high-density phase dominated by start-stop waves [13, 14, 20].

Therefore it would be interesting to obtain exact results for this model in certain special cases or to apply at least systematic approximation methods. The former will be discussed in this chapter, the latter in the next one.

In the so-called *mean-field-approximation* one neglects spatial correlations and the distance to car ahead and the the kind of this car is chosen at random according to a probability distribution which is taken from the car statistics at a given timestep. With this simplification it is possible to write down iteration equations for the car statistics (i.e. number of cars with a certain velocity). From this one can directly derive the fundamental diagram as a function of the maximum velocity and the deceleration probability [17].

A rather simple final expression for the flow $f(c,p)$ as a function of the density $c$ and the deceleration probability $p$ is obtained for the special case $v_{max} = \infty$ since the complete set of equations for the cars with different velocities simplifies considerably. The final result reads

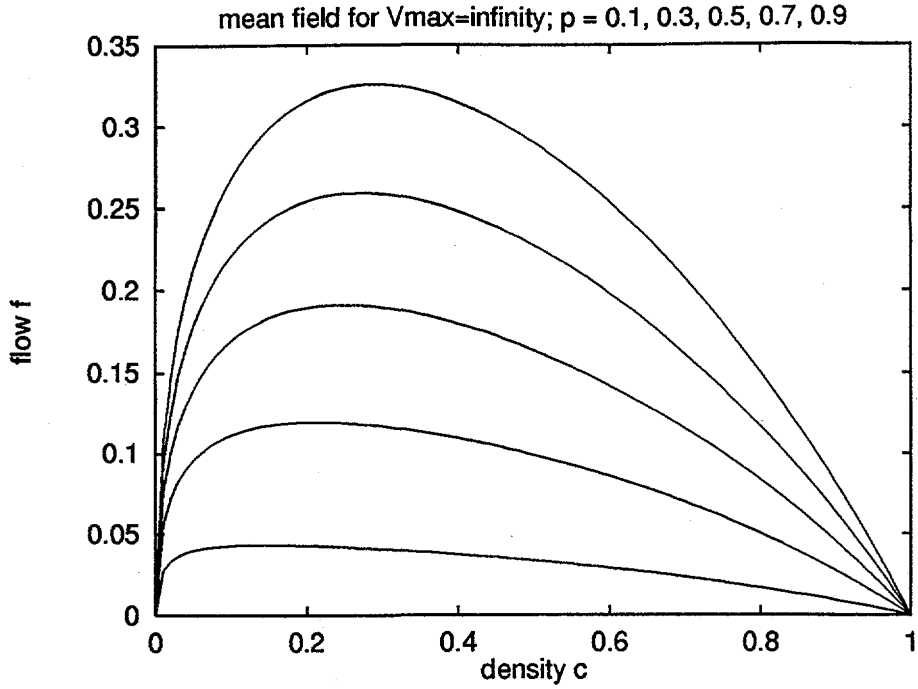


Figure 1: Flow for $v_{max} = \infty$ in mean field approximation. From bottom to top, the randomization parameter $p$ is 0.1, 0.3, 0.5, 0.7, and 0.9.

$$f(c,p) = qcd\left[1 + \sum_{n=1}^{\infty} d^{2n} \prod_{l=0}^{n-1} (p + qd^l)\right]$$

where we have introduced $q = 1 - p$ and $d = 1 - c$. In Fig. 1 the fundamental diagram is shown for five different values of $p$. The curves shown are trivial upper bounds for the flow at any finite maximum velocity (this in fact is true only for one-lane traffic!).

Inspecting the diagram more accurate two remarks have to be made: 1) The slope at the origin ($c \sim 0$) is infinite which means that already a very small number of cars gives rise to a macroscopic amount of flow. This in fact is unrealistic and only an artifact of the calculation due to setting $v_{max}$ to infinity. 2) The absolute value of the maximum of the flow (1.2 for $p = 0.5$) is much too low when compared to real data (more than 0.5). This is due to the procedure of the mean-field approximation where the spatial correlations are omitted. Therefore a car with high velocity has with the same probability a slow car in front of it as a slow car itself. But it is know that traffic is clustering in the sense that one can divide the cars into clusters of slow and fast cars only [8].

## 5 Approximative methods

In order to include correlations in space we have used to different approximation schemes. One is called *cluster-approximation* and takes into account correlations over a certain distance $n$ exactly and longer distances again in a statistical way ("$n$-cluster method") [16, 17, 21]. For $n = 1$ one recovers the mean-field theory of the last section.

The problem with this approximation scheme is that one has to close the set of equations which is possible only using conditional probabilities for the regions beyond the custers. This makes the practical calculations quite involved. A second and more serious problem is that the number of equations grows with $v_{max}$ and $n$ as $(v_{max} + 1)^n$, yielding for $v_{max} = 5$ in the lowest order beyond mean-field theory ($n = 2$) a number of 36 equations! Therefore one is restricted to low

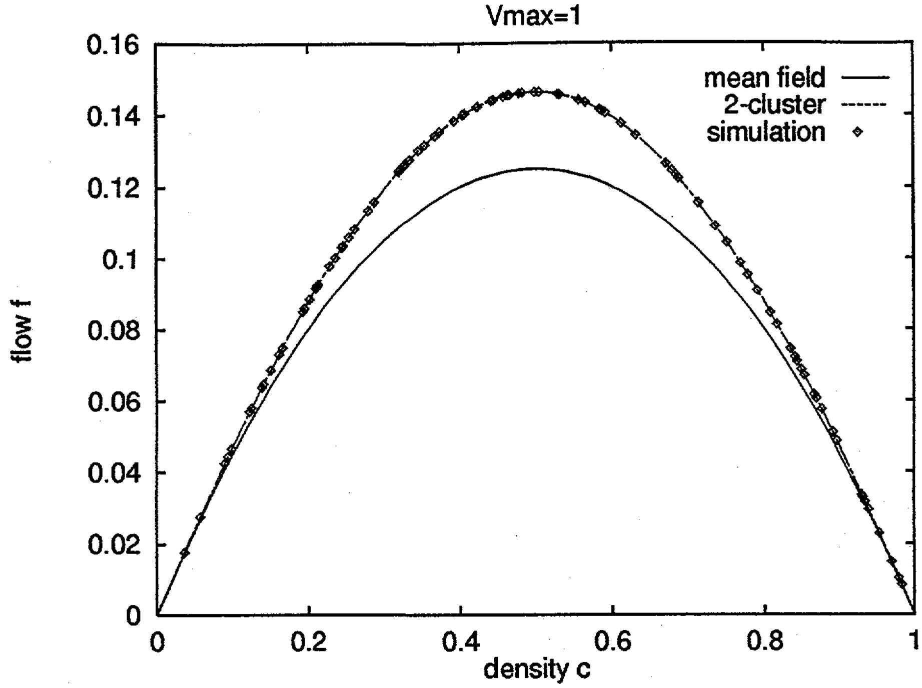


Figure 2: Convergence of the $n$-cluster approximations to the simulation result for the case $v_{max} = 1$ and $p = 0.5$. Already the 2-cluster approximation is exact.

values of $v_{max}$ for larger clusters.

Surprisingly we found that the $n = 2$-cluster approximation gives the exact result in the case $v_{max} = 1$ (which can be proved exactly, see [17]) given by

$$f(p, c) = \frac{1 - \sqrt{1 - 4qc(1-c)}}{2}.$$

It can be seen that the fundamental diagram is symmetric with respect to $c = 0.5$ which is a common feature of general models with $v_{max} = 1$ due to the "particle-hole" symmetry (which means that driving a car to the right is the same as driving a free site ("hole") to the left). In practice the fundamental diagrams are by no means symmetric and therefore higher maximum velocities are necessary. It should also be mentioned that in the case of random updates (e.g. in the asymmetric exclusion process) the mean-field approximation already gives the exact result. This means that no correlations at all exist in this model, a completely unrealistic situation. For synchronous update cars attract each other with a force over two lattice sites ("bunching") which can be seen in real traffic.

We have performed the cluster approximation for $v_{max} = 2$ up to a cluster length of 5 (with $3^5 = 243$ equations). The huge number of equations has to be generated by computer-algebra and than be solved numerically. The results are shown in Fig. 3. The difference between the calculations for $n = 4$ and 5 are less than 1% and the latter data fit quite well the simulation results. On the other hand one should note that the result does *not* become exact even for $n = 5$ reflecting the fact that in this model longe range correlations indeed exist (as should be expected). This is in contrast to the model with $v_{max} = 1$ and supports, beside the asymmetricity of the fundamental diagram, the necessity of maximum velocities $v_{max} > 1$.

Finally we have introduced a new kind of approximation called *car-oriented mean-field theory* [18]. Here one takes into account explicitely the distance of a car to the car directly ahead as an additional parameter. It can be shown that this approximation leads for $v_{max} = 1$ directly

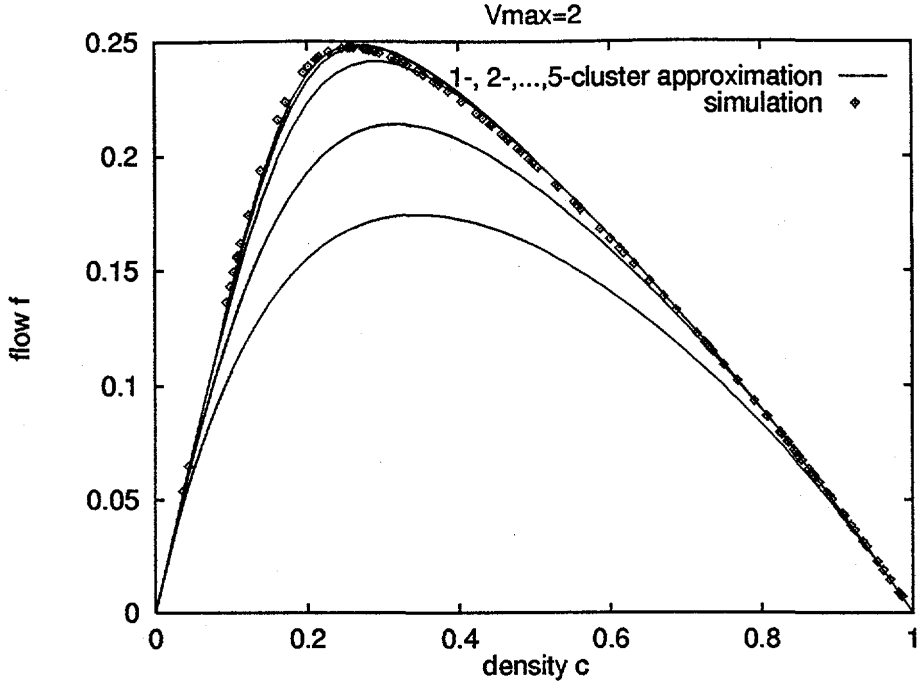


Figure 3: Convergence of the $n$-cluster approximations to the simulation result for $v_{max} = 2$ and $p = 0.5$. Already the 5-cluster approximation gives a good fit of the simulation data.

to the exact result. For higher $v_{max}$ this yields a significant improvement of the fundamental diagram in comparison with the simulations but the analytical theory is quite involved. Hopefully it will be possible also to treat nonequilibrium situations with this kind of approximation.

## 6 Summary and outlook

In this article we have discussed results for a new kind of cellular automaton model for the description of traffic flow. Although the definition of the model is quite simple it is capable of describing real traffic data in a reasonable way. Due to its discreteness in space and time it is an useful tool for efficient large scale simulations of traffic networks. Furthermore analytical investigations show that the complex behaviour of the model contains all properties which one would expect from analyzing data taken from real traffic.

There are several possibilities to extend the applicability of the model. First of all it is by no means necessary to restrict traffic to cars with the same maximum velocity. It is possible to assign to each car its own parameters $v_{max}$ and $p$. But with the geometry chosen above (single-lane traffic on a ring) the slowest car determines the flow of the whole system since passing is not allowed.

Therefore the mixing of different types of cars only makes sense if one includes passing through additional rules or one increases the number of lanes. In two-lane traffic the lane-changing behaviour is crucial and several assumptions can be made [22, 23, 24]. The simplest way is to ask the cars to change in an intermediate timestep and than to treat the lanes separately as one-lane traffic. Different properties of the car drivers can be taken into account, e.g. how far they look back before changing the lane etc. It can be seen that asymmetric rules with different probabilities for left-right and right-left changes are necessary for realistic traffic data.

In comparison to the Follow-the-Leader theory this CA model goes one step further by discretizing space and time and by simplifying the interaction between the cars considerably.

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