LBL--32249

DE92 040378

Simple Formulae for the Optimization of the FEL Gain Length Including the Effects of Emittance, Betatron Oscillations and Energy Spread*

Yong Ho Chin

Accelerator & Fusion Research Division
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

Presented at the Fourteenth International Free Electron Laser Conference, Kobe, Japan, August 23-28, 1992 and to be published in the Proceedings.

* This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Simple Formulae for the Optimization of the FEL Gain Length Including the Effects of Emittance, Betatron Oscillations and Energy Spread*

Yong Ho Chin

Lawrence Berkeley Laboratory,
University of California
Berkeley, CA 94720

Abstract

Simple analytical formulae are presented for a quick optimization of the Free Electron Laser (FEL) gain length for given values of radiation wavelength, electron beam current, normalized transverse emittance and energy spread. The optimization parameters include the gap size of the wiggler, the wiggler period and the betatron wavelength (in the case of external focusing). The method is based on the handy formulae for the FEL gain of a Gaussian beam [1] including the effects of energy spread, emittance, and betatron oscillations of the electron beam. We have found a simple relation between the minimum FEL gain length and the optimum betatron wavelength for given energy spread, emittance, and gap size of the wiggler. When the emittance is about the radiation wavelength divided by 4π and the energy spread is negligible, this relation shows that the gain length is optimized if the betatron wavelength is chosen so that the betatron phase advances by a half radian in the gain length.

I. Introduction

Recently, we have developed a three-dimensional (3-D) FEL theory in the high gain regime before saturation based upon the Maxwell-Vlasov equation, including the effects of energy spread, transverse emittance, betatron focusing and oscillations of the electron beam, and the diffraction and guiding of the radiation field [1,2]. We have presented a dispersion relation for the FEL gain of a Gaussian beam and its approximate version

* This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

that can be used as handy formulae for a quick estimate of the growth rate. Comparison of the growth rates obtained by solving the dispersion relation with those obtained by Yu, Krinsky and Gluckstern's variational method [3], and by simulations using the code TDA [4] for the waterbag and the Gaussian transverse electron distributions showed good agreement [2]. They disagree (by about 20% at most) only in the limiting case when the Rayleigh range is much longer than the one-dimensional gain length. This disagreement is due to the fact that only a first-order truncated version of the exact dispersion relation is used in the computation. Using the above (dispersion-relation based) analytical tool, we can now not only carry out the optimization of FEL parameters much more easily and quickly than with numerical simulations, but also can obtain a physical insight into the dependence of the FEL performance on many parameters.

Kim [5] and Yu [6] have derived formulae for a rough estimate of the power at saturation and the wiggler length required to yield saturation through self-amplified spontaneous emission, and proposed a procedure to design parameters so as to maximize the saturated power. In this paper, we deal with the optimization of the power gain length rather than the saturated power. For given values of radiation wavelength, electron beam current, normalized emittance, and energy spread, both the saturated power and the gain length become functions only of the gap size of the wiggler, the wiggler period, and the focusing strength (in the case of natural focusing, the third parameter drops). For a fixed gap size, the total wiggler length to obtain saturation in the case when the saturated power is maximized is substantially longer than that required in the case when the gain length is minimized. However, the saturated power is less in the latter case. This drawback can be countered by employing a tapered wiggler after saturation in the uniform parameter wiggler.

II. Optimization Procedure

The growth rate of the fundamental guided mode can be expressed in a scaled form using four dimensionless scaling parameters. One form of such a scaling relation convenient when the total beam current is constant is

$$\frac{Re(q)}{k_w D} = F(2k_1\varepsilon_x, \frac{\sigma_\gamma}{D}, \frac{k_\beta}{k_w D}, \frac{k-k_1}{k_1 D}), \tag{1}$$

where $Re(q)$ is the growth rate in the exponential growth regime. The power gain length L_G can be calculated from the above growth rate function as

$$\frac{1}{L_G} = 2k_w D \cdot \frac{Re(q)}{k_w D}. \tag{2}$$

The dispersion relation for the Gaussian beam, for instance, can be written in the above scaling form as

$$1 = \frac{i}{4\sqrt{2\pi}}\frac{\frac{k_\beta}{k_w D}}{2k_1\varepsilon_x}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{-\frac{x^2}{2}}x^3 e^{-\frac{t^2}{2}}dx dt}{(\frac{q}{k_w D}+2i\frac{\sigma_\gamma}{D}t-\frac{i}{4}2k_1\varepsilon_x\frac{k_\beta}{k_w D}x^2)^2}$$
$$\times\int_0^{\infty}\frac{e^{-x^2}x dx}{\frac{q}{k_w D}+i\frac{k-k_1}{k_1 D}+i\frac{x^2}{2k_1\varepsilon_x}\frac{k_\beta}{k_w D}}, \qquad (3)$$

where we apply a rule that the integral signs in the multiple integrals are paired with the differential signs from inside to outside. Here, $k_w = 2\pi/\lambda_w$ is the wiggler wave number, $k_1 = 2k_w\gamma_r^2/(1+K^2)$ is the resonant radiation wave number corresponding to the resonant energy γ_r of the reference electron in units of its rest mass mc^2, c is the speed of light, K is the rms value of the wiggler parameter, ε_x is the rms transverse emittance of the electron beam, σ_γ is the rms relative energy spread, k_β is the betatron wave number, and $k = \omega/c$ is the wave number of the radiation field. The quantity D is the scaling parameter defined by

$$D = \sqrt{\frac{8}{\gamma_r}\frac{K^2}{1+K^2}\frac{I_0}{I_A}}\ [\mathrm{JJ}], \qquad (4)$$

where I_0 is the total beam current, $I_A = ec/r_e \approx 17.05$kA is the Alfvén current, e is the electron charge, and r_e is the electron classical radius. For a planar wiggler, the Bessel factor [JJ] is given by

$$[\mathrm{JJ}] = \mathrm{J}_0(\frac{K^2}{2(1+K^2)}) - \mathrm{J}_1(\frac{K^2}{2(1+K^2)}), \qquad (5)$$

where $\mathrm{J}_m(x)$ is the Bessel function. For a helical wiggler, [JJ]=1. We have found the following parameterized formulae for the scaled growth rate of a beam with the Gaussian transverse and Gaussian energy distributions which agree well with the exact solutions of the dispersion relation, Eq. (3) [1,2]:

$$\log\frac{Re(q)}{k_w D} = -\ (0.759+0.238\chi+0.0139\chi^2)$$
$$\times\left\{1+(2k_1\varepsilon_x\frac{k_\beta}{k_w D})^2/(0.149+0.0268\log\frac{k_\beta}{k_w D})\right.$$
$$\left.+\ (44.03+3.32\chi+5.45\chi^2)\cdot\left[(\frac{\sigma_\gamma}{D})^2-0.713(\frac{\sigma_\gamma}{D})^4+68.65(\frac{\sigma_\gamma}{D})^6\right]\right\}^{\frac{1}{2}}$$
$$\text{for}\quad 2k_1\varepsilon_x\frac{k_w D}{k_\beta}\geq 0.05\ \text{ and }\ \frac{k_\beta}{k_w D}\leq 1, \qquad (6)$$

and

$$\frac{Re(q)}{k_w D} = \left[0.0628 - 0.219\chi - 0.000568\chi^2\right]^{\frac{1}{2}}$$

$$\times \exp\left[-\frac{(2k_1\varepsilon_x \frac{k_\beta}{k_w D})^2}{(1.091 + 0.1345\frac{k_\beta}{k_w D})} - (\ 11.92 + 2.202\chi + 0.1414\chi^2\ )\ (\frac{\sigma_\gamma}{D})^2\right]$$

$$\text{for} \quad 2k_1\varepsilon_x\frac{k_w D}{k_\beta} < 0.05 \ \text{ or } \ \frac{k_\beta}{k_w D} > 1, \tag{7}$$

where

$$\chi = \log(2k_1\varepsilon_x\frac{k_w D}{k_\beta}). \tag{8}$$

Let us now consider a procedure for the optimization of the power gain length. We assume that the required radiation wave length λ_1, the electron beam current I_0, the normalized emittance $\varepsilon_0 = \varepsilon_x\gamma$, and the rms relative energy spread σ_γ are all fixed. Under these conditions, the power gain length becomes a function only of the wiggler period λ_w and the gap size of the wiggler, g. All the other FEL parameters can be determined from these two parameters in the following five steps:

1) For a given wiggler type, there is a relation between λ_w and the peak wiggler magnetic field B_0. For example, for a Nd-Fe-B wiggler ($g/\lambda_w \leq 0.722$),

$$B_0(T) = 3.44\exp[-5.00\left(\frac{g}{\lambda_w}\right) + 1.54\left(\frac{g}{\lambda_w}\right)^2]. \tag{9}$$

The rms value of the wiggler parameter, K, is given by

$$K = \frac{0.934}{d}\lambda_w(\text{cm})B_0(T), \tag{10}$$

where $d = \sqrt{2}$ for a planar wiggler, and d=1 for a helical wiggler.

2) From the resonant condition, the electron energy corresponding to the radiation wave length λ_1 is determined by

$$\gamma_r = \sqrt{\frac{\lambda_w}{2\lambda_1}\frac{1+K^2}{K^2}}. \tag{11}$$

3) The scaling parameter D can be calculated from Eqs. (4) and (5).

4) The emittance ε_x at the energy γ_r is determined to be

$$\varepsilon_x = \frac{\varepsilon_0}{\gamma_r}. \tag{12}$$

5) In the case of natural focussing with equal focusing on the x- and y-planes by parabolic pole faces, the betatron wave number is given by

$$k_\beta = \frac{K k_w}{\sqrt{2}\gamma_r}. \tag{13}$$

Now, the gain length L_G can be obtained by substituting these parameters into Eqs. (6), (7) and (2).

The enhancement of the growth rate may be achieved by employing the external focusing. In this case, the value of k_β which maximizes the scaled growth rate $Re(q)/(k_w D)$ can be found easily by applying an optimization program to Eqs. (6) and (7). The result can be conveniently parameterized in the following form:

$$\begin{aligned}\left(\frac{k_\beta}{k_w D}\right)_{opt} &= 0.2123 \cdot (2k_1\varepsilon_x)^{-1.277} \cdot \exp(-0.03513\xi^2) \\ &\times \exp\left\{(23.51 + 8.865\xi + 1.125\xi^2) \cdot \left[(\frac{\sigma_\gamma}{D})^2 - 3.914(\frac{\sigma_\gamma}{D})^4\right]\right\},\end{aligned} \tag{14}$$

where

$$\xi = \log(2k_1\varepsilon_x). \tag{15}$$

Similarly, the maximized growth rate can be parameterized in the following form:

$$\begin{aligned}\left(\frac{Re(q)}{k_w D}\right)_{opt} &= 0.2504 \cdot (2k_1\varepsilon_x)^{-0.7368} \cdot \exp(-0.09159\xi^2) \\ &\times \exp\left\{\left[-3.925 - 14.027(2k_1\varepsilon_x) + 0.1329(2k_1\varepsilon_x)^2\right] \cdot (\frac{\sigma_\gamma}{D})^2\right\}.\end{aligned} \tag{16}$$

The above formulae agree well with the exact solutions obtained from Eqs. (6) and (7) within 5 % in the range where $\sigma_\gamma/D \lesssim 0.4$ and $0.04 \lesssim 2k_1\varepsilon_x \lesssim 10$.

It may be interesting to point out that $(Re(q)/(k_w D))_{opt}$ and $(k_\beta/(k_w D))_{opt}$ hold a rough relation

$$\left(\frac{Re(q)}{k_w D}\right)_{opt} + 2\left(\frac{\sigma_\gamma}{D}\right) \approx (2k_1\varepsilon_x)^{\frac{2}{3}}\left(\frac{k_\beta}{k_w D}\right)_{opt}. \tag{17}$$

In terms of L_G, the above relation can be rewritten as

$$\frac{1}{2(L_G)_{opt}} + 2k_w\sigma_\gamma \approx (2k_1\varepsilon_x)^{\frac{2}{3}}(k_\beta)_{opt}. \tag{18}$$

A strong resemblance can be observed between Eq. (17) and the denominator of the first integral over x and t in Eq. (3):

$$\frac{q}{k_w D} + 2i\frac{\sigma_\gamma}{D}t - \frac{i}{4}2k_1\varepsilon_x\frac{k_\beta}{k_w D}x^2, \tag{19}$$

when $t = 1$ and $x = 2$. The term (19) represents the Landau damping due to the longitudinal velocity variation by the energy spread and the transverse emittance. When the betatron focusing is increased for a given emittance, the reduced beam size tends to increase the growth rate, while the increased longitudinal velocity spread tends to reduce the growth rate at the same time. These two conflicting tendencies get balanced at the focusing strength $(k_\beta/(k_w D))_{opt}$. Equation (17) suggests that this balancing point that gives the maximum growth rate is actually the point where the value of the growth rate (in units of k_w) becomes comparable to the longitudinal velocity spread (in units of c). This relation can be used for a rough estimate of the optimum focusing strength once a desirable power gain length is given. One interesting conclusion from this relation is that the gain length is optimized if the betatron wavelength is chosen so that the betatron phase advances by a half radian in the gain length when $2k_1\varepsilon_x \approx 1$ and the energy spread is negligible.

As an example of the optimization, we consider the following case: the required radiation wavelength λ_1 is $50\mathring{A}$, and the electron beam parameters are $I_0 = 1000A$, $\varepsilon_0 = \varepsilon_x\gamma = 1.5 \times 10^{-6}$(m-rad), and $\sigma_\gamma = 2.2 \times 10^{-4}$. We assume a planar wiggler made of Nd-Fe-B magnetic blocks with parabolic pole faces providing equal natural focusing on the x- and y-planes. At first, we take into account only the natural focusing. Figure 1 shows the power gain length L_G as a function of the wiggler period λ_w for several gap sizes g. We find that L_G has its minimum values at $\lambda_w \approx 1.06$ cm, 1.5 cm and 1.88 cm for $g = 2$ mm, 4 mm, and 6 mm, respectively. The resonant electron energy, the wiggler field, and the rms wiggler parameter at these minimum points are $\gamma_r = 1338$, 1734, 1986.5, $B(T) = 1.346$, 1.012, 0.827, and $K = 0.889$, 1.002, 1.038, for $g = 2$ mm, 4 mm, and 6 mm, respectively.

Now, we consider the effects of the external focusing on the gain length. Equation (14) is used to obtain the strength of the external focusing which minimizes the gain length for given g and λ_w. In Figs. 2 and 3, we plot the power gain length L_G and the betatron wave length λ_β as functions of the wiggler period λ_w for several gap sizes g. It is found that the gain length is reduced only slightly (by about 20% at most) at the minimum points, but the reduction factor increases as λ_w increases. For a small λ_w, the optimum λ_β for the external focusing is similar to that of the natural focusing. For a large λ_w, it slowly decreases as λ_w increases, in contrast to the natural focusing case where λ_β is an increasing function of λ_w. The gain length increases more slowly as a function of λ_w than in the natural focusing case for $\lambda_w \gtrsim 2.0$ cm.

III. Minimum Gain Length

We can now proceed to a further optimization of the wiggler parameters. The wiggler period $(\lambda_w)_{opt}$ for the minimum power gain length L_G for a given g satisfies the equation

$$\frac{d}{d\lambda_w}(\frac{1}{L_G}) = \frac{d}{d\lambda_w}(2k_w D \cdot \frac{Re(q)}{k_w D}) = 0. \tag{20}$$

Among the variables in the above equation, k_w and $Re(q)/(k_w D)$ are monotonously decreasing and increasing functions of λ_w, respectively. Only the scaling parameter D has a structure with a single peak over λ_w for each g. Therefore, Eq. (20) can be well approximated by $dD/d\lambda_w = 0$. Substituting Eq. (4) into Eq. (20), we have

$$\begin{aligned} -\frac{[\mathrm{JJ}]}{4} \quad + \quad & \left\{\frac{2-K^2}{2(1+K^2)}[\mathrm{JJ}] - \Theta\frac{K^2}{(1+K^2)^2}\left[\frac{1}{2}\left(\mathrm{J}_0(\frac{K^2}{2(1+K^2)}) - \mathrm{J}_2(\frac{K^2}{2(1+K^2)})\right)\right.\right. \\ & \left.\left. + \mathrm{J}_1(\frac{K^2}{2(1+K^2)})\right]\right\} \times \left\{1 - \left[c_2(\frac{g}{\lambda_w}) + 2c_3(\frac{g}{\lambda_w})^2\right]\right\} = 0, \end{aligned} \tag{21}$$

where we have assumed that the wiggler parameter can be expressed as

$$K = c_1\lambda_w \exp\left[c_2(\frac{g}{\lambda_w}) + c_3(\frac{g}{\lambda_w})^2\right]. \tag{22}$$

Here, the parameter $\Theta = 1$ for a planar wiggler, and $\Theta = 0$ for a helical wiggler. It should be noticed that Eq. (21) involves only λ_w and g as variables. The numerical solutions of Eq. (21) for the planar wiggler with Nd-Fe-B magnetic blocks case are shown in Fig. 4 by the solid curve. The value of $(\lambda_w)_{opt}$ for g =2 mm, 4mm, and 6 mm are about 0.99 cm, 1.42 cm, and 1.78 cm, respectively. They agree well with the corresponding λ_w for the minimum L_G shown in Fig. 1. The broken curve is obtained by fitting the exact curve using the following polynomials: For a planar wiggler, we have

$$(\lambda_w)_{opt} = 3.819 \times 10^{-3} + 3.160 \times 10^{-3}g - 1.619 \times 10^{-4}g^2 + 4.910 \times 10^{-5}g^3, \tag{23}$$

and for a helical wiggler,

$$(\lambda_w)_{opt} = 3.411 \times 10^{-3} + 3.081 \times 10^{-3}g - 1.591 \times 10^{-4}g^2 + 4.842 \times 10^{-5}g^3, \tag{24}$$

where g is in units of millimeter. These approximate solutions agree with the exact solutions to within about 6 % over the range $1 \leq g \leq 15$ mm. Now, the minimum gain length is a function only of g. Once g is chosen, the optimum λ_w can be estimated from Eqs. (23) and (24), and the minimum gain length and all the other FEL parameters can be calculated quickly through the optimization procedure described in Sec. II.

IV. Conclusions

We have presented simple formulae and optimization procedure for a quick estimate of the FEL parameters which minimize the power gain length. These formulae include the formulae for the optimum strength of the external focusing for the maximized growth rate (Eq. (14)) and for the maximized growth rate itself (Eq. (16)), and the one for the optimum wiggler period for the minimum gain length (Eqs. (23) and (24)). Our numerical example shows that the minimum gain length was reduced only slightly by employing the external focusing. It helps to maintain a slow growth of the gain length as the wiggler period increases. Typically, the optimum wiggler period for the maximum saturated power is several times longer than that for the minimum gain length. Since the saturated power is proportional to $(1/L_G)^4$, the external focusing may be much more effective in the enhancement of the saturated power than the gain length.

References

[1] Y. H. Chin, K. -J. Kim and M. Xie, Nucl. Instrum. Methods, **A318,** 481 (1992).

[2] Y. H. Chin, K. -J. Kim and M. Xie, LBL-32329 (1992), and to be published in Phys. Rev. A.

[3] L. -H. Yu, S. Krinsky, and R. L. Gluckstern, Phys. Rev. Lett., **64**, 3011 (1990).

[4] T. M. Tran and J. S. Wurtele, Comput. Phys. Commun. **54**, 263 (1989).

[5] K. -J. Kim, LBL-31969 (1992), and presented at the Workshop on 4th Generation Light Source, Stanford, California (1992).

[6] L. -H. Yu, Phys. Rev. A, **44**, 5178 (1991).

Figure Captions

FIG. 1. The power gain length L_G as a function of wiggler period λ_w for several gap sizes g with the natural focusing only.

FIG. 2. The power gain length L_G as a function of wiggler period λ_w for several gap sizes g with the optimized external focusing.

FIG. 3. The optimum betatron wavelength $(\lambda_\beta)_{opt}$ as a function of wiggler period λ_w for several gap sizes g.

FIG. 4. The optimum wiggler period $(\lambda_w)_{opt}$ for the minimum gain length as a function of the gap size g.

Fig. 1

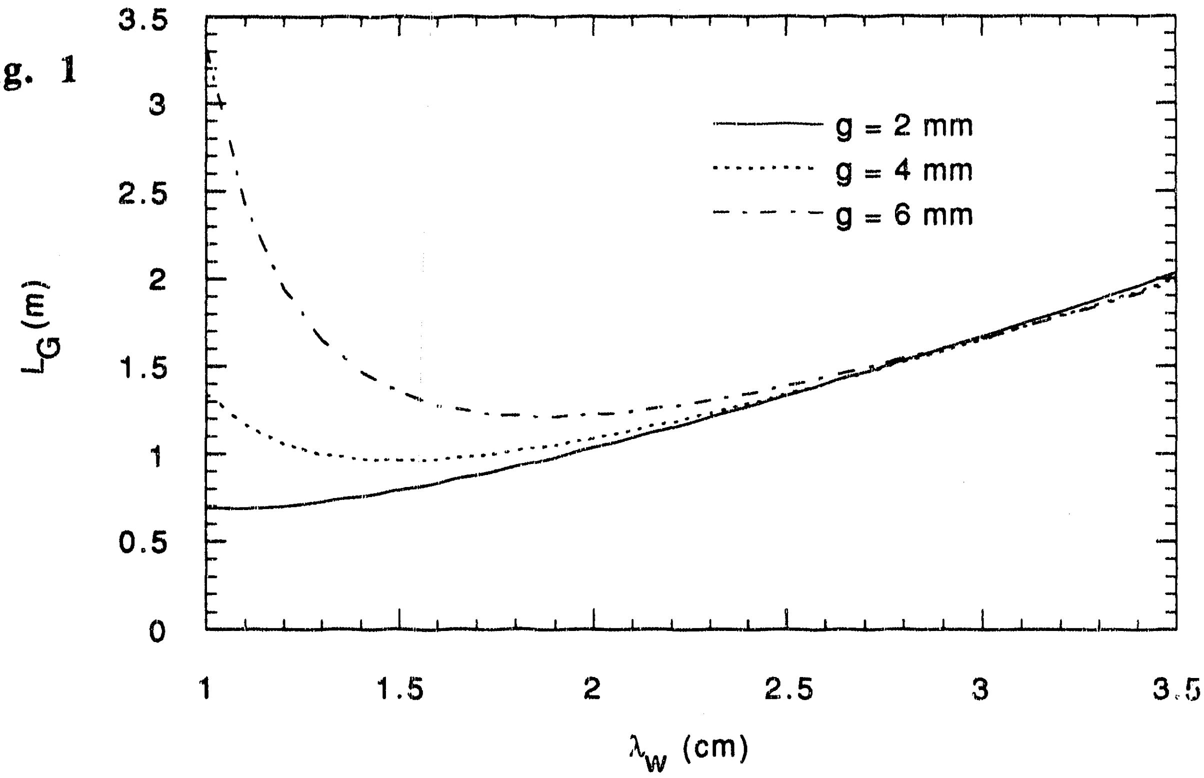

3.5
3
2.5
2
1.5
1
0.5
0
L_G (m)
g = 2 mm
g = 4 mm
g = 6 mm
1
1.5
2
2.5
3
3.5
λ_w (cm)

Fig. 2

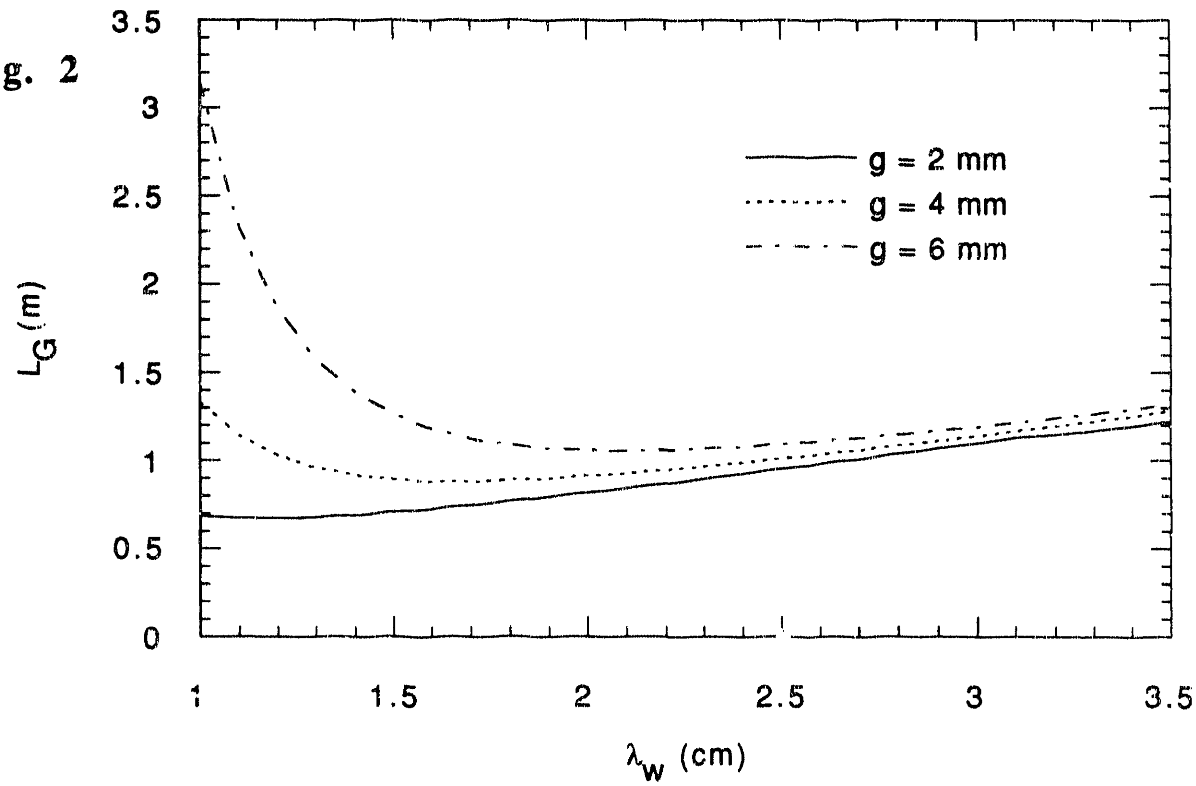
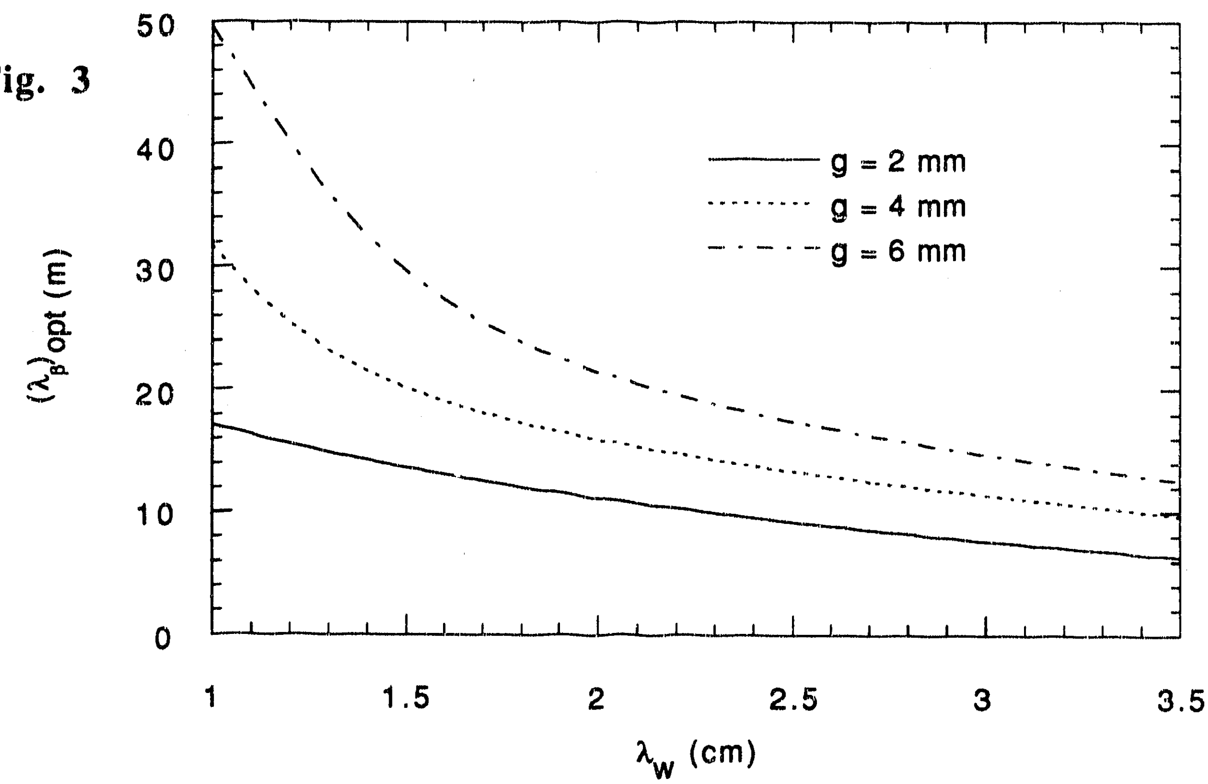
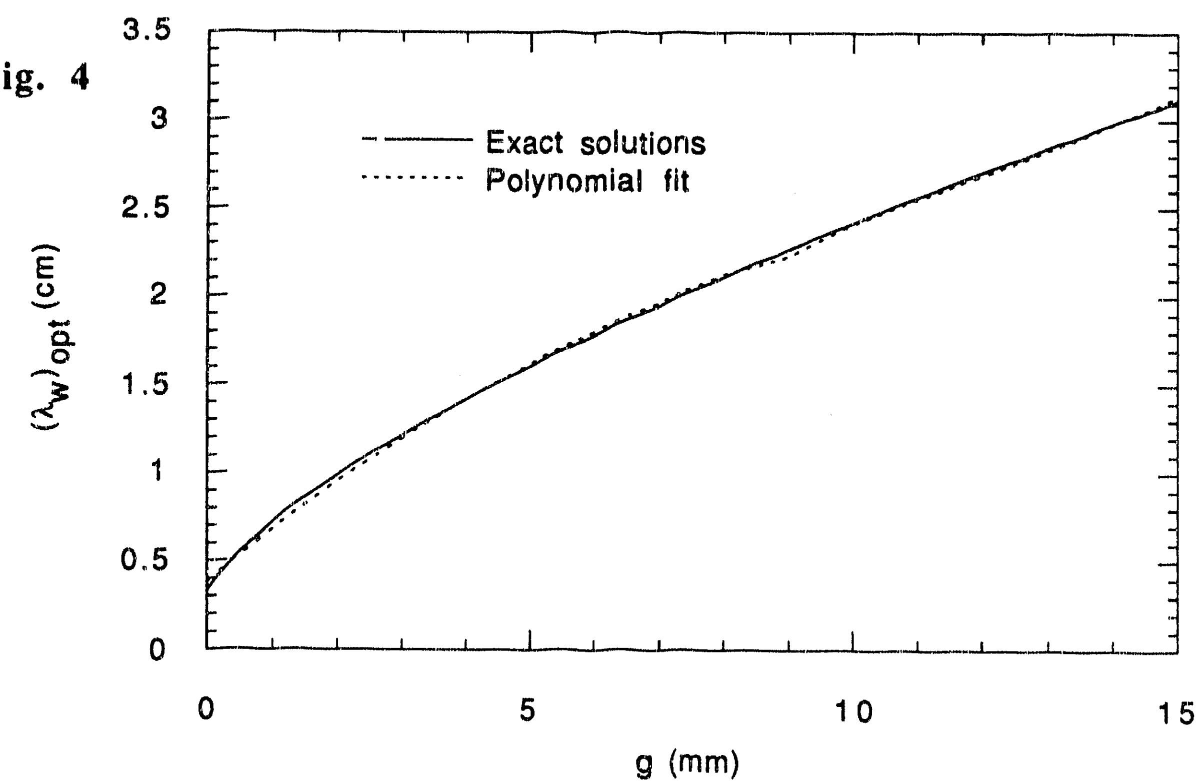

3.5
3
2.5
2
1.5
1
0.5
0
L_G (m)
g = 2 mm
g = 4 mm
g = 6 mm
1
1.5
2
2.5
3
3.5
λ_w (cm)

Fig. 3



50
40
30
20
10
0
1
1.5
2
2.5
3
3.5
$(\lambda_\beta)_{opt}$ (m)
λ_w (cm)
g = 2 mm
g = 4 mm
g = 6 mm

Fig. 4



3.5
3
2.5
2
1.5
1
0.5
0
5
10
15
$(\lambda_w)_{opt}$ (cm)
g (mm)
Exact solutions
Polynomial fit

END

DATE FILMED

11/12/92