ANL/RE-93/2

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THE EXACT SOLUTIONS TO THE DYNAMIC RESPONSE OF TANKS CONTAINING TWO LIQUIDS

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February 1993

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ABSTRACT

The exact solution to the dynamic response of circular cylindrical tanks containing two liquids, considering the gravitational (g) effect at the interface of the two liquids, is presented. Only rigid tanks were studied. The solution is expressed as the superposition of the so-called impulsive and convective solutions. The results are compared with those obtained by neglecting the gravitational effect at the interface to elucidate the g effect and with those of the tanks containing only one liquid to elucidate the effect of the interaction between two liquids. The response functions examined include the hydrodynamic pressure, base shear, base moments, sloshing motions at surface and at the interface of two liquids and the associated sloshing frequencies. It is found that there are two natural frequencies associated with each sloshing mode in contrast to only one frequency associated with each sloshing mode if the g effect at the interface is neglected; also, the convective pressure has a jump at the interface of two liquids, whereas the impulsive pressure is continuous at the interface. Further, it is shown that in a tank containing two liquids the maximum sloshing wave height may increase significantly, and the fundamental frequency of the sloshing motion is lower than that of an identical tank filled with only one liquid. Additionally, the well-known mechanical model for tanks containing one liquid is generalized for tanks containing two liquids.

I. INTRODUCTION

The sloshing motion in a liquid containing tank has been the subject of numerous studies in the past 30 years. An excellent review of this topic can be found in Ref. 1. Most of the previous studies were focused on the tank containing only one liquid. However, due to the application of electrorefining for recovery and purification of materials, there is a need to study the dynamic response of tanks containing liquids with different densities. An exploratory study on the dynamic response of tanks containing two liquids was investigated by Tang and Chang [2] and Tang [3]. Those studies show that the dynamic response of a tank containing two liquids is quite different from that of an identical tank containing only one liquid and that the solutions obtained based on the assumption that the tank is filled with only one liquid can be far off from the true solutions. Especially, the sloshing wave height may increase significantly in a tank that contains two liquids. However, the solutions presented in Refs. 2 and 3 were based on the assumption that the hydrodynamic pressure is continuous at the interface of two liquids. In other words, the gravitational (g) effect at the interface of two liquids is neglected; as a result, the solutions are considered to be approximate, and there is still a need to reanalyze the problem and to assess the importance of the g effects. This report is intended to be responsive to this need.

The objectives of this report are: (1) to present the exact solutions for the dynamic response of rigid tanks that contain two liquids, which consider the g effect at the interface; and (2) to present numerical results with which the importance of the g effect can be evaluated and (3) to assess the accuracy and identify the range of applicability of the approximate solutions given in Refs. 2 and 3. In addition to the sloshing motions at the surface and at the interface and the associated sloshing frequencies, the response functions examined include the hydrodynamic pressure on the tank wall, the base shear and moments at sections immediately above and below the tank base plate. Each of these response functions is expressed as the sum of two components, the impulsive and convective components. The impulsive component of solution is defined to be the part of the solution that is proportional to the base excitation, i.e., the impulsive component has the base excitation as its time function. The convective component of the solution is the remaining part of the solution. Specifically, this component is associated with the liquid sloshing motion, and has the pseudoacceleration functions corresponding to the sloshing wave motion as its time function. The division of the response into impulsive and convective components is necessary because it is essential to the approach used by Tang [4], Veletsos and Tang [5] and Veletsos and Yang [6] in the analysis of the flexible tanks.

In the presentation, the solutions obtained by the equations presented in this report including the g effect will be referred as the "exact" solutions, whereas the solutions obtained by the equations presented in Refs. 2 and 3 excluding the g effect will be referred as the "approximate" solutions. It will be shown in this report that for the impulsive component the exact solution is identical to that of the approximate solution, so the emphasis of this report will be placed on the convective component of the solution. Note that in this report, only the linear response is considered.

II. SYSTEM DESCRIPTION

The tank-liquids system investigated is shown in Fig. 1. It is a ground-supported upright circular cylindrical tank of radius R which is filled with two liquids to a total height of H. The lower portion liquid, identified as Liquid I, has heavier mass density, ρ_1, and the upper portion liquid, identified as Liquid II, has lighter mass density, ρ_2. The heights of Liquid I and II are H_1 and H_2, respectively. The tank wall is assumed to be of uniform thickness and clamped to a rigid base. Both liquids are considered to be incompressible and inviscid. The response of the liquids is assumed to be linear.

Let r, θ, z_1 denote the radial, circumferential, and vertical axial coordinates of a point in the Liquid I, and let r, θ, and z_2 be the corresponding coordinates for a point in Liquid II as shown in Fig. 1. The origins of the two coordinate systems are at the central axis of the cylindrical tank.

The base motion experienced by the tank is a horizontal acceleration, denoted by $\ddot{x}(t)$, acting in the direction along the θ=0 coordinate axis. The temporal variation of $\ddot{x}(t)$ can be arbitrary.

III. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Given the conditions that the liquids are incompressible and inviscid, the hydrodynamic pressures induced at Liquid I and Liquid II, denoted by p_1 and p_2 respectively, must satisfy the Laplace equations

$$\nabla^2 p_1 = 0 \tag{1a}$$

in the region $0 \le r \le R$, $0 \le \theta \le 2\pi$, and $0 \le z_1 \le H_1$, and

$$\nabla^2 p_2 = 0 \tag{1b}$$

in the region $0 \le r \le R$, $0 \le \theta \le 2\pi$, and $0 \le z_2 \le H_2$.

The liquid acceleration at an arbitrary point along n-direction is given by

$$a_n = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial n} \tag{2}$$

for points in Liquid I, and

$$a_n = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial n} \tag{3}$$

for points in Liquid II.

The boundary conditions for Liquid I are:

(a) The vertical acceleration of Liquid I at the tank base must equal zero, i.e.,

$$\left.\frac{\partial p_1}{\partial z_1}\right|_{z_1=0} = 0 \tag{4a}$$

(b) The radial acceleration of Liquid I adjacent to the tank wall must equal the acceleration of the tank wall, i.e.,

$$-\frac{1}{\rho_1}\left.\frac{\partial p_1}{\partial r}\right|_{r=R} = \ddot{x}(t)\cos\theta \text{ , and} \tag{4b}$$

(c) The value of p_1 at $r=0$ is finite.

The boundary conditions for Liquid II are:

(a) The radial acceleration along the tank wall is given by

$$-\frac{1}{\rho_2}\left.\frac{\partial p_2}{\partial r}\right|_{r=R} = \ddot{x}(t)\cos\theta \ , \tag{4c}$$

(b) At free surface, the linearized boundary condition is

$$\left(\frac{\partial^2 p_2}{\partial t^2} + g\,\frac{\partial p_2}{\partial z_2}\right)_{z_2=H_2} = 0 \tag{4d}$$

where g is the gravitational acceleration, and

(c) The value of p_2 at r=0 is finite.

The boundary conditions at the interface of two liquids are as follows.

(a) Continuity of vertical accelerations, i.e.,

$$-\frac{1}{\rho_1}\left.\frac{\partial p_1}{\partial z_1}\right|_{z_1=H_1} = -\frac{1}{\rho_2}\left.\frac{\partial p_2}{\partial z_2}\right|_{z_2=0} \tag{4e}$$

and

(b) Kinematic and pressure conditions:

b.1 Kinematic condition. If $\eta(r,\theta,t)$ represents the height of the small disturbance at the interface above the still interface level, the $\eta(r,\theta,t)$ is related to p_1 by

$$\frac{\partial^2 \eta}{\partial t^2} = -\frac{1}{\rho_1}\left.\frac{\partial p_1}{\partial z_1}\right|_{z_1=H_1} \tag{5}$$

b.2 Pressure condition. If the g effect is considered for the interface motion, there is a discontinuity of the hydrodynamic pressure with the amount of $(\rho_1 - \rho_2)g\eta$ at the interface. Therefore,

$$p_1\big|_{z_1=H_1} - p_2\big|_{z_2=0} = (\rho_1 - \rho_2)g\eta \tag{6}$$

Eliminating η between Eqs. (5) and (6) and making use of the Eq. (4e), one obtains the following equation for the interface boundary conditions in addition to Eq. (4e).

$$\left(\frac{\partial^2 p_1}{\partial t^2} + g\,\frac{\partial p_1}{\partial z_1}\right)\Bigg|_{z_1=H_1} = \left(\frac{\partial^2 p_2}{\partial t^2} + g\,\frac{\partial p_2}{\partial z_2}\right)\Bigg|_{z_2=0} \tag{4f}$$

Also,

p_1 and p_2 are finite at r=0 (4g)

The solutions for p_1 and p_2 are expressed as the sum of the impulsive component and convective component, i.e.,

$$p_1 = p_1^i + p_2^c \tag{7}$$

and

$$p_2 = p_2^i + p_2^c \tag{8}$$

where the superscript i = impulsive component; and the superscript c = convective component.

The impulsive component of the hydrodynamic pressure p_1^i and p_2^i are taken to be the solutions that satisfy

$$\nabla^2 p_1^i = 0 \tag{9a}$$

and

$$\nabla^2 p_2^i = 0 \tag{9b}$$

and the following boundary conditions:

$$\frac{\partial p_1^i}{\partial z_1}\bigg|_{z_1=0} = 0 \tag{10a}$$

$$\frac{\partial p_1^i}{\partial r}\bigg|_{r=R} = -\rho_1 \ddot{x}(t)\cos\theta \tag{10b}$$

$$\frac{\partial p_2^i}{\partial r}\bigg|_{r=R} = -\rho_2 \ddot{x}(t)\cos\theta \tag{10c}$$

$$p_2^i\big|_{z_2=H_2} = 0 \tag{10d}$$

$$-\frac{1}{\rho_1}\frac{\partial p_1^i}{\partial z_1}\bigg|_{z_1=H_1} = -\frac{1}{\rho_2}\frac{\partial p_2^i}{\partial z_2}\bigg|_{z_2=0} \tag{10e}$$

$$p_1^i\big|_{z_1=H_1} = p_2^i\big|_{z_2=0} \tag{10f}$$

p_1^i and p_2^i are finite at r=0 (10g)

and the convective component of the hydrodynamic pressure p_1^c and p_2^c are taken to be the solutions that satisfy

$$\nabla^2 p_1^c = 0 \tag{11a}$$

and

$$\nabla^2 p_2^c = 0 \tag{11b}$$

and the following boundary conditions:

$$\frac{\partial p_1^c}{\partial z_1}\bigg|_{z_1=0} = 0 \tag{12a}$$

$$-\frac{1}{\rho_1}\frac{\partial p_1^c}{\partial r}\bigg|_{r=R} = 0 \tag{12b}$$

$$-\frac{1}{\rho_2}\frac{\partial p_2^c}{\partial r}\bigg|_{r=R} = 0 \tag{12c}$$

$$\left(\frac{\partial p_2^c}{\partial t^2} + g\frac{\partial p_2^c}{\partial z_2}\right)\bigg|_{z_2=H_2} = -g\frac{\partial p_2^i}{\partial z_2}\bigg|_{z_2=H_2} \tag{12d}$$

$$-\frac{1}{\rho_1}\frac{\partial p_1^c}{\partial z_1}\bigg|_{z_1=H_1} = -\frac{1}{\rho_2}\frac{\partial p_2^c}{\partial z_2}\bigg|_{z_2=0} \tag{12e}$$

$$\left(\frac{\partial^2 p_1^c}{\partial t^2} + g\frac{\partial p_1^c}{\partial z_1} + g\frac{\partial p_1^i}{\partial z_1}\right)\bigg|_{z_1=H_1} = \left(\frac{\partial^2 p_2^c}{\partial t^2} + g\frac{\partial p_2^c}{\partial z_2} + g\frac{\partial p_2^i}{\partial z_2}\right)\bigg|_{z_2=0} \tag{12f}$$

p_1^c and p_2^c are finite at r=0 (12g)

It can be shown easily that the sum of the above division of the hydrodynamic pressure satisfies the Eqs. (1a) and (1b) and the boundary conditions, Eqs. (4a) to (4g). One also notices that the above division of hydrodynamic pressure effectively assumes that the impulsive pressure is continuous at the interface, see Eq. (10f), and the gravitational force causes only the convective pressure to have a jump at the interface, see Eq. (12f). The above division is correct only if p_1^i and p_2^i have the time function of $\ddot{x}(t)$, and p_1^c and p_2^c have the time functions of the pseudoaccelerations associated with the sloshing motion.

The impulsive components of the hydrodynamic pressure defined by Eqs. (9a) and (9b) with the boundary conditions, Eqs. (10a) to (10g) are the same as those presented in Refs. 2 and 3, in which p_1^i and p_2^i are given by

$$p_1^i(r,\theta,z_1,t) = \left[-\frac{r}{R} + \sum_{n=1}^{\infty}\frac{G_n}{\Delta_n}A_n\cosh\left(\lambda_n\frac{z_1}{R}\right)\frac{J_1\left(\lambda_n\frac{r}{R}\right)}{J_1(\lambda_n)}\right]\rho_1 R\ddot{x}(t)\cos\theta \tag{13a}$$

and

$$p_2^i(r,\theta,z_2,t) = \left[-\frac{r}{R} + \sum_{n=1}^{\infty} \frac{G_n}{\Delta_n} \left[B_n \cosh\left(\lambda_n \frac{z_2}{R}\right) + C_n \sinh\left(\lambda_n \frac{z_2}{R}\right)\right] \frac{J_1\left(\lambda_n \frac{r}{R}\right)}{J_1(\lambda_n)} \right] \rho_2 R \ddot{x}(t) \cos\theta$$

(13b)

where J_1 is the Bessel function of the first kind and order 1, λ_n = the nth zero of $J'_1(x)$, the first derivative of J_1, and

$$A_n = \alpha + (1-\alpha)\cosh\beta_{2n} \tag{14a}$$

$$B_n = \cosh\beta_{1n} + (\alpha-1)\sinh\beta_{1n}\sinh\beta_{2n} \tag{14b}$$

$$C_n = \left[\alpha + (1-\alpha)\cosh\beta_{2n}\right]\sinh\beta_{1n} \tag{14c}$$

$$G_n = \frac{2}{\lambda_n^2 - 1} \tag{14d}$$

$$\Delta_n = \cosh\beta_{1n}\,\cosh\beta_{2n} + \alpha\,\sinh\beta_{1n}\,\sinh\beta_{2n} \tag{14e}$$

in which $\beta_{1n} = \lambda_n \frac{H_1}{R}$, $\beta_{2n} = \lambda_n \frac{H_2}{R}$, and $\alpha = \rho_2/\rho_1$.

The extensive numerical data for the impulsive components have been reported in Tang and Chang [2] and Tang [3]. Therefore, no more attention will be given to the impulsive components hereinafter. However, it is worthwhile to point out that the solutions given by Eqs. (13a) and (13b), are obtained by the eigenfunction expansion to the variable r. There is another approach to solve the same problem by the eigenfunction expansion to the variables z_1 and z_2 as that presented in Ref. 7. The latter approach is also used in Ref. 4 in a study for flexible tanks.

The report now turns its attention to the convective components of the solution. The derivation for p_1^c and p_2^c are given in Appendix A. The solutions for these functions can be cast into the forms given as follows.

$$p_1^c(r,\theta,z_1,t) = \left[\sum_{n=1}^{\infty}\sum_{k=1}^{2} C_{nk}^{I}(r,z_1)A_{jK}(t)\right]\rho_1 R\cos\theta \tag{15}$$

and

$$p_2^c(r,\theta,z_2,t) = \left[\sum_{n=1}^{\infty}\sum_{k=1}^{2} C_{nk}^{II}(r,z_2)A_{jk}(t)\right]\rho_2 R\cos\theta \tag{16}$$

where superscripts I and II denote Liquids I and II, respectively, and the expressions for functions $C_{nk}^{I}(r,z_1)$ and $C_{nk}^{II}(r,z_2)$ are given in Appendix A; the functions $A_{nk}(t)$, k=1,2, are the pseudoacceleration functions for the nth sloshing mode of vibration, $J_1(\lambda_n\ r/R)$, and are defined by

$$A_{nk}(t) = \omega_{nk}\int_0^t \ddot{x}(t)\ \sin(\omega_{nk}(t-\tau))\,d\tau, \tag{17}$$

in which ω_{nk} = the natural frequency associated with the nth sloshing mode of vibration. Introducing the notations, Λ_{nk}, for nondimensional coefficients that are related to ω_{nk} by the equation

$$\Lambda_{nk} = \frac{\omega_{nk}^2 R}{\lambda_n g} \tag{18}$$

It is shown in Appendix A that Λ_{nk}, k=1,2, are the roots of the characteristic equation given by

$$a\Lambda_n^2 - b\Lambda_n + c = 0 \tag{19}$$

where

$$a = 1 + \alpha\ \tanh\beta_{1n}\ \tanh\beta_{2n} \tag{20a}$$

$$b = \tanh\beta_{1n} + \tanh\beta_{2n} \tag{20b}$$

and

$$c = (1-\alpha)\tanh\beta_{1n}\ \tanh\beta_{2n} \tag{20c}$$

It can be shown that the discriminant $D = b^2 - 4ac > 0$ for $\alpha > 0$; therefore, Eq. (19) has two real and unequal roots. Explicitly, these roots are given by

$$\Lambda_{n1} = \frac{b + \sqrt{D}}{2a} \tag{21}$$

and

$$\Lambda_{n2} = \frac{b - \sqrt{D}}{2a} \tag{22}$$

Obviously, $\Lambda_{n1} > \Lambda_{n2}$; hence, $\omega_{n1} > \omega_{n2}$. The reasons for this reverse order numbering are:

1. For the case of $\alpha = 1$ (two liquids have the same densities), $\Lambda_{n1} = \tanh\left(\lambda_n \frac{H}{R}\right)$, the identical expression for the case of one liquid presented in Ref. 1, and $\Lambda_{n2} = 0$.

2. Generally speaking, ω_{11} is always the dominant frequency of the surface sloshing motion, whereas for some cases, identified later, ω_{12} is the dominant frequency for the interface sloshing motion. Since the surface motion has more engineering implication, the dominant frequency of the surface motion is numbered first.

3. It will be seen later that for the convective pressure exerted on tank wall the one associated with ω_{11} does not change sign, whereas the one associated with ω_{12} changes its sign when it cross the interface of two liquids.

The results presented above show that there are two natural frequencies associated with each sloshing mode of vibration. This finding is in sharp contrast to that presented in Refs. 2 and 3 in which only one frequency is found associated with each sloshing mode of vibration. This is the direct consequence of the g effect considered at the interface of two liquids.

IV. PRESENTATION OF RESULTS

Similar to the approximate solutions presented in Refs. 2 and 3, the exact solutions are controlled by three parameters, i.e., H/R, H_2/H_1 and α. In the numerical results presented below, those obtained by the equations presented herein are identified as "Exact', whereas those obtained from the equations presented in Refs. 2 and 3 are identified as "Approx.".

A. Sloshing Frequencies

It is convenient for the purpose of presentation to express the natural frequency as

$$\omega_{nk} = \Lambda^{*}_{nk} \sqrt{\frac{g}{R}} \tag{23}$$

where

$$\Lambda^{*}_{nk} = \sqrt{\Lambda_{nk}\,\lambda_{n}} \tag{24}$$

The values of Λ_{nk} for n=1,2 and k=1,2 for different values of control parameters, H/R and α, and for two values of H_2/H_1 = 0.5 and 2.0 are listed in Tables I and II, respectively. They are identified under "Exact". Examining these two tables, one can see that for the same values of H/R and H_2/H_1, the values of Λ_{nk} decreases with decreasing value of α. One also notices that the values listed in Table I are identical to their counterparts in Table II. This is due to the fact that the characteristic Eq. (19) is symmetric with respect to H_1 and H_2; i.e., for H_1 = c1 and H_2 = c2, Eq. (19) has the same roots as those for H_1 = c2 and H_2 = c1. The values of H_2/H_1 = 0.5 and 2 are chosen purposely herein to emphasize the fact that though for such cases the natural frequencies of the sloshing motion are the same, other response quantities are not. Also, listed in these tables are the approximate solutions for Λ_{12} and Λ_{21}, identified under "Approx.". Comparing the "Exact" with the "Approx." one can see that the approximate solutions yield accurate results for larger values of H/R, H_2/H_1 and α considered, but the accuracy deteriorates for low values of H/R, H_2/H_1 and α.

B. Surface and Interface Sloshing Displacements

The surface sloshing wave height, d(r,θ,t), of an arbitrary point at the liquid surface is determined from

$$p_2^c\Big|_{z_2=H_2} = \rho_2 g d(r,\theta,t) \tag{25}$$

The expression for computing the maximum value of the surface sloshing wave height is obtained by evaluating Eq. (25) for $d(r,\theta,t)$ at $r=R$ and $\theta=0$, and this result is expressed as

$$d(R,0,t) = \sum_{n=1}^{\infty} \sum_{k=1}^{2} d_{nk} \frac{A_{nk}(t)}{g} R \tag{26}$$

The sloshing wave height at the interface of two liquids, $\eta(r,\theta,t)$, can be determined from

$$\eta(r,\theta,t) = \frac{1}{(\rho_1 - \rho_2)g} \left(p_1^c \Big|_{z_1=H_1} - p_2^c \Big|_{z_2=0} \right) \tag{27}$$

which is obtained from Eq. (6) by eliminating the impulsive pressure. Note that for the case of $\alpha=1$, $\eta(r,\theta,t)$ cannot be determined from Eq. (27). The expression for computing the maximum value of the interface wave height is obtained by evaluating Eq. (27) at $r=R$ and $\theta=0$. The result is expressed as

$$\eta(R,0,t) = \sum_{n=1}^{\infty} \sum_{k=1}^{2} \eta_{nk} \frac{A_{nk}(t)}{g} R \tag{28}$$

d_{nk} and η_{nk} are dimensionless coefficients that depend on the control parameters. The values of d_{nk} for n=1,2 and k=1,2 and η_{nk} for n =1 and k = 1,2 for the same values of control parameters considered in Tables I and II are listed in Tables III and IV. They are identified under "Exact". From Tables III and IV, one can see that for the same values of H/R and H_2/H_1, the values of d_{11} and d_{21} increase as the value of α decreases. Also the results show that d_{11} and η_{11} have the same sign, whereas d_{12} and η_{12} have the opposite sign which indicates that for n=1 and k=1 mode the surface and interface are in phase, whereas for n=1 and k=2 mode they are out of phase. Also, one notices that for H_2/H_1 =2 for all values of H/R and α considered, the value for η_{12} is greater than that for η_{11}. For these cases, the interface sloshing motion has the dominant frequency of ω_{12} which is smaller than ω_{11}, the dominant frequency for the surface sloshing motion. This means that the interface sloshing motion vibrates in a different frequency as that of the surface sloshing motion. A numerical example shows this phenomenon can be found in Ref. 8.

Examining the data in Tables III and IV more critically, one can find that $d_{11} + d_{12}$ = 0.837, $d_{21} + d_{22}$ = 0.073 and $\eta_{11} + \eta_{12}$ = 0.837 irrespective of the values of H/R, H_2/H_1 and α

considered. This is not a coincidence. In effect, it can be shown that for each sloshing mode, the following equations hold.

$$\sum_{k=1}^{2} d_{nk} = \frac{2}{\lambda_n^2 - 1} \tag{29}$$

and

$$\sum_{k=1}^{2} \eta_{nk} = \frac{2}{\lambda_n^2 - 1} \tag{30}$$

For a limiting case, if all natural frequencies of the sloshing motion are very high in comparison with the dominant frequency of $\ddot{x}(t)$, all the pseudoacceleration functions $A_{nk}(t)$ will reduce to $\ddot{x}(t)$. Then, by making use of Eqs. (29) and (30), Eqs. (26) and (28) can be rewritten as

$$d(R,0,t) = \left(\sum_{n=1}^{\infty}\sum_{k=1}^{2} d_{nk}\right)\frac{\ddot{x}(t)R}{g}$$

$$= \left(\sum_{n=1}^{\infty}\frac{2}{\lambda_n^2-1}\right)\frac{\ddot{x}(t)R}{g} \tag{31}$$

and

$$\eta(R,0,t) = \left(\sum_{n=1}^{\infty}\sum_{k=1}^{2} \eta_{nk}\right)\frac{\ddot{x}(t)R}{g}$$

$$= \left(\sum_{n=1}^{\infty}\frac{2}{\lambda_n^2-1}\right)\frac{\ddot{x}(t)R}{g} \tag{32}$$

By making use the following identity

$$\sum_{n=1}^{\infty}\frac{2}{\lambda_n^2-1} = 1 \tag{33}$$

Eqs. (31) and (32) can be further reduce to

$$d(R,0,t) = \frac{\ddot{x}(t)R}{g} \tag{34}$$

and

$$\eta(R,0,t) = \frac{\ddot{x}(t)R}{g} \tag{35}$$

If $\ddot{x}(t) = A$, a uniform acceleration, Eq. (34) gives the exact same expression as that in Ref. 9 for computing the maximum rise of the liquid surface in a tank containing one liquid. Since Eqs. (34) and (35) hold irrespective of the α, a tank containing two liquids accelerated by a uniform acceleration will have the same rise for the liquid surface and interface which is physically sound.

Also listed in Tables III and IV are the approximate solutions for d_{11} and d_{21}. Comparing with the exact solutions, one can see that in Table III the approximate solutions are larger than those of the exact for $H/R < 1$ and $\alpha < 0.75$, and in Table IV the approximate solutions are in general fairly accurate. However, one should be reminded that the sloshing wave height depends not only on these coefficients but also on the pseudoacceleration functions. A numerical example given in Ref. 8 shows that the sloshing wave height in a typical tank used in reprocess of spent fuels [10] obtained by both solutions are in good agreement.

C. Convective Pressure

The convective pressure exerted on the tank wall is conveniently expressed in the form

$$p^{c}(\theta,z,t) = \left[\sum_{n=1}^{\infty}\sum_{k=1}^{2} C_{nk}(z)A_{nk}(t)\right]\rho_1 R\cos\theta \tag{36}$$

where $C_{nk}(z)$ is given by

$$C_{nk}(z) = C^{I}_{nk}(r,z_1)\Big|_{r=R} \qquad \text{for } 0 \le z \le H_1 \tag{37a}$$

$$C_{nk}(z) = \alpha\, C^{II}_{nk}(r,z_2)\Big|_{r=R} \qquad \text{for } H_1 \le z \le H \tag{37b}$$

The coordinate z used in Eq. (36) is related to z_1 and z_2 by the equations

$$z = z_1 \quad \text{for} \quad 0 \leq z \leq H_1 \tag{38a}$$

and

$$z = z_2 + H_1 \quad \text{for} \quad H_1 \leq z \leq H \tag{38b}$$

The distributions of C_{nk}, n=1, k=1,2 for α = 0.25 and 0.75 are shown in Fig. 2 for a broad tank, H/R = 0.5 for H_2/H_1 = 0.5, and in Fig. 3 for a tall tank, H/R =3, for H_2/H_1 = 0.5. For H_2/H_1 = 2, the curves are shown in Figs. 4 and 5 which are the counterparts of the curves in Figs. 2 and 3, respectively. The dashed curves in these figures are the corresponding approximate solutions. Note that there is no counterpart of $C_{12}(z)$ in the approximate solutions. From these figures, one can see that the exact solution of convective pressure has a jump at the interface while the approximate one does not, and, as one would have expected, this jump is larger for smaller value of α. As a result, the approximate solution always underestimates the convective pressure in Liquid I. One also notices that in these figures, $C_{12}(z)$ changes its sign when it cross the interface, and, surprisingly, the magnitude of $C_{12}(z)$ at top of Liquid I may be several times larger than that of $C_{11}(z)$ for the tall tank, H/R=3 and H_2/H_1 =2. Since the approximate solution is not able to predict the sharp increase of the convective pressure at the interface; hence, it should not be used in the computation of the convective pressure.

D. Convective Base Shear

The convective component of base shear, $Q^c(t)$, is given by

$$Q^c(t) = \int_0^{2\pi}\int_0^{H_1} p_1^c\Big|_{r=R} R\cos\theta \, dz_1 \, d\theta + \int_0^{2\pi}\int_0^{H_2} p_2^c\Big|_{r=R} R\cos\theta \, dz_2 \, d\theta \tag{39}$$

Substituting Eqs. (15) and (16) into Eq. (39) and performing the integration, one obtains the expression for $Q^c(t)$ which is given by

$$Q^c(t) = \left[\sum_{n=1}^{\infty} \sum_{k=1}^{2} S_{nk}^{I} A_{nk}(t)\right] M_{\ell 1} + \left[\sum_{n=1}^{\infty} \sum_{k=1}^{2} S_{nk}^{II} A_{nk}(t)\right] M_{\ell 2} \tag{40}$$

in which $M_{\ell 1} = \rho_1 \pi H_1 R^2$ = total liquid mass of Liquid I; $M_{\ell 2} = \rho_2 \pi R_2 H^2$ = total liquid mass of Liquid II; and S_{nk}^{I} and S_{nk}^{II} = dimensionless coefficients depend on the values of H/R, H_2/H_1 and α. Since Eq. (39) involves only simple integrations of the hyperbolic functions, the expressions for S_{nk}^{I} and S_{nk}^{II} are not given herein.

To study the effect of two liquids on the total base shear, it is also desirous to have an expression that can be used for comparison with an identical tank that contains one liquid. Therefore, $Q^c(t)$ is also expressed as

$$Q^c(t) = \left[\sum_{n=1}^{\infty} \sum_{k=1}^{2} r_{nk}^{s} A_{nk}(t)\right] M_{\ell}^{1} \tag{41}$$

in which $M_{\ell}^{1} = \pi \rho_1 H R^2$ = total liquid mass if the tank is filled with Liquid I; and r_{nk}^{s} = dimensionless coefficient related to S_{nk}^{I} and S_{nk}^{II} by the equation

$$r_{nk}^{s} = \frac{S_{nk}^{I} H_1 + \alpha S_{nk}^{II} H_2}{H} \tag{42}$$

The values of S_{nk}^{I}, S_{nk}^{II}, and r_{nk}^{s} for n=1 and k =1,2 for the same values of control parameters considered for Tables I and II are listed in Tables V and VI, from which one notes that the values of S_{11}^{I}, S_{11}^{II} and r_{11}^{s} decrease as the value of H/R increases. The values of S_{12}^{I} and r_{12}^{s} increase as the value of H/R increases for H_2/H_1 = 0.5, but for the case of H_2/H_1 = 2 they have the opposite trend. The values of S_{12}^{II} are negative for all the cases considered; this is due to the fact that the function $C_{12}(z)$ has negative value in Liquid II. Also, presented in Tables V and VI are the approximate solutions for S_{11}^{I}, S_{11}^{II} and r_{11}^{s}. One can see from these tables that the accuracy of the approximate solutions is poor for low values of H/R and α, and the accuracy improves for larger values of H/R and α.

E. Convective Base Moments

The convective component of base moment at a section immediately above the tank base is given by

$$M^c(t) = \int_0^{2\pi}\int_0^{H} p_1^c\Big|_{r=R} R z_1 \cos\theta \, dz_1 \, d\theta + \int_0^{2\pi}\int_0^{H_2} p_2^c\Big|_{r=R} R(z_2+H_1)\cos\theta \, dz_2 \, d\theta \tag{43}$$

and the result is expressed as

$$M^c(t) = \left[\sum_{n=1}^{\infty}\sum_{k=1}^{2} C_{nk}^{MI} A_{nk}(t)\right] M_{\ell 1} H_1$$

$$+ \left[\sum_{n=1}^{\infty}\sum_{k=1}^{2} C_{nk}^{MII} A_{nk}(t)\right] M_{\ell 2} H_2 \tag{44}$$

Again, since Eq. (43) involves only the simple integration of the hyperbolic functions, the expressions for C_{nk}^{IM} and C_{nk}^{IIM} are not given herein. Equation (44) is also rewritten as

$$M^c(t) = \left[\sum_{n=1}^{\infty}\sum_{k=1}^{2} r_{nk}^{M} A_{nk}(t)\right] M_\ell^1 H \tag{45}$$

in which r_{nk}^{M} = a dimensionless coefficient defined by

$$r_{nk}^{M} = C_{nk}^{MI}\left(\frac{H_1}{H}\right)^2 + \alpha\, C_{nk}^{MII}\left(\frac{H_2}{H}\right)^2 \tag{46}$$

The convective component of the base moment induced by the pressure exerted on the tank base is denoted by $\Delta M^c(t)$ which is computed by the following equation

$$\Delta M^c(t) = \int_0^{2\pi}\int_0^{R} p_1^c\Big|_{z_1=0} r^2 \cos\theta \, dr \, d\theta \tag{47}$$

and may be expressed in the form as

$$\Delta M^{c}(t) = \left[\sum_{n=1}^{\infty} \sum_{k=1}^{2} \Delta C_{nk}^{M} A_{nk}(t)\right] M_{l1} H_1 \tag{48}$$

in which ΔC_{nk}^{M} = a dimensionless coefficient depends on the values of H/R, H_2/H_1 and α. Eq. (48) is also expressed as

$$\Delta M^{c}(t) = \left[\sum_{n=1}^{\infty} \sum_{k=1}^{2} \Delta r_{nk}^{M} A_{nk}(t)\right] M_{l}^{1} H \tag{49}$$

to compare the result with that corresponding to $\alpha = 1$. The coefficient Δr_{nk}^{M} is related to ΔC_{nk}^{M} by the equation

$$\Delta r_{nk}^{M} = \Delta C_{nk}^{M} \left(\frac{H_1}{H}\right)^2 \tag{50}$$

The values of C_{nk}^{IM}, C_{nk}^{IIM}, r_{nk}^{M}, ΔC_{nk}^{M} and Δr_{nk}^{M} for n=1 and k=1,2 for the same values of the control parameters considered in Tables V and VI are presented in Tables VII and VIII. Also, the approximate solutions for C_{11}^{IM}, C_{11}^{IIM}, r_{11}^{M} are listed in Tables VI and VIII for comparison. The same trend found in Tables V and VI for base shear are found in Tables VII and VIII for the base moments.

V. EQUIVALENT MECHANICAL MODEL

The well-known equivalent mechanical model presented by Housner [11] for tanks containing one liquid can be generalized herein to represent tanks containing two liquids. This model is shown in Fig. 6. The rigidly attached mass, m_0, is located at a height, h_0, from the tank bottom representing the impulsive component of the response, and the nkth elastically supported mass, m_{nk}, is located at a height, h_{nk}, from the base representing the convective component of the response The heights, h_0 and h_{nk}, are used to evaluate the base moment at a section immediately above the tank base. By changing h_0 to h'_0 and h_{nk} to h'_{nk}, the model may also be used to evaluate the base moment at a section immediately below the tank base. The parameters of the model, m_0, m_{nk}, h_0, h_{nk}, h'_0, h'_{nk}, and k_{nk} are given as follows.

The impulsive component of base shear, denoted by $Q^i(t)$, may be computed from Eq. (39) by replacing p_1^c and p_2^c by p_1^i and p_2^i given by Eqs. (13a) and (13b) and perform the integrations. The result may be expressed as

$$Q^i(t) = r_{0s}^i M_\ell^1 \ddot{x}(t) \tag{51}$$

where r_{0s}^i = dimensionless coefficient depending on H/R, H_2/H_1 and α. The numerical results for r_{0s}^i are available in Refs. 2 and 3. The impulsive component of base moments, denoted by $M^i(t)$ for the moment at a section immediately above the tank base and by $\Delta M^i(t)$ for the moment induced by the pressure exerted on the tank base, may be computed from Eqs. (43) and (47) by replacing the convective pressure by the corresponding impulsive pressure and perform the integration. The results are then expressed as

$$M^i(t) = r_{0M}^i M_\ell^1 H \ddot{x}(t) \tag{52}$$

and

$$\Delta M^i(t) = \Delta r_{0M}^i M_\ell^1 H \ddot{x}(t) \tag{53}$$

The numerical results for the dimensionless coefficients, r_{0M}^i and Δr_{0M}^i, are available in Refs. 2 and 3.

The parameters for the model are then defined as follows.

$$m_o = r_{0s}^i M_\ell^1 \tag{54}$$

$$h_o = \left(\frac{r_{0M}^i}{r_{0s}^i} \right) H \tag{55}$$

$$h'_o = \frac{\left(r^i_{0M} + \Delta r^i_{0M}\right)}{r^i_{0s}} H \tag{56}$$

$$m_{nk} = r^s_{nk} M^1_\ell \tag{57}$$

$$k_{nk} = \omega^2_{nk} m_{nk} \tag{58}$$

$$h_{nk} = \left(\frac{r^M_{nk}}{r^s_{nk}}\right) H \tag{59}$$

and

$$h'_{nk} = \frac{\left(r^M_{nk} + \Delta r^M_{nk}\right)}{r^s_{nk}} H \tag{60}$$

It is easy to show that the mechanical model yields the same base shear and moments as those of the original tank-liquid system. The numerical results for h_0, h'_0, h_{11}, h'_{11}, h_{12}, and h'_{12} are given in Tables IX and X for the same values of control parameters considered in Tables I and II. Note that for some cases the values of h_{12} and h'_{12} are negative.

VI. CONCLUSIONS

A comprehensive study on the dynamic response of tanks containing two liquids is presented. The effect of gravitation at the interface is considered. Extensive numerical results are presented from which the gravitation and two-liquid interaction effects at the interface are identified. These two effects are summarized as follows.

A. Effect of Gravitation at the Interface

This effect is drawn from the comparison of the exact solutions with the approximate solutions.

1. There are two natural frequencies, ω_{nk}, k=1, 2, associated with the nth sloshing mode of vibration. The surface and interface sloshing motions are in phase for k =1 and out phase for k =2. For some cases, the ω_{11} is the dominant frequency of the surface sloshing motion while ω_{12} is the dominant frequency of the interface sloshing motion.

2. It changes the distribution of the convective component of hydrodynamic pressure causing a discontinuity at the interface. General speaking, it increases the convective pressure at Liquid I so is the base shear and moments. The convective pressure corresponding to the k =2 mode changes its sign when it crosses the interface, and, surprisingly, at the top of Liquid I it maybe several times larger than that of the k=1 mode for tall tanks with high values of H_2/H_1.

3. The approximate solutions may yield accurate results for the sloshing wave height, base shear and moments for $\alpha > 0.5$ and $H/R > 1$, but it should not be used to compute the convective pressure distribution.

4. It has no effect on the impulsive component of the hydrodynamic pressure. This component is continuous at the interface of two liquids. The exact and approximate solutions for the impulsive component of response are identical.

5. Even though, the approximate solution may not give accurate results for the pressure and base shear and moments, it may still be used to produce accurate results for the sloshing wave height.

B. <u>Effect of Two Liquids Interaction</u>

This effect is drawn based on the comparison of the results for two liquids with those for one liquid.

1. It increases the number of control parameters. For tanks containing one liquid the response is controlled by one parameter, H/R, whereas for tanks containing two liquids the response is controlled by three parameters, H/R, H_2/H_1 and ρ_2/ρ_1.

2. It increases the sloshing wave height and decreases the sloshing frequency. For certain combination of the control parameters, the sloshing wave height may increase significantly; therefore, it may leads to an unconservative result to compute the response of a tank containing two liquids based on the assumption of one liquid.

ACKNOWLEDGMENTS

The authors wish to thank Ralph Seidensticker of Argonne National Laboratory for helpful discussions during the performance of this work. This work was performed in the Engineering Mechanics Program of the Reactor Engineering Division of Argonne National Laboratory under the auspices of the U.S. Department of Energy under contract W-31-109-Eng-38.

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Table I. Sloshing Frequency Coefficient for $H_2/H_1 = 0.5$

$\frac{H}{R}$	Λ^*_{11} Exact	Λ^*_{11} Approx.	Λ^*_{12} Exact	Λ^*_{21} Exact	Λ^*_{21} Approx.	Λ^*_{22} Exact
$\alpha = 0.25$						
0.3	0.138	0.111	0.073	0.340	0.290	0.204
0.5	0.170	0.140	0.094	0.363	0.331	0.244
1.0	0.205	0.178	0.127	0.367	0.361	0.278
1.5	0.214	0.196	0.145	0.367	0.366	0.284
2.0	0.216	0.205	0.155	0.367	0.367	0.284
2.5	0.216	0.210	0.161	0.367	0.367	0.285
3.0	0.216	0.213	0.164	0.367	0.367	0.285
$\alpha = 0.5$						
0.3	0.145	0.127	0.057	0.346	0.316	0.157
0.5	0.177	0.157	0.072	0.365	0.346	0.186
1.0	0.208	0.192	0.097	0.367	0.364	0.208
1.5	0.214	0.204	0.110	0.367	0.367	0.211
2.0	0.216	0.210	0.117	0.367	0.367	0.212
2.5	0.216	0.213	0.121	0.367	0.367	0.212
3.0	0.216	0.214	0.123	0.367	0.367	0.212
$\alpha = 0.75$						
0.3	0.149	0.141	0.039	0.350	0.336	0.105
0.5	0.181	0.172	0.049	0.365	0.357	0.124
1.0	0.210	0.202	0.065	0.367	0.366	0.137
1.5	0.215	0.210	0.073	0.367	0.367	0.139
2.0	0.216	0.213	0.077	0.367	0.367	0.139
2.5	0.216	0.215	0.079	0.367	0.367	0.139
3.0	0.216	0.215	0.080	0.367	0.367	0.139
$\alpha = 1.0$						
0.3	0.153	0.153	0	0.353	0.353	0
0.5	0.184	0.184	0	0.366	0.366	0
1.0	0.211	0.211	0	0.367	0.367	0
1.5	0.215	0.215	0	0.367	0.367	0
2.0	0.216	0.216	0	0.367	0.367	0
2.5	0.216	0.216	0	0.367	0.367	0
3.0	0.216	0.216	0	0.367	0.367	0

Table II. Sloshing Frequency Coefficient for $H_2/H_1 = 2.0$

$\frac{H}{R}$	Λ^*_{11} Exact	Λ^*_{11} Approx.	Λ^*_{12} Exact	Λ^*_{21} Exact	Λ^*_{21} Approx.	Λ^*_{22} Exact
$\alpha = 0.25$						
0.3	0.138	0.135	0.073	0.340	0.335	0.204
0.5	0.170	0.167	0.094	0.363	0.360	0.244
1.0	0.205	0.202	0.127	0.367	0.367	0.278
1.5	0.214	0.212	0.145	0.367	0.367	0.284
2.0	0.216	0.215	0.155	0.367	0.367	0.284
2.5	0.216	0.216	0.161	0.367	0.367	0.285
3.0	0.216	0.216	0.164	0.367	0.367	0.285
$\alpha = 0.5$						
0.3	0.145	0.142	0.057	0.346	0.342	0.157
0.5	0.177	0.173	0.072	0.365	0.363	0.186
1.0	0.208	0.206	0.097	0.367	0.367	0.208
1.5	0.214	0.213	0.110	0.367	0.367	0.211
2.0	0.216	0.215	0.117	0.367	0.367	0.212
2.5	0.216	0.216	0.121	0.367	0.367	0.212
3.0	0.216	0.216	0.123	0.367	0.367	0.212
$\alpha = 0.75$						
0.3	0.149	0.148	0.039	0.350	0.348	0.105
0.5	0.181	0.179	0.049	0.365	0.364	0.124
1.0	0.210	0.208	0.065	0.367	0.367	0.137
1.5	0.215	0.214	0.073	0.367	0.367	0.139
2.0	0.216	0.216	0.077	0.367	0.367	0.139
2.5	0.216	0.216	0.079	0.367	0.367	0.139
3.0	0.216	0.216	0.080	0.367	0.367	0.139
$\alpha = 1.0$						
0.3	0.153	0.153	0	0.353	0.353	0
0.5	0.184	0.184	0	0.366	0.366	0
1.0	0.211	0.211	0	0.367	0.367	0
1.5	0.215	0.215	0	0.367	0.367	0
2.0	0.216	0.216	0	0.367	0.367	0
2.5	0.216	0.216	0	0.367	0.367	0
3.0	0.216	0.216	0	0.367	0.367	0

Table III. Surface and Interface Displacement Coefficients for $H_2/H_1 = 0.5$

$\frac{H}{R}$	d_{11} Exact	d_{11} Approx.	d_{12} Exact	d_{21} Exact	d_{21} Approx.	d_{22} Exact	η_{11} Exact	η_{12} Exact
$\alpha = 0.25$								
0.3	1.151	1.642	-0.314	0.105	0.128	-0.032	0.652	0.185
0.5	1.164	1.591	-0.328	0.109	0.111	-0.036	0.637	0.200
1.0	1.213	1.421	-0.376	0.101	0.088	-0.028	0.571	0.266
1.5	1.251	1.262	-0.414	0.087	0.079	-0.014	0.470	0.367
2.0	1.249	1.144	-0.412	0.079	0.075	-0.006	0.358	0.479
2.5	1.208	1.059	-0.371	0.075	0.074	-0.003	0.259	0.578
3.0	1.147	0.998	-0.311	0.074	0.073	-0.001	0.181	0.655
$\alpha = 0.5$								
0.3	0.983	1.242	-0.147	0.087	0.101	-0.014	0.594	0.243
0.5	0.989	1.219	-0.152	0.088	0.093	-0.015	0.575	0.262
1.0	1.005	1.142	-0.169	0.083	0.081	-0.010	0.493	0.344
1.5	1.009	1.064	-0.173	0.078	0.076	-0.005	0.387	0.450
2.0	0.996	1.004	-0.159	0.075	0.074	-0.002	0.288	0.549
2.5	0.972	0.959	-0.135	0.074	0.074	-0.001	0.209	0.628
3.0	0.946	0.926	-0.109	0.073	0.073	-0.000	0.150	0.687
$\alpha = 0.75$								
0.3	0.894	0.952	-0.057	0.078	0.085	-0.006	0.563	0.274
0.5	0.896	0.945	-0.060	0.078	0.081	-0.006	0.540	0.297
1.0	0.901	0.919	-0.064	0.076	0.076	-0.003	0.451	0.386
1.5	0.900	0.893	-0.063	0.074	0.074	-0.002	0.349	0.488
2.0	0.893	0.874	-0.056	0.074	0.074	-0.001	0.259	0.578
2.5	0.883	0.861	-0.046	0.073	0.073	-0.000	0.190	0.647
3.0	0.874	0.852	-0.037	0.073	0.073	-0.000	0.139	0.698
$\alpha = 1.0$								
0.3	0.837	0.837	0	0.073	0.073	0	--	--
0.5	0.837	0.837	0	0.073	0.073	0	--	--
1.0	0.837	0.837	0	0.073	0.073	0	--	--
1.5	0.837	0.837	0	0.073	0.073	0	--	--
2.0	0.837	0.837	0	0.073	0.073	0	--	--
2.5	0.837	0.837	0	0.073	0.073	0	--	--
3.0	0.837	0.837	0	0.073	0.073	0	--	--

Table IV. Surface and Interface Displacement Coefficients for $H_2/H_1 = 2.0$

$\frac{H}{R}$	d_{11}		d_{12}	d_{21}		d_{22}	η_{11}	η_{12}
	Exact	Approx.	Exact	Exact	Approx.	Exact	Exact	Exact
$\alpha = 0.25$								
0.3	1.139	1.106	-0.302	0.097	0.091	-0.024	0.173	0.664
0.5	1.132	1.089	-0.295	0.091	0.084	-0.019	0.166	0.671
1.0	1.097	1.026	-0.260	0.078	0.075	-0.005	0.136	0.701
1.5	1.043	0.961	-0.207	0.074	0.073	-0.001	0.097	0.740
2.0	0.983	0.912	-0.146	0.073	0.073	-0.000	0.062	0.775
2.5	0.930	0.880	-0.094	0.073	0.073	-0.000	0.036	0.801
3.0	0.893	0.861	-0.056	0.073	0.073	-0.000	0.021	0.816
$\alpha = 0.5$								
0.3	0.977	0.998	-0.140	0.084	0.084	-0.011	0.224	0.613
0.5	0.973	0.987	-0.136	0.081	0.079	-0.008	0.211	0.626
1.0	0.952	0.948	-0.115	0.075	0.074	-0.002	0.162	0.675
1.5	0.922	0.908	-0.085	0.073	0.073	-0.000	0.107	0.729
2.0	0.893	0.880	-0.056	0.073	0.073	-0.000	0.065	0.772
2.5	0.871	0.861	-0.034	0.073	0.073	-0.000	0.037	0.800
3.0	0.856	0.850	-0.020	0.073	0.073	-0.000	0.021	0.816
$\alpha = 0.75$								
0.3	0.892	0.910	-0.055	0.077	0.078	-0.004	0.250	0.587
0.5	0.890	0.905	-0.053	0.076	0.076	-0.003	0.234	0.603
1.0	0.880	0.886	-0.043	0.074	0.074	-0.001	0.172	0.664
1.5	0.868	0.868	-0.031	0.073	0.073	-0.000	0.111	0.726
2.0	0.856	0.855	-0.020	0.073	0.073	-0.000	0.065	0.771
2.5	0.848	0.847	-0.012	0.073	0.073	-0.000	0.037	0.800
3.0	0.843	0.843	-0.007	0.073	0.073	-0.000	0.021	0.816
$\alpha = 1.0$								
0.3	0.837	0.837	0	0.073	0.073	0	--	--
0.5	0.837	0.837	0	0.073	0.073	0	--	--
1.0	0.837	0.837	0	0.073	0.073	0	--	--
1.5	0.837	0.837	0	0.073	0.073	0	--	--
2.0	0.837	0.837	0	0.073	0.073	0	--	--
2.5	0.837	0.837	0	0.073	0.073	0	--	--
3.0	0.837	0.837	0	0.073	0.073	0	--	--

Table V. Coefficients of Convective Components of Base Shear for $H_2/H_1 = 0.5$

$\frac{H}{R}$	S_{11}^{I} Exact	S_{11}^{I} Approx.	S_{12}^{I} Exact	S_{11}^{II} Exact	S_{11}^{II} Approx.	S_{12}^{II} Exact	r_{11}^{s} Exact	r_{11}^{s} Approx.	r_{12}^{s} Exact
$\alpha = 0.25$									
0.3	0.727	0.380	0.058	1.114	1.611	-0.313	0.578	0.388	0.013
0.5	0.647	0.325	0.061	1.071	1.514	-0.323	0.521	0.343	0.014
1.0	0.419	0.183	0.074	0.944	1.206	-0.359	0.358	0.223	0.020
1.5	0.250	0.096	0.090	0.830	0.936	-0.382	0.236	0.142	0.028
2.0	0.145	0.051	0.101	0.723	0.737	-0.375	0.157	0.096	0.036
2.5	0.084	0.029	0.105	0.618	0.595	-0.343	0.108	0.069	0.041
3.0	0.049	0.017	0.102	0.525	0.492	-0.302	0.077	0.052	0.043
$\alpha = 0.50$									
0.3	0.724	0.566	0.046	0.948	1.209	-0.146	0.641	0.579	0.006
0.5	0.626	0.480	0.048	0.903	1.139	-0.152	0.568	0.510	0.007
1.0	0.372	0.267	0.057	0.775	0.930	-0.169	0.377	0.333	0.010
1.5	0.208	0.141	0.064	0.666	0.752	-0.176	0.249	0.219	0.013
2.0	0.117	0.077	0.066	0.575	0.618	-0.169	0.174	0.154	0.016
2.5	0.068	0.044	0.064	0.497	0.518	-0.155	0.128	0.116	0.017
3.0	0.041	0.026	0.060	0.432	0.442	-0.136	0.099	0.091	0.017
$\alpha = 0.75$									
0.3	0.731	0.673	0.024	0.860	0.966	-0.058	0.702	0.690	0.001
0.5	0.617	0.564	0.025	0.813	0.910	-0.060	0.615	0.603	0.002
1.0	0.346	0.308	0.029	0.690	0.757	-0.067	0.403	0.395	0.002
1.5	0.188	0.164	0.030	0.593	0.633	-0.069	0.273	0.267	0.003
2.0	0.105	0.091	0.030	0.515	0.539	-0.066	0.199	0.195	0.004
2.5	0.062	0.053	0.028	0.452	0.465	-0.061	0.154	0.152	0.004
3.0	0.038	0.032	0.026	0.399	0.407	-0.055	0.125	0.123	0.004
$\alpha = 1.0$									
0.3	0.740	0.740	0	0.803	0.803	0	0.761	0.761	0
0.5	0.612	0.612	0	0.756	0.756	0	0.660	0.660	0
1.0	0.329	0.329	0	0.638	0.638	0	0.432	0.432	0
1.5	0.176	0.176	0	0.550	0.550	0	0.301	0.301	0
2.0	0.099	0.099	0	0.483	0.483	0	0.227	0.227	0
2.5	0.059	0.059	0	0.428	0.428	0	0.182	0.182	0
3.0	0.036	0.036	0	0.382	0.382	0	0.151	0.151	0

Table VI. Coefficients of Convective Component of Base Shear for $H_2/H_1 = 2.0$

	S_{11}^{I}		S_{12}^{I}	S_{11}^{II}		S_{12}^{II}	r_{11}^{s}		r_{12}^{s}
$\frac{H}{R}$	Exact	Approx.	Exact	Exact	Approx.	Exact	Exact	Approx.	Exact
$\alpha = 0.25$									
0.3	0.385	0.252	0.416	1.078	1.050	-0.303	0.308	0.259	0.088
0.5	0.337	0.212	0.412	0.981	0.953	-0.296	0.276	0.230	0.088
1.0	0.200	0.111	0.392	0.707	0.678	-0.269	0.184	0.150	0.086
1.5	0.103	0.050	0.362	0.503	0.479	-0.232	0.118	0.097	0.082
2.0	0.050	0.022	0.327	0.374	0.357	-0.194	0.079	0.067	0.077
2.5	0.024	0.010	0.290	0.291	0.281	-0.162	0.056	0.050	0.070
3.0	0.011	0.005	0.255	0.237	0.231	-0.136	0.043	0.040	0.062
$\alpha = 0.50$									
0.3	0.545	0.447	0.231	0.918	0.941	-0.142	0.488	0.463	0.030
0.5	0.460	0.371	0.230	0.829	0.849	-0.140	0.430	0.406	0.030
1.0	0.245	0.185	0.222	0.596	0.607	-0.130	0.280	0.264	0.031
1.5	0.115	0.082	0.206	0.436	0.441	-0.115	0.184	0.174	0.030
2.0	0.053	0.036	0.185	0.337	0.338	-0.099	0.130	0.125	0.029
2.5	0.024	0.016	0.163	0.272	0.272	-0.085	0.099	0.096	0.026
3.0	0.011	0.008	0.143	0.227	0.227	-0.073	0.079	0.078	0.023
$\alpha = 0.75$									
0.3	0.649	0.602	0.102	0.833	0.851	-0.056	0.633	0.627	0.006
0.5	0.533	0.491	0.102	0.749	0.766	-0.055	0.552	0.547	0.006
1.0	0.265	0.238	0.098	0.543	0.553	-0.052	0.360	0.356	0.007
1.5	0.119	0.105	0.091	0.407	0.412	-0.047	0.243	0.241	0.007
2.0	0.053	0.046	0.080	0.322	0.324	-0.041	0.179	0.177	0.006
2.5	0.024	0.021	0.070	0.264	0.266	-0.036	0.140	0.140	0.006
3.0	0.011	0.010	0.061	0.223	0.224	-0.031	0.115	0.115	0.005
$\alpha = 1.0$									
0.3	0.728	0.728	0	0.778	0.778	0	0.761	0.761	0
0.5	0.584	0.584	0	0.698	0.698	0	0.660	0.660	0
1.0	0.276	0.276	0	0.511	0.511	0	0.432	0.432	0
1.5	0.121	0.121	0	0.391	0.391	0	0.301	0.301	0
2.0	0.054	0.054	0	0.314	0.314	0	0.227	0.227	0
2.5	0.024	0.024	0	0.261	0.261	0	0.182	0.182	0
3.0	0.011	0.011	0	0.222	0.222	0	0.151	0.151	0

Table VII. Coefficients of Convective Component of Base Moments for $H_2/H_1 = 0.5$

$\frac{H}{R}$	C_{11}^{IM}		C_{12}^{IM}	C_{11}^{IIM}		C_{12}^{IIM}	r_{11}^{M}		r_{12}^{M}
	Exact	Approx.	Exact	Exact	Approx.	Exact	Exact	Approx.	Exact
$\alpha = 0.25$									
0.3	0.368	0.192	0.029	2.791	4.032	-0.782	0.241	0.197	-0.009
0.5	0.333	0.168	0.032	2.691	3.795	-0.808	0.223	0.180	-0.008
1.0	0.233	0.102	0.041	2.399	3.045	-0.899	0.170	0.130	-0.007
1.5	0.151	0.058	0.054	2.135	2.383	-0.957	0.127	0.092	-0.002
2.0	0.095	0.034	0.066	1.878	1.894	-0.937	0.095	0.068	0.003
2.5	0.059	0.020	0.074	1.622	1.544	-0.853	0.071	0.052	0.009
3.0	0.037	0.013	0.076	1.388	1.290	-0.744	0.055	0.041	0.013
$\alpha = 0.5$									
0.3	0.366	0.286	0.023	2.376	3.027	-0.366	0.295	0.295	-0.010
0.5	0.322	0.247	0.025	2.270	2.859	-0.380	0.269	0.269	-0.010
1.0	0.206	0.148	0.031	1.971	2.356	-0.421	0.201	0.197	-0.009
1.5	0.126	0.085	0.038	1.715	1.923	-0.437	0.151	0.145	-0.007
2.0	0.077	0.051	0.043	1.495	1.594	-0.418	0.117	0.111	-0.004
2.5	0.048	0.031	0.045	1.304	1.349	-0.380	0.094	0.089	-0.001
3.0	0.030	0.020	0.045	1.143	1.162	-0.337	0.077	0.073	0.001
$\alpha = 0.75$									
0.3	0.369	0.340	0.012	2.155	2.419	-0.144	0.344	0.353	-0.007
0.5	0.318	0.290	0.013	2.046	2.288	-0.150	0.312	0.320	-0.007
1.0	0.192	0.171	0.016	1.757	1.922	-0.166	0.232	0.236	-0.007
1.5	0.114	0.099	0.018	1.525	1.624	-0.170	0.178	0.179	-0.006
2.0	0.069	0.060	0.020	1.339	1.396	-0.162	0.142	0.143	-0.005
2.5	0.043	0.037	0.020	1.185	1.217	-0.148	0.118	0.118	-0.003
3.0	0.028	0.024	0.020	1.056	1.075	-0.133	0.100	0.100	-0.002
$\alpha = 1.0$									
0.3	0.374	0.374	0	2.012	2.012	0	0.390	0.390	0
0.5	0.315	0.315	0	1.903	1.903	0	0.352	0.352	0
1.0	0.183	0.183	0	1.626	1.626	0	0.262	0.262	0
1.5	0.106	0.106	0	1.416	1.416	0	0.205	0.205	0
2.0	0.065	0.065	0	1.255	1.255	0	0.168	0.168	0
2.5	0.041	0.041	0	1.123	1.123	0	0.143	0.143	0
3.0	0.027	0.027	0	1.012	1.012	0	0.124	0.124	0

Table VII. Coefficients of Convective Component of Base Moments for H_2/H_1 = 0.5 (Cont'd)

$\frac{H}{R}$	ΔC_{11}^{M}		ΔC_{12}^{M}	Δr_{11}^{M}		Δr_{12}^{M}
	Exact	Approx.	Exact	Exact	Approx.	Exact
α = 0.25						
0.3	5.243	2.742	0.418	2.330	1.218	0.186
0.5	1.615	0.811	0.153	0.718	0.361	0.068
1.0	0.219	0.096	0.039	0.097	0.042	0.017
1.5	0.044	0.017	0.016	0.020	0.008	0.007
2.0	0.010	0.004	0.007	0.005	0.002	0.003
2.5	0.003	0.001	0.003	0.001	0.000	0.001
3.0	0.001	0.000	0.001	0.000	0.000	0.001
α = 0.5						
0.3	5.220	4.082	0.329	2.320	1.814	0.146
0.5	1.561	1.197	0.120	0.694	0.532	0.053
1.0	0.194	0.139	0.030	0.086	0.062	0.013
1.5	0.037	0.025	0.011	0.016	0.011	0.005
2.0	0.008	0.005	0.005	0.004	0.002	0.002
2.5	0.002	0.001	0.002	0.001	0.001	0.001
3.0	0.001	0.000	0.001	0.000	0.000	0.000
α = 0.75						
0.3	5.270	4.852	0.172	2.342	2.157	0.076
0.5	1.539	1.406	0.062	0.684	0.625	0.028
1.0	0.181	0.161	0.015	0.080	0.072	0.007
1.5	0.033	0.029	0.005	0.015	0.013	0.002
2.0	0.007	0.006	0.002	0.003	0.003	0.001
2.5	0.002	0.002	0.001	0.001	0.001	0.000
3.0	0.001	0.000	0.000	0.000	0.000	0.000
α = 1.0						
0.3	5.337	5.337	0	2.372	2.372	0
0.5	1.527	1.527	0	0.679	0.679	0
1.0	0.172	0.172	0	0.076	0.076	0
1.5	0.031	0.031	0	0.014	0.014	0
2.0	0.007	0.007	0	0.003	0.003	0
2.5	0.002	0.002	0	0.001	0.001	0
3.0	0.000	0.000	0	0.000	0.000	0

Table VIII. Coefficients of Convective Component of Base Moments for $H_2/H_1 = 2.0$

$\frac{H}{R}$	C_{11}^{IM}		C_{12}^{IM}	C_{11}^{IIM}		C_{12}^{IIM}	r_{11}^{M}		r_{12}^{M}
	Exact	Approx.	Exact	Exact	Approx.	Exact	Exact	Approx.	Exact
$\alpha = 0.25$									
0.3	0.193	0.126	0.209	1.086	1.057	-0.302	0.142	0.132	-0.010
0.5	0.170	0.107	0.207	1.001	0.970	-0.294	0.130	0.120	-0.010
1.0	0.103	0.057	0.202	0.756	0.721	-0.262	0.095	0.086	-0.007
1.5	0.055	0.027	0.193	0.567	0.535	-0.218	0.069	0.062	-0.003
2.0	0.028	0.012	0.181	0.440	0.417	-0.173	0.052	0.048	0.001
2.5	0.014	0.006	0.168	0.355	0.340	-0.136	0.041	0.038	0.004
3.0	0.007	0.003	0.155	0.297	0.288	-0.108	0.034	0.032	0.005
$\alpha = 0.5$									
0.3	0.273	0.224	0.116	0.926	0.948	-0.141	0.236	0.236	-0.019
0.5	0.232	0.187	0.116	0.849	0.867	-0.138	0.214	0.213	-0.018
1.0	0.126	0.095	0.114	0.642	0.651	-0.125	0.157	0.155	-0.015
1.5	0.061	0.044	0.110	0.495	0.496	-0.105	0.117	0.115	-0.011
2.0	0.029	0.020	0.103	0.398	0.397	-0.086	0.092	0.090	-0.008
2.5	0.014	0.010	0.094	0.332	0.331	-0.069	0.075	0.075	-0.005
3.0	0.007	0.005	0.086	0.285	0.283	-0.056	0.064	0.064	0.003
$\alpha = 0.75$									
0.3	0.326	0.302	0.051	0.841	0.860	-0.056	0.316	0.320	-0.013
0.5	0.269	0.247	0.051	0.769	0.785	-0.055	0.286	0.289	-0.013
1.0	0.136	0.122	0.051	0.587	0.596	-0.050	0.211	0.212	-0.011
1.5	0.063	0.056	0.048	0.463	0.467	-0.043	0.161	0.162	-0.009
2.0	0.030	0.026	0.045	0.381	0.382	-0.035	0.130	0.130	-0.007
2.5	0.014	0.012	0.041	0.323	0.324	-0.029	0.109	0.109	-0.005
3.0	0.007	0.006	0.037	0.280	0.281	-0.024	0.094	0.094	-0.004
$\alpha = 1.0$									
0.3	0.365	0.365	0	0.786	0.786	0	0.390	0.390	0
0.5	0.295	0.295	0	0.717	0.717	0	0.352	0.352	0
1.0	0.142	0.142	0	0.553	0.553	0	0.262	0.262	0
1.5	0.064	0.064	0	0.445	0.445	0	0.205	0.205	0
2.0	0.030	0.030	0	0.371	0.371	0	0.168	0.168	0
2.5	0.014	0.014	0	0.318	0.318	0	0.143	0.143	0
3.0	0.007	0.007	0	0.278	0.278	0	0.124	0.124	0

Table VIII. Coefficients of Convective Component of Base Moments for $H_2/H_1 = 2.0$ (Cont'd)

$\frac{H}{R}$	ΔC_{11}^M Exact	ΔC_{11}^M Approx.	ΔC_{12}^M Exact	Δr_{11}^M Exact	Δr_{11}^M Approx.	Δr_{12}^M Exact
$\alpha = 0.25$						
0.3	11.307	7.384	12.200	1.256	0.820	1.356
0.5	3.519	2.220	4.302	0.391	0.247	0.478
1.0	0.498	0.276	0.978	0.055	0.031	0.109
1.5	0.106	0.052	0.373	0.012	0.006	0.041
2.0	0.026	0.012	0.171	0.003	0.001	0.019
2.5	0.007	0.003	0.085	0.001	0.000	0.009
3.0	0.002	0.001	0.045	0.000	0.000	0.005
$\alpha = 0.5$						
0.3	15.996	13.119	6.768	1.777	1.458	0.752
0.5	4.808	3.873	2.400	0.534	0.430	0.267
1.0	0.610	0.462	0.554	0.068	0.051	0.062
1.5	0.118	0.085	0.212	0.013	0.009	0.024
2.0	0.027	0.019	0.097	0.003	0.002	0.011
2.5	0.007	0.005	0.048	0.001	0.001	0.005
3.0	0.002	0.001	0.025	0.000	0.000	0.003
$\alpha = 0.75$						
0.3	19.051	17.670	2.993	2.117	1.963	0.333
0.5	5.576	5.133	1.063	0.620	0.570	0.118
1.0	0.660	0.593	0.245	0.073	0.066	0.027
1.5	0.122	0.108	0.093	0.014	0.012	0.010
2.0	0.028	0.024	0.042	0.003	0.003	0.005
2.5	0.007	0.006	0.021	0.001	0.001	0.002
3.0	0.002	0.002	0.011	0.000	0.000	0.001
$\alpha = 1.0$						
0.3	21.346	21.346	0	2.372	2.372	0
0.5	6.110	6.110	0	0.679	0.679	0
1.0	0.688	0.688	0	0.076	0.076	0
1.5	0.124	0.124	0	0.014	0.014	0
2.0	0.028	0.028	0	0.003	0.003	0
2.5	0.007	0.007	0	0.001	0.001	0
3.0	0.002	0.002	0	0.000	0.000	0

Table IX. Quantities in Mechanical Model for Tank-Liquid System, $H_2/H_1 = 0.5$

$\frac{H}{R}$	$\frac{h_o}{H}$	$\frac{h'_o}{H}$	$\frac{h_{11}}{H}$	$\frac{h'_{11}}{H}$	$\frac{h_{12}}{H}$	$\frac{h'_{12}}{H}$
$\alpha = 0.25$						
0.3	0.333	2.646	0.417	4.452	-0.692	14.084
0.5	0.332	1.497	0.428	1.807	-0.610	4.289
1.0	0.334	0.713	0.475	0.746	-0.335	0.541
1.5	0.338	0.516	0.537	0.620	-0.085	0.167
2.0	0.343	0.443	0.602	0.631	0.096	0.183
2.5	0.348	0.411	0.663	0.673	0.219	0.253
3.0	0.352	0.395	0.716	0.719	0.304	0.318
$\alpha = 0.50$						
0.3	0.366	2.618	0.460	4.081	-1.674	22.619
0.5	0.365	1.476	0.475	1.697	-1.504	6.436
1.0	0.368	0.717	0.533	0.762	-0.979	0.386
1.5	0.374	0.534	0.606	0.671	-0.550	-0.168
2.0	0.381	0.470	0.674	0.695	-0.261	-0.129
2.5	0.387	0.443	0.731	0.738	-0.069	-0.017
3.0	0.392	0.430	0.776	0.778	0.063	0.085
$\alpha = 0.75$						
0.3	0.386	2.622	0.490	3.825	-4.523	47.388
0.5	0.386	1.467	0.507	1.620	-4.091	12.656
1.0	0.389	0.719	0.575	0.774	-2.855	-0.066
1.5	0.397	0.546	0.650	0.704	-1.926	-1.157
2.0	0.405	0.487	0.715	0.732	-1.326	-1.061
2.5	0.412	0.464	0.766	0.771	-0.931	-0.828
3.0	0.419	0.454	0.804	0.806	-0.658	-0.615
$\alpha = 1.0$						
0.3	0.400	2.637	0.512	3.629	--	--
0.5	0.400	1.464	0.533	1.561	--	--
1.0	0.404	0:721	0.606	0.782	--	--
1.5	0.413	0.555	0.681	0.727	--	--
2.0	0.423	0.500	0.742	0.755	--	--
2.5	0.431	0.480	0.787	0.791	--	--
3.0	0.439	0.472	0.820	0.822	--	--

Table X. Quantities in Mechanical Model for Tank-Liquid System, $H_2/H_1 = 2$

$\frac{H}{R}$	$\frac{h_o}{H}$	$\frac{h'_o}{H}$	$\frac{h_{11}}{H}$	$\frac{h'_{11}}{H}$	$\frac{h_{12}}{H}$	$\frac{h'_{12}}{H}$
$\alpha = 0.25$						
0.3	0.337	2.563	0.461	4.539	-0.118	15.250
0.5	0.337	1.479	0.472	1.890	-0.110	5.334
1.0	0.336	0.731	0.518	0.818	-0.078	1.188
1.5	0.332	0.531	0.584	0.684	-0.034	0.470
2.0	0.328	0.448	0.658	0.695	0.011	0.259
2.5	0.324	0.405	0.727	0.740	0.051	0.187
3.0	0.322	0.380	0.781	0.786	0.084	0.164
$\alpha = 0.50$						
0.3	0.373	2.569	0.484	4.128	-0.625	24.739
0.5	0.373	1.462	0.499	1.742	-0.597	8.306
1.0	0.373	0.725	0.559	0.801	-0.490	1.515
1.5	0.375	0.542	0.635	0.707	-0.370	0.406
2.0	0.377	0.474	0.706	0.730	-0.267	0.108
2.5	0.379	0.442	0.764	0.772	-0.187	0.018
3.0	0.381	0.425	0.806	0.809	-0.125	-0.004
$\alpha = 0.75$						
0.3	0.389	2.602	0.500	3.845	-2.097	52.268
0.5	0.389	1.462	0.518	1.639	-2.001	16.901
1.0	0.392	0.722	0.586	0.790	-1.669	2.469
1.5	0.397	0.549	0.663	0.719	-1.344	0.227
2.0	0.404	0.489	0.729	0.746	-1.089	-0.334
2.5	0.410	0.464	0.779	0.784	-0.895	-0.484
3.0	0.415	0.452	0.815	0.817	-0.747	-0.506
$\alpha = 1.0$						
0.3	0.400	2.637	0.512	3.629	--	--
0.5	0.400	1.464	0.533	1.561	--	--
1.0	0.404	0.721	0.606	0.782	--	--
1.5	0.413	0.555	0.681	0.727	--	--
2.0	0.423	0.500	0.742	0.755	--	--
2.5	0.431	0.480	0.787	0.791	--	--
3.0	0.439	0.472	0.820	0.822	--	--

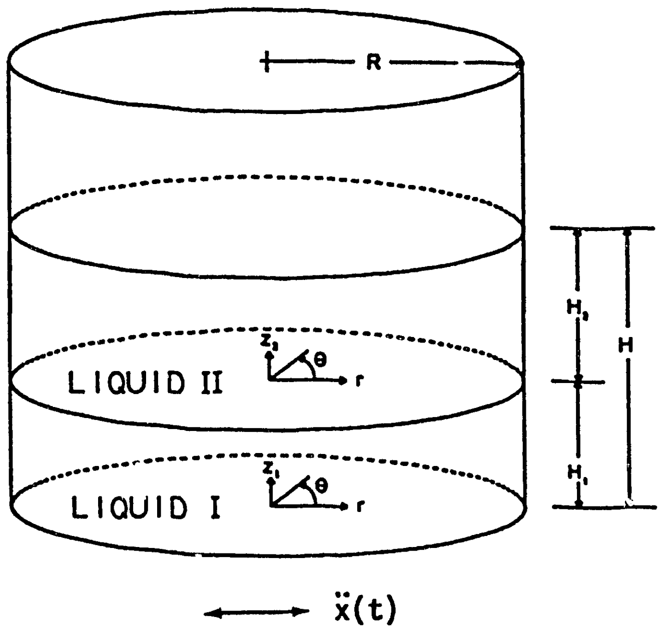

Fig. 1. System Considered

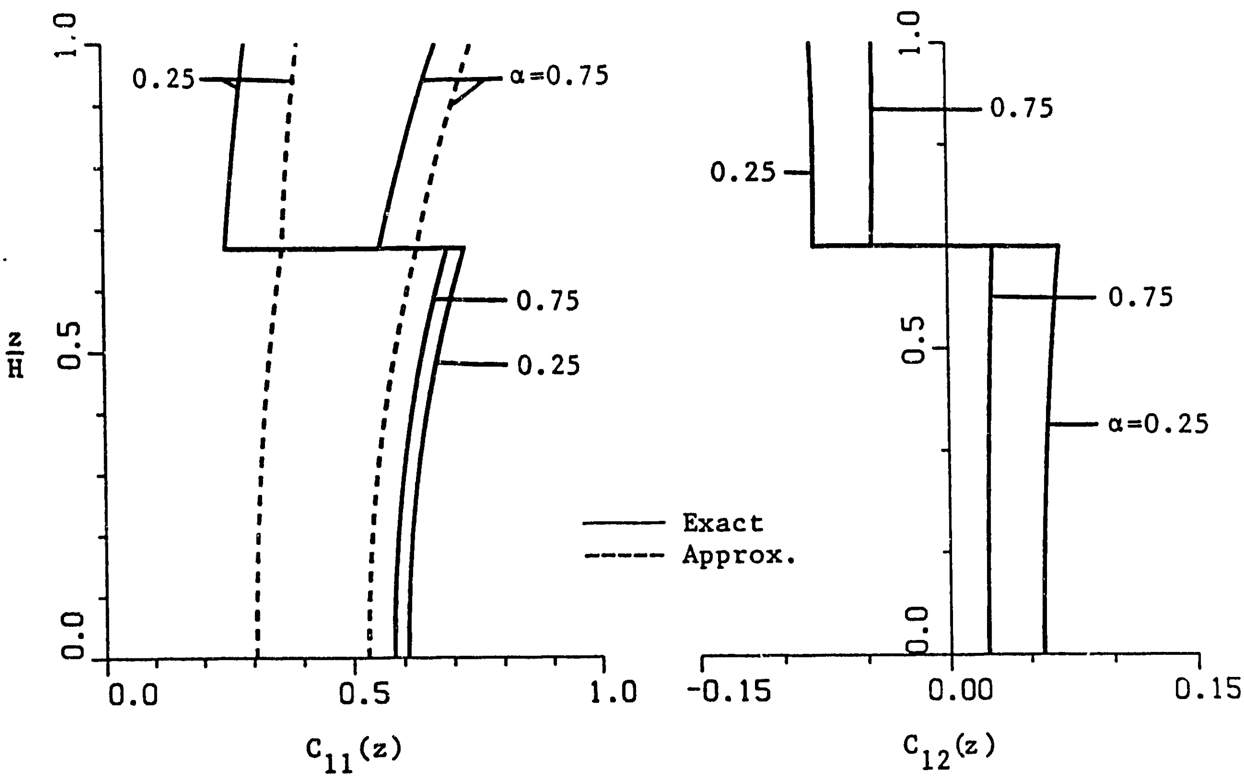

Fig. 2. Convective Pressure Exerted on Tank Wall with H/R = 0.5, H_2/H_1 = 0.

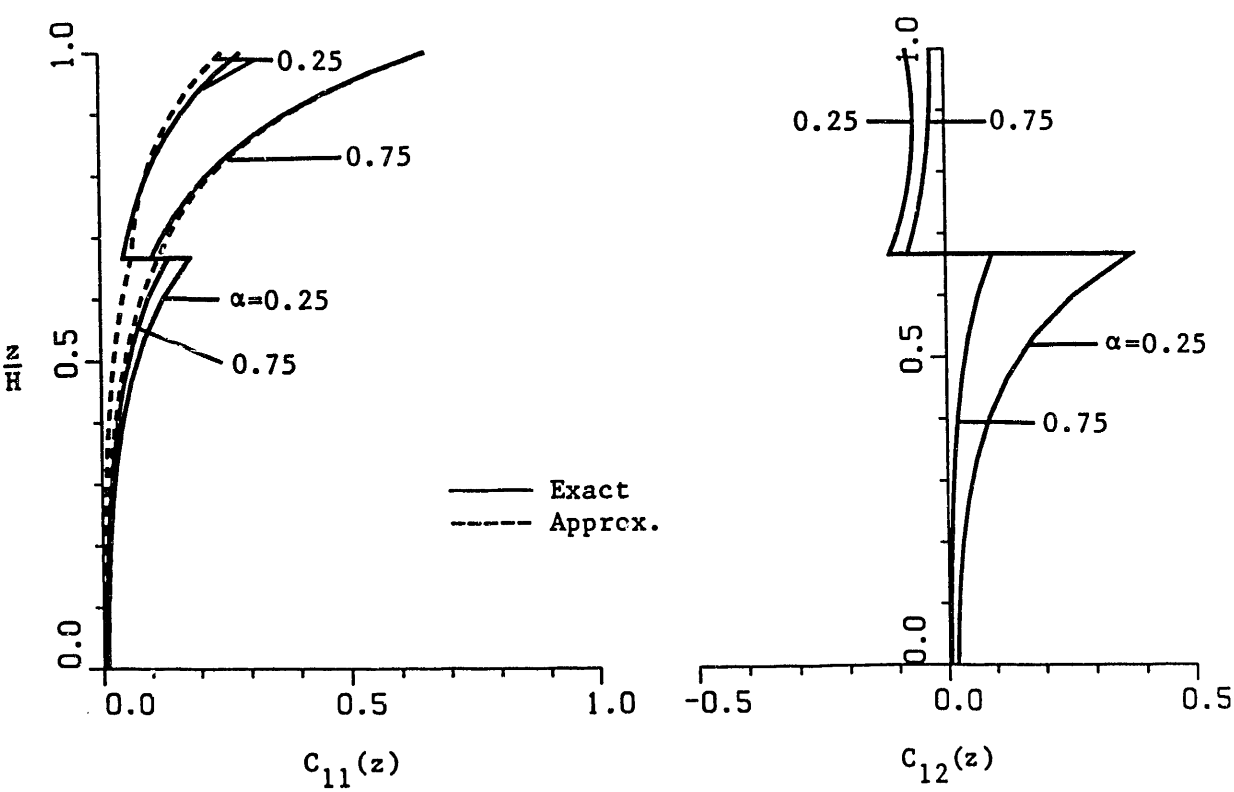

Fig. 3. Convective Pressure Exerted on Tank Wall with H/R = 3, H_2/H_1 = 0.5

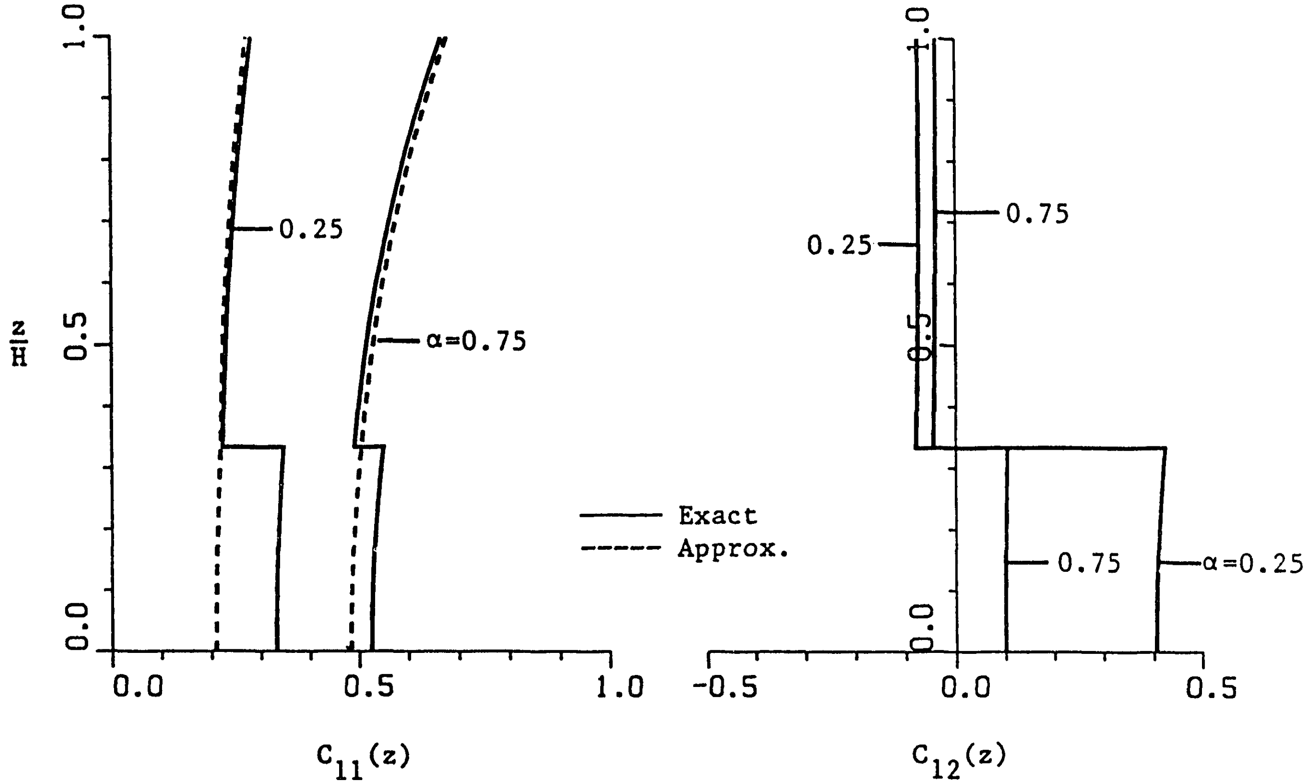

Fig. 4. Convective Pressure Exerted on Tank Wall with H/R = 0.5, H_2/H_1 = 2

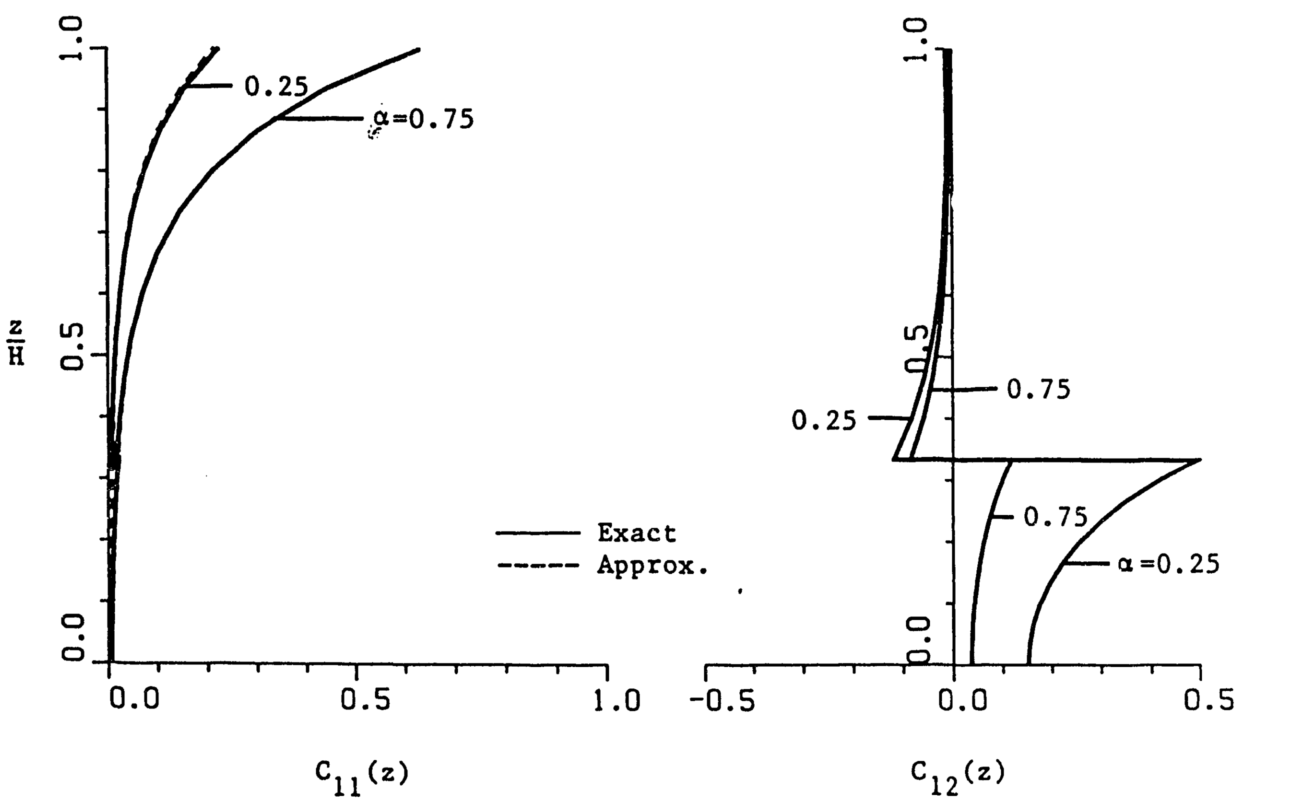

Fig. 5. Convective Pressure Exerted on Tank Wall with H/R = 3, H_2/H_1 = 2

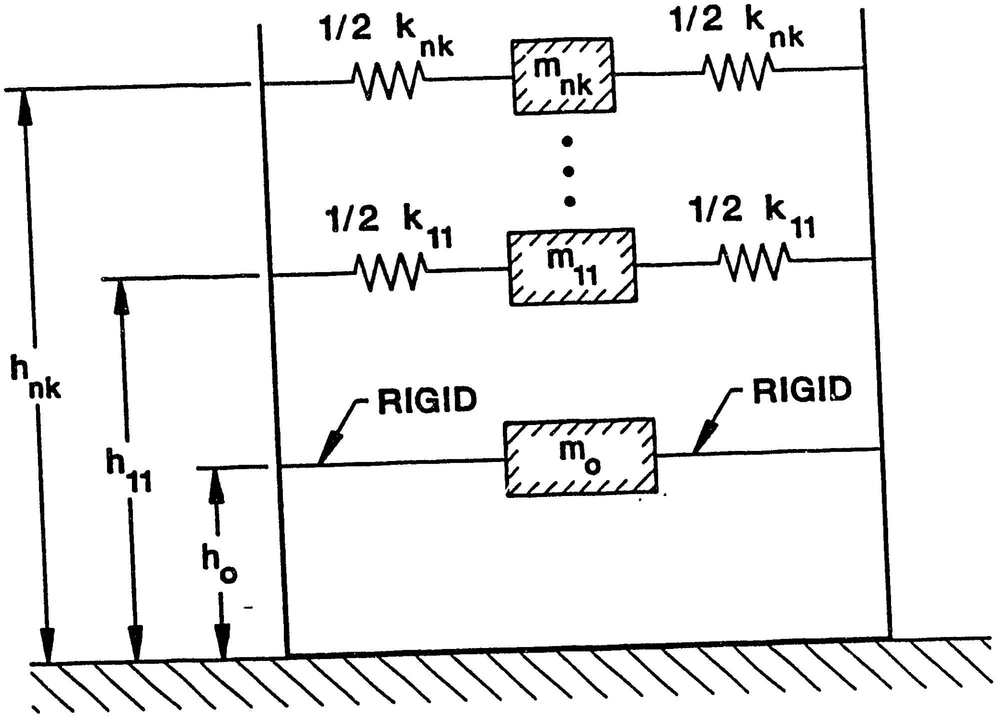

Fig. 6. Equivalent Mechanical Model

APPENDIX A

Solutions for Convective Component

The method of separation of variables is employed to solve Eqs. (11a) and (11b), and the integration constants are determined from the boundary conditions. Satisfying Eqs. (12a), (12b), (12c), (12e) and (12g), the function p_1^c takes the form

$$p_1^c(r,0,z_1,t) = \left[\sum_{n=1}^{\infty} D_n(t)\cosh\left(\lambda_n \frac{z_1}{R}\right)\frac{J_1\left(\lambda_n \frac{r}{R}\right)}{J_1(\lambda_n)}\right]\rho_1 R\cos\theta \tag{61a}$$

and p_2^c takes the form

$$p_2^c(r,\theta,z_2,t) = \left[\sum_{n=1}^{\infty}\left(E_n(t)\cosh\left(\lambda_n \frac{z_2}{R}\right) + D_n(t)\sinh\beta_{1n}\sinh\left(\lambda_n \frac{z_2}{R}\right)\right)\right.$$

$$\left.\frac{J_1\left(\lambda_n \frac{r}{R}\right)}{J_1(\lambda_n)}\right]\rho_2 R\cos\theta \tag{61b}$$

in which $D_n(t)$ and $E_n(t)$ = integration functions that can be determined by satisfying Eqs. (12d) and (12f). Substituting Eqs. (13a), (13b), (61a) and (61b) into Eq. (12d), one obtains a differential equation

$$\sinh\beta_{1n}\sinh\beta_{2n}\,\ddot{D}_n + \frac{g\lambda_n}{R}\sinh\beta_{1n}\cosh\beta_{2n}\,D_n + \cosh\beta_{2n}\,\ddot{E}_n$$

$$+ \frac{g\lambda_n}{R}\sinh\beta_{2n}\,E_n = -\frac{g\lambda_n}{R}\frac{G_n}{\Delta_n}\left(B_n\sinh\beta_{2n} + C_n\cosh\beta_{2n}\right)\ddot{x}(t) \tag{62}$$

and Substituting Eqs. (13b) and (61b) into Eq. (12f), one obtains another differential equation

$$\cosh\beta_{1n}\,\ddot{D}_n + (1-\alpha)\,\frac{g\lambda_n}{R}\,\sinh\beta_{1n}\,D_n - \alpha\ddot{E}_n = -\frac{g\lambda_n}{R}\,\frac{G_n}{\Delta_n}\,(A_n\,\sinh\beta_{1n} - \alpha C_n)\,\ddot{x}(t) \qquad (63)$$

then, Eqs. (62) and (63) can be solved for $D_n(t)$ and $E_n(t)$.

Taking the Laplace transformation on both sides of Eqs. (62) and (63) and assuming the homogeneous initial conditions for $D_n(t)$ and $E_n(t)$, one obtains two algebraic equations for determination of the Laplace transforms of $D_n(t)$ and $E_n(t)$. The required solutions for $D_n(t)$ and $E_n(t)$ are then obtained by finding the inverse Laplace transforms; then replacing $D_n(t)$ and $E_n(t)$ in Eqs. (61a) and (61b) with the results obtained, one can cast the final expressions into the form presented in Eqs. (15) and (16).

The expressions for $C^{I}_{nk}(r, z_1)$ and $C^{II}_{nk}(r,z_2)$ for k=1 and 2 are given as follows.

$$C^{I}_{n1}(r,z_1) = -\frac{G_n}{\Delta_n}\,\frac{1}{\Lambda_{n1}}\,x_1\,\cosh\left(\lambda_n\,\frac{z_1}{R}\right)\frac{J_1\left(\lambda_n\,\frac{r}{R}\right)}{J_1(\lambda_n)} \qquad (64a)$$

$$C^{I}_{n2}(r,z_1) = \frac{-G_n}{\Delta_n}\cdot\frac{1}{\Lambda_{n2}}\,y_1\,\cosh\left(\lambda_n\,\frac{z_1}{R}\right)\frac{J_1\left(\lambda_n\,\frac{r}{R}\right)}{J_1(\lambda_n)} \qquad (64b)$$

$$C^{II}_{n1}(r,z_2) = -\frac{G_n}{\Delta_n}\cdot\frac{1}{\Lambda_{n1}}\left[x_2\cosh\left(\lambda_n\,\frac{z_2}{R}\right) + x_1\,\sinh\beta_{1n}\sin\left(\lambda_n\,\frac{z_2}{R}\right)\right]$$

$$\frac{J_1\left(\lambda_n\,\frac{r}{R}\right)}{J_1(\lambda_n)} \qquad (65a)$$

$$C^{II}_{n2}(r,z_2) = -\frac{G_n}{\Delta_n}\cdot\frac{1}{\Lambda_{n2}}\left[y_2\cosh\left(\lambda_n\,\frac{z_2}{R}\right) + y_1\,\sinh\beta_{1n}\sin\left(\lambda_n\,\frac{z_2}{R}\right)\right]$$

$$\frac{J_1\left(\lambda_n\,\frac{r}{R}\right)}{J_1(\lambda_n)} \qquad (65b)$$

where

$$x_1 = \frac{\Lambda_{n1}\, q_1 - q_2}{\Lambda_{n1} - \Lambda_{n2}} \tag{66a}$$

$$y_1 = \frac{q_2 - \Lambda_{n2}\, q_1}{\Lambda_{n1} - \Lambda_{n2}} \tag{66b}$$

$$x_2 = \frac{\Lambda_{n1}\, q_3 - q_4}{\Lambda_{n1} - \Lambda_{n2}} \tag{66c}$$

and

$$y_2 = \frac{q_4 - \Lambda_{n2}\, q_3}{\Lambda_{n1} - \Lambda_{n2}} \tag{66d}$$

in which

$$q_1 = \frac{1}{\Delta_n^*}\left[A_n \tanh\beta_{1n} + \alpha B_n \frac{\tanh\beta_{2n}}{\cosh\beta_{1n}}\right] \tag{67a}$$

$$q_2 = \frac{1}{\Delta_n^*}\left[A_n \tanh\beta_{1n} \tanh\beta_{2n} - \alpha C_n \frac{\tanh\beta_{2n}}{\cosh\beta_{1n}}\right] \tag{67b}$$

$$q_3 = \frac{1}{\Delta_n^*}\left[B_n \tanh\beta_{2n} - A_n \sinh\beta_{1n} \tanh\beta_{1n} \tanh\beta_{2n}\right] + C_n \tag{67c}$$

and

$$q_4 = \frac{1}{\Delta_n^*}\left[B_n (1-\alpha)\tanh\beta_{1n} \tanh\beta_{2n} + C_n \tanh\beta_{1n} - A_n \sinh\beta_{1n} \tanh\beta_{1n}\right] \tag{67d}$$

and

$$\Delta_n^* = 1 + \alpha \tanh\beta_{1n} \tanh\beta_{2n} \tag{68}$$

The characteristic equation for the natural frequencies can be obtained from the associated eigenvalue problem of Eqs. (62) and (63). Letting $\ddot{x}(t) = 0$ and $D_n(t) = D_n e^{iwt}$, $E_n(t) = E_n e^{iwt}$ in Eqs. (62) and (63), one obtains two homogeneous equations which are expressed by

$$-\omega^2 \begin{bmatrix} \sinh\beta_{1n} \sinh\beta_{2n} & \cosh\beta_{2n} \\ \cosh\beta_{1n} & -\alpha \end{bmatrix} \begin{Bmatrix} D_n \\ E_n \end{Bmatrix}$$

$$+ \frac{g\lambda_n}{R} \begin{bmatrix} \sinh\beta_{1n} \cosh\beta_{2n} & \sinh\beta_{2n} \\ (1-\alpha)\sinh\beta_{1n} & 0 \end{bmatrix} \begin{Bmatrix} D_n \\ E_n \end{Bmatrix} = 0 \tag{69}$$

It is easy to show that the determinant of Eq. (69) is Eq. (19), the characteristic equation.

END

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