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SLOSHING RESPONSE OF LAYERED LIQUIDS IN RIGID TANKS

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ABSTRACT

The sloshing action of layered liquids in rigid cylindrical and long rectangular tanks is investigated considering both their free vibrational characteristics and their response to a horizontal component of base shaking. Special attention is given to the maximum surface displacement induced by the base motion. The analysis is formulated for systems with N superimposed layers of different thicknesses and densities, and it is illustrated by a numerical example. In addition, comprehensive numerical data are presented for two-layered and some three-layered systems which elucidate the underlying response mechanisms and the effects and relative importance of the numerous parameters involved. It is shown that for each horizontal natural mode of vibration, there are N distinct vertical modes, the frequencies of which are lower than the natural frequency of a homogeneous liquid of the same total depth. It is further shown that the maximum surface sloshing displacement of the base-excited layered system is typically larger than of the corresponding homogeneous system, and that the results for the long rectangular and the cylindrical tanks are quite similar.

TABLE OF CONTENTS

LIST OF FIGURES

Figure **Page**

LIST OF TABLES

Table **Page**

EXECUTIVE SUMMARY

The sloshing action of layered liquids in rigid cylindrical and long rectangular tanks is investigated considering both their free vibrational characteristics and their response to a horizontal component of base shaking. Special attention is given to the maximum surface displacement induced by the base motion. The analysis is formulated for systems with N superimposed layers of different thicknesses and densities, and it is illustrated by a numerical example. In addition, comprehensive numerical data are presented for two-layered and some three-layered systems which elucidate the underlying response mechanisms and the effects and relative importance of the numerous parameters involved.

The principal conclusions of the study may be summarized as follows:

1. For a liquid with N homogeneous layers, there is an infinite number of horizontal natural modes of vibration, and corresponding to each such mode, there are N distinct vertical modes. The latter modes have from zero to $N-1$ points of zero crossings, and their frequencies are lower than the corresponding frequency of a uniform liquid of the same total depth.

2. For a specified horizontal mode of vibration, the natural frequencies of a two-layered system are, respectively, higher and lower than those computed considering the two liquid layers to act independently.

3. The natural modes of the layered liquid satisfy simple orthogonality relations that are identified in the text.

4. The maximum surface sloshing displacement of a base-excited layered system is generally greater than that induced in a homogeneous system of the same total depth. The increase is significant, however, only when the densities of individual layers differ substantially. The increased response is associated with the fact that, in addition to the lateral component of shaking, the base of the top layer is subjected to a rocking motion associated with the sloshing action of the interface.

5. For large-capacity tanks subjected to earthquake-ground motions, the fundamental mode of vibration is the dominant contributor to the surface sloshing

displacements of the liquid. Furthermore, the contribution of the higher horizontal modes is typically larger than that of the higher vertical modes.

6. For the 2-layered system considered in the illustrative example, the maximum surface displacement along the tank wall was found to range from 3.96 times the maximum ground displacement when the densities of the two layers were considered to be equal, to 5.3 times the maximum ground displacement when the density of the top layer was taken as one-tenth that of the lower layer.

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SECTION 1

INTRODUCTION

Whereas the response to earthquakes of tanks containing a uniform, homogeneous liquid has been the subject of numerous studies in recent years, there is a paucity of information concerning the corresponding response of tanks containing layers of liquid of different densities. For an overview of the previous contributions, the reader is referred to the state-of-the-art report by Veletsos [1] and to the references of more recent publications by Haroun [2], Haroun and Badawi [3], Lau and Zeng [4], Malhotra et al [5] and Veletsos et al [6, 7, 8].

Current interest in the response of tanks with layered liquids is motivated by two factors: (1) Many waste storage tanks in nuclear facilities contain two or more layers of liquid or liquid-like material of different densities; and (2) recent processing for the recovery and decontamination of discharge fuel materials is typically carried out in tanks containing two-layered liquids [9].

The only known study of the sloshing response of tanks with a layered liquid is the one reported recently by Tang et al [10], who examined the free vibrational characteristics and the surface sloshing action of a two-layered liquid in a rigid, circular cylindrical tank subjected to a horizontal component of base shaking. The solutions presented, however, are based on an incorrect characterization of the pressure condition at the interface of the two liquids, and the accuracy of the reported expressions and numerical results is questionable.

The objectives of this paper are: (1) To reformulate the analysis of the problem, making use of the correct interface condition and considering the general case of a system with N homogeneous liquid layers of different thicknesses and mass densities; and (2) through comprehensive parametric studies of systems with two and three layers, to elucidate the underlying response mechanism and the effects and relative importance of the parameters involved.

In addition to circular cylindrical tanks, long rectangular tanks are examined, and the interrelationship of the responses of the two systems is identified. The response quantities investigated include the natural modes of vibration of the liquid, the associated frequencies, and the sloshing motions induced by a horizontal component of base shaking.

For all the solutions reported, the tanks are presumed to be rigid. However, inasmuch as the sloshing action of the liquid is normally associated with significantly longer periods of vibration than the dominant periods of the earthquake ground motions, based on previous analyses of tanks with a homogeneous liquid [1, 7, 8], the results are expected to be also applicable to flexible tanks that are either rigidly or flexibly supported at the base.

SECTION 2

SYSTEMS AND FUNDAMENTAL RELATIONS

2.1 Systems Considered

The systems investigated are shown in Fig. 2.1. They are rigid, vertical tanks that are filled to a height H with two or more layers of liquid of different thicknesses and densities. The tanks are either rectangular, of width $2R$ in one direction and infinite extent in the normal direction as shown in part (a) of the figure, or cylindrical, with a circular cross section of radius R as shown in part (b) of the figure, and they are presumed to be anchored to a rigid moving base. The liquids are considered to be incompressible, irrotational and inviscid, and only linear actions are examined.

The liquid layers are numbered sequentially starting with 1 at the lowermost or bottom layer and terminating with N at the uppermost or top layer. The mass density and height of the j th layer are denoted by ρ_j and H_j, respectively. The values of ρ_j are considered to decrease with increasing j. Points within the j th layer of the long rectangular system are defined by the local Cartesian coordinates, x and z_j, shown in part (a) of Fig. 2.1, and those for the cylindrical system are defined by the cylindrical coordinates, r, θ, z_j, shown in part (b) of the figure.

The ground motion is considered to be horizontal and uniform and to be directed along the x- or $\theta = 0$ coordinate axis. The acceleration of the ground motion at any time, t, is denoted by $\ddot{x}_g(t)$, and the corresponding velocity and displacement are denoted by $\dot{x}_g(t)$ and $x_g(t)$, respectively.

2.2 Fundamental Relations

The flow field in the j th layer must satisfy Laplace's Eq.,

$$\nabla^2 \phi_j = 0 \tag{1}$$

in which ϕ_j = a velocity potential function of time and the position coordinates, and the operator ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \tag{2}$$

in the rectangular coordinate system, and by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z_j^2} \tag{3}$$

in the cylindrical coordinate system. If v_{jn} is the instantaneous value of the velocity of an arbitrary particle in the j th layer in the direction of a generalized n-coordinate, then

$$v_{jn} = -\frac{\partial \phi_j}{\partial n} \tag{4}$$

and the corresponding hydrodynamic pressure is

$$p_j = \rho_j \frac{\partial \phi_j}{\partial t} \tag{5}$$

The solution of Eq. (1) must satisfy the following boundary conditions:

1. At the tank base, the vertical component for the liquid velocity must vanish; accordingly,

$$\left(\frac{\partial \phi_1}{\partial z_1}\right)_{z_1=0} = 0 \tag{6}$$

2. Along the tank wall, the radial or normal velocity component of both the tank and liquid must equal the corresponding component of the ground motion. For the long rectangular system, this requires that

$$\left(-\frac{\partial \phi_j}{\partial x}\right)_{x=\pm R} = \dot{x}_g(t) \tag{7}$$

whereas for the cylindrical system, it requires that

$$\left(-\frac{\partial \phi_j}{\partial r}\right)_{r=R} = \dot{x}_g(t)\, cos\theta \tag{8}$$

3. At the free liquid surface, the following linearized pressure boundary condition must be satisfied

$$\left(\dot{\phi}_N - g\, d_N\right)_{z_N=H_N} = 0 \tag{9}$$

where d_N represents the vertical surface displacement, a dot superscript denotes differentiation with respect to time, and g = the gravitational acceleration. The origin of this equation is identified under item 4.

4. At the interface of a pair of layers, the vertical velocity of the liquid must be continuous; accordingly,

$$\left(\frac{\partial \phi_j}{\partial z_j}\right)_{z_j=H_j} = \left(\frac{\partial \phi_{j+1}}{\partial z_{j+1}}\right)_{z_{j+1}=0} \tag{10}$$

Additionally, the total pressure (hydrodynamic plus the increment due to the vertical displacement at the interface) must be continuous. If d_j represents the instantaneous vertical displacement of an arbitrary point at the upper interface of the j th layer measured from the position of static equilibrium, then assuming that the displacements are small and that the inertia of the interfacial wave is negligible, the pressure condition may be written in the form indicated in Lamb [11], as

$$\rho_j \left(\dot{\phi}_j\right)_{z_j=H_j} - \rho_j \, g \, d_j = \rho_{j+1} \left(\dot{\phi}_{j+1}\right)_{z_{j+1}=0} - \rho_{j+1} \, g \, d_j \tag{11}$$

Eq. (9) may be deduced from Eq. (11) merely by letting $j = N$ and $\rho_{j+1} = 0$.

It is clear from Eq. (11) that while the total pressures are continuous, the hydrodynamic components are discontinuous at the interfaces of layers of different densities. In the studies of Tang et al [10], the contribution of the pressure increment due to the interfacial displacement was not considered, and the hydrodynamic component of the pressure was taken as continuous. The consequences of this approximation are identified in later sections.

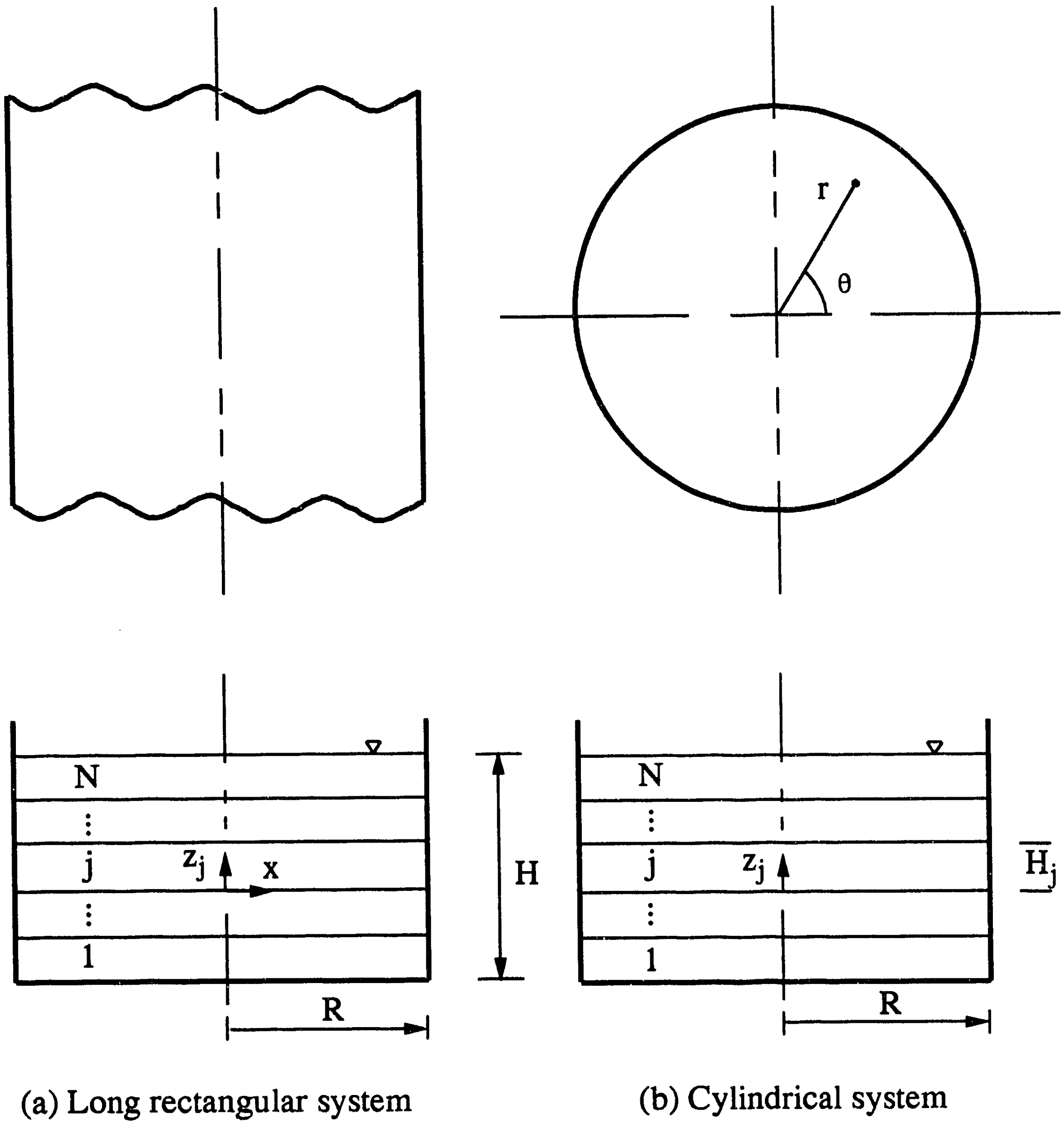

Figure 2.1 Systems considered

SECTION 3

METHOD OF ANALYSIS

The solution of Eq. (1) is obtained in a manner analogous to that employed by Bauer [12] and Abramson [13] in their studies of tanks containing a homogeneous liquid, by the superposition of two component solutions as

$$\phi_j = \chi_j + \psi_j \tag{12}$$

In this expression, χ_j = a velocity potential function associated with the rigid body motion of the tank walls, and ψ_j = a corresponding function providing for the relative motion of the contained liquid and the tank walls. The function χ_j represents the solution obtained when both the upper and lower surfaces of the j th liquid layer are rigidly capped, whereas ψ_j represents a corrective solution which accounts for the difference between the actual and fully constrained conditions at these boundaries. It is important to realize that these component solutions are different from the so-called impulsive and convective solutions used by Housner [14], Veletsos et al [1, 6, 15] and Haroun and Housner [16] in their studies of tanks with homogeneous liquids.

3.1 Solution for χ_j

For the long rectangular system,

$$\frac{\partial \chi_j}{\partial x} = -\dot{x}_g(t) \tag{13}$$

whereas for the cylindrical system,

$$\frac{\partial \chi_j}{\partial r} = -\dot{x}_g(t)\, cos\theta \tag{14}$$

On integrating these expressions, one obtains

$$\chi_j = -\dot{x}_g(t)\, x \tag{15}$$

for the rectangular system, and

$$\chi_j = -\dot{x}_g(t)\, r\, cos\theta \tag{16}$$

for the cylindrical system.

It is observed that the functions χ_j are independent of the physical properties of the liquid layers, and will henceforth be denoted by χ. Furthermore, considering that Eqs. (15) and (16) are independent of the vertical coordinates z_j, it follows from Eqs. (4) and (5) that χ is associated with no vertical velocities or displacements but represents simply a finite-sized pressure field which increases linearly in the horizontal direction.

3.2 Solution for ψ_j

The function ψ_j must satisfy Laplaces's Eq. (1), the solution of which may be obtained by the method of separation of variables as follows. For the long rectangular system,

$$\psi_j = Z(z_j)\,X(x)\,T(t) \tag{17}$$

and for the cylindrical system,

$$\psi_j = Z(z_j)\,\tilde{R}(r)\,T(t)\,cos\theta \tag{18}$$

in which X, $\tilde{R}$, Z and T are functions of x, r, z_j and t, respectively.

Inasmuch as the boundary conditions along the walls are satisfied exactly by the potential function χ, the corresponding conditions for ψ_j are zero at these boundaries. On substituting Eqs. (17) and (18) into Laplace's Eq. and making use of the homogeneous boundary conditions along the walls, the following expressions are obtained for ψ_j. For the long rectangular system,

$$\psi_j = \sum_{m=1}^{\infty} \left[P_{m,j}(t)\,cosh\lambda_m\eta_j + Q_{m,j}(t)\,sinh\lambda_m\eta_j\right]\,sin\lambda_m\xi \tag{19}$$

in which $\xi = x/R$, $\eta_j = z_j/R$,

$$\lambda_m = (2m-1)\,\frac{\pi}{2} \tag{20}$$

and $P_{m,j}(t)$ and $Q_{m,j}(t)$ are time-dependent coefficients that must be determined from the conditions at the lower and upper boundaries of the j th layer. These boundaries will henceforth be referred to as the $(j-1)$ th and j th interfaces, respectively. The corresponding expression for the cylindrical tank is

$$\psi_j = \sum_{m=1}^{\infty} \left[P_{m,j}(t)\,cosh\lambda_m\eta_j + Q_{m,j}(t)\,sinh\lambda_m\eta_j\right]\,J_1\left(\lambda_m\xi\right)\,cos\theta \tag{21}$$

in which ξ now stands for the normalized radial distance, r/R; J_1 = the Bessel function of the first kind and first order; and λ_m = the m th zero of the first derivative of J_1, i.e., the m th root of $J_1'(\lambda) = 0$. The first three of these roots are

$$\lambda_1 = 1.841 \qquad \lambda_2 = 5.331 \qquad \lambda_3 = 8.536 \tag{22}$$

Note that the meaning of ξ and the values of λ_m, $P_{m,j}(t)$ and $Q_{m,j}(t)$ are different in Eqs. (19) and (21).

Before proceeding to the formulation of the equations of motion, it should be noted that if Eq. (12) is substituted into Eq. (11) and the resulting terms are rearranged and normalized with respect to ρ_1, the pressure condition for the j th interface may be expressed in terms of the potential functions ψ_j and χ as

$$\frac{\rho_j}{\rho_1}\,\dot{\psi}_j - \frac{\rho_{j+1}}{\rho_1}\,\dot{\psi}_{j+1} - \left[\frac{\rho_j}{\rho_1} - \frac{\rho_{j+1}}{\rho_1}\right] g\, d_j = -\left[\frac{\rho_j}{\rho_1} - \frac{\rho_{j+1}}{\rho_1}\right] \dot{\chi} \tag{23}$$

3.3 Equations of Motion for System

In formulating the equations of motion for the multi-layered system, it is desirable to use as generalized coordinates the modal values of the vertical displacements at the junctions of the tank wall and the interfaces of the liquid layers, rather than the quantities $P_{m,j}$ and $Q_{m,j}$. To this end, let $D_{m,j}(t)$ be the displacement at the intersection of the j th liquid interface and the wall when the system is vibrating in its m th horizontal mode of vibration. For the cylindrical tanks, for which these displacements are functions of the circumferential coordinate θ, $D_{m,j}(t)$ refers to the value at $\theta = 0$. The sloshing displacement $d_j(\xi, t)$ for an arbitrary point of the j th interface may then be expressed as follows. For the long rectangular system,

$$d_j(\xi, t) = \sum_{m=1}^{\infty} D_{m,j}(t)\,\frac{sin\lambda_m\xi}{sin\lambda_m} \tag{24}$$

and for the cylindrical system,

$$d_j(\xi, \theta, t) = \sum_{m=1}^{\infty} D_{m,j}(t)\,\frac{J_1(\lambda_m\xi)}{J_1(\lambda_m)}\,cos\theta \tag{25}$$

It should be recalled that the values of λ_m for the rectangular system are defined by Eq. (20), whereas those for the cylindrical system are defined by the roots of $J_1'(\lambda)$ = 0.

In order to relate $P_{m,j}(t)$ and $Q_{m,j}(t)$ to $D_{m,j}(t)$, the vertical velocities of the liquid at the j th and $(j-1)$ th interfaces evaluated from Eq. (4) are equated to those obtained by differentiating with respect to time the interfacial displacements defined by Eqs. (24) and (25). On solving the resulting equations and back substituting, the potential function ψ_j may be rewritten as

$$\psi_j = -\sum_{m=1}^{\infty} \frac{R}{\lambda_m}\left[\frac{\dot{D}_{m,j}(t)\,cosh\lambda_m\eta_j - \dot{D}_{m,j-1}(t)\,cosh\lambda_m(\alpha_j - \eta_j)}{sinh\lambda_m\alpha_j}\right]\frac{sin\lambda_m\xi}{sin\lambda_m} \tag{26}$$

for the long rectangular system, and as

$$\psi_j = -\sum_{m=1}^{\infty} \frac{R}{\lambda_m} \left[\frac{\dot{D}_{m,j}(t)\, cosh\lambda_m \eta_j - \dot{D}_{m,j-1}(t)\, cosh\lambda_m(\alpha_j - \eta_j)}{sinh\lambda_m \alpha_j} \right] \frac{J_1(\lambda_m \xi)}{J_1(\lambda_m)} cos\theta \tag{27}$$

for the cylindrical system. In these expressions, $\alpha_j = H_j/R$.

The equation of motion for the j th interface of the rectangular system may now be derived from Eq. (23) by substituting the expressions for χ, d_j and ψ_j defined by Eqs. (15), (24) and (26), respectively. The left-hand member of the expression obtained in this manner involves an infinite sum of horizontal sinusoidal modes. On multiplying both sides of this expression by $sin\lambda_m\xi$ and integrating from 0 to 1, all but one of the terms on the left side cancel because of the orthogonality of the trigonometric functions involved, and the equation reduces to

$$A_{j,j-1}\,\ddot{D}_{m,j-1} + A_{j,j}\,\ddot{D}_{m,j} + A_{j,j+1}\,\ddot{D}_{m,j+1} + \frac{\lambda_m g}{R}\,B_{j,j}\,D_{m,j} = -\epsilon_m\,\lambda_m\,c_j\,\ddot{x}_g(t) \tag{28}$$

where

$$A_{j,j} = \frac{\rho_j}{\rho_1}\, coth\lambda_m\alpha_j + \frac{\rho_{j+1}}{\rho_1}\, coth\lambda_m\alpha_{j+1} \tag{29}$$

$$A_{j,j-1} = -\frac{\rho_j}{\rho_1}\frac{1}{sinh\lambda_m\alpha_j} \tag{30}$$

$$A_{j,j+1} = -\frac{\rho_{j+1}}{\rho_1}\frac{1}{sinh\lambda_m\alpha_{j+1}} \tag{31}$$

$$B_{j,j} = c_j = \frac{\rho_j}{\rho_1} - \frac{\rho_{j+1}}{\rho_1} \tag{32}$$

and ϵ_m is a dimensionless factor defined by

$$\epsilon_m = \frac{2}{\lambda_m^2} \tag{33}$$

It is shown later that the factor ϵ_m appears in the expression for the surface sloshing motion of a homogeneous liquid, and to highlight its meaning, is kept separately from λ_m. Note that $D_{m,0} = 0$; hence, both $D_{m,j-1}$ and $\ddot{D}_{m,j-1}$ in Eq. (28) vanish for $j =$ 1.

The equation of motion for the j th interface of the cylindrical system is obtained similarly by substituting Eqs. (16), (25) and (27) into Eq. (23). The two sides of the resulting expression are then multiplied by $\xi\, J_1(\lambda_m\xi)\, d\xi$ and integrated from 0 to 1. Because of the orthogonality of the Bessel functions, the infinite summation of terms again reduces to Eq. (28) with $A_{j,j-1}$, $A_{j,j}$, $A_{j,j+1}$, $B_{j,j}$ and c_j defined, as before, by

Eqs. (29) through (32), except that the values of λ_m are different in the two cases. Additionally, the factor ϵ_m for the cylindrical system is given by

$$\epsilon_m = \frac{2}{\lambda_m^2 - 1} \tag{34}$$

rather than by Eq. (33).

The complete set of equations for the multilayered system is obtained by repeated application of Eq. (28) to all interfaces. The resulting set of equations may be written as

$$[A]\,\{\ddot{D}_m\} + \frac{\lambda_m\, g}{R}\,[B]\,\{D_m\} = -\epsilon_m\,\lambda_m\,\{c\}\,\ddot{x}_g(t) \tag{35}$$

where $\{D_m\}$ and $\{c\}$ are vectors of size N, the j th elements of which are $D_{m,j}$ and c_j, respectively; $[A]$ is a tri-diagonal, symmetric matrix of size $N \times N$, for which the elements of the j th row are given by Eqs. (29), (30) and (31); $[B]$ is a diagonal matrix of the same size, with its j th element given by Eq. (32); and ϵ_m is defined by Eq. (33) for the long rectangular system and by Eq. (34) for the cylindrical system.

SECTION 4

FREE VIBRATION

The equations for free vibration are deduced from Eq. (35) by setting its right-hand member equal to zero. The solution of these equations is obtained in the usual manner by letting

$$\{D_m(t)\} = \{\hat{D}_m\}\, e^{i\omega_m t} \tag{36}$$

and solving the resulting characteristic value problem,

$$[B]\,\{\hat{D}_m\} = \frac{\omega_m^2\, R}{\lambda_m\, g}\,[A]\,\{\hat{D}_m\} \tag{37}$$

in which $i = \sqrt{-1}$, and ω_m = the circular frequency associated with the m th horizontal mode of vibration. For the long rectangular system, the latter mode is defined by the function $sin\lambda_m\xi$, whereas for the cylindrical system, it is defined by the function $J_1(\lambda_m\xi)$.

It is clear from Eq. (37) that, for each horizontal mode of vibration, there exist N vertical modes, each associated with a distinct frequency. This fundamental fact was not revealed in the solutions presented by Tang et al [10], which led to a single frequency and a single vertical mode of vibration for each value of m.

The n th circular natural frequency of the system for the m th horizontal mode of vibration is denoted by ω_{mn}, the corresponding vector of interfacial displacement amplitudes is denoted by $\{\hat{D}_{mn}\}$, and the j th element of the latter vector is denoted by $\hat{D}_{mn,j}$. The ordering of these frequencies and modes is identified later. The characteristic vectors are real-valued and satisfy the orthogonality relations

$$\{\hat{D}_{mr}\}^T\,[A]\,\{\hat{D}_{ms}\} \;= 0 \tag{38}$$

and

$$\{\hat{D}_{mr}\}^T\,[B]\,\{\hat{D}_{ms}\} \;= 0 \tag{39}$$

for $r \neq s$. Furthermore, both $[A]$ and $[B]$ can be shown to be positive definite, ensuring that all natural frequencies are real and positive.

4.1 Two-Layered System

For the special case of a two-layered system, for which the matrices $[A]$ and $[B]$ in Eq. (37) are of size 2 × 2, the resulting frequency equation, after multiplying through by $tanh\,\lambda_m\alpha_1\,tanh\,\lambda_m\alpha_2$, becomes

$$\left(1+\frac{\rho_2}{\rho_1}\,tanh\lambda_m\alpha_1\,tanh\lambda_m\alpha_2\right)\omega_m^4-(tanh\lambda_m\alpha_1+tanh\lambda_m\alpha_2)\,\frac{\lambda_m g}{R}\,\omega_m^2$$
$$+\left(1-\frac{\rho_2}{\rho_1}\right)\left(\frac{\lambda_m g}{R}\right)^2 tanh\lambda_m\alpha_1\,tanh\lambda_m\alpha_2=0 \qquad (40)$$

With the natural frequencies of the system, ω_{m1} and ω_{m2}, determined from this equation, the ratio of the interfacial to the surface modal displacement amplitudes for the (mn)th mode of vibration is determined from Eq. (37) to be

$$\frac{\hat{D}_{mn,1}}{\hat{D}_{mn,2}}=cosh\lambda_m\alpha_2-\frac{\lambda_m g}{\omega_{mn}^2 R}\,sinh\lambda_m\alpha_2 \qquad (41)$$

Finally, the orthogonality relation defined by Eq. (39) can be written as

$$\left(1-\frac{\rho_2}{\rho_1}\right)\hat{D}_{m1,1}\,\hat{D}_{m2,1}+\frac{\rho_2}{\rho_1}\,\hat{D}_{m1,2}\,\hat{D}_{m2,2}=0 \qquad (42)$$

Provided one uses the appropriate values of λ_m as previously indicated, Eqs. (40), (41) and (42) are applicable to both the long rectangular and the cylindrical systems. Incidentally, with the appropriate reinterpretation of the meaning of the various symbols, Eqs. (40) and (41) can be shown to be identical to those presented by Lamb [11] for the sloshing frequencies and the associated modal ratios of two superposed liquids flowing in a long rectangular channel.

For a homogeneous liquid with $\rho_2/\rho_1 = 1$, on neglecting the trivial solution of zero frequency, Eq. (40) yields the well known expression (e.g., Reference 1) for the mth circular natural frequency of sloshing motion,

$$\omega_m = C_m\sqrt{\frac{\lambda_m g}{R}} \qquad (43)$$

in which

$$C_m=\sqrt{tanh\left(\frac{\lambda_m H}{R}\right)} \qquad (44)$$

Furthermore, for the limiting case of $\rho_2/\rho_1 = 0$, which corresponds either to a system without the upper layer or to one with a very heavy, practically immobile lower layer, the two frequencies reduce, as they should, to those obtained from Eq. (43) for homogeneous liquids with depths H_1 and H_2, respectively.

4.2 Numerical Solutions for Sloshing Frequencies and Modes

The circular natural frequency corresponding to the m th horizontal and n th vertical mode of vibration may conveniently be expressed in a generalized form of Eq. (43) as

$$\omega_{mn} = C_{mn} \sqrt{\frac{\lambda_m g}{R}} \tag{45}$$

in which C_{mn} is a dimensionless factor that depends on the tank shape and slenderness, H/R, the number, relative thicknesses and relative densities of the liquid layers, and, of course, on the order of the frequency or mode under consideration. As already indicated, the values of λ_m in this expression are defined by Eq. (20) for the long rectangular system, and by Eq. (22) for the cylindrical system.

4.2.1 Two-Layered Systems. The frequency coefficients C_{11} and C_{12} for two-layered liquids in long rectangular tanks are presented in Fig. 4.1, and those for the corresponding cylindrical systems are shown in Fig. 4.2. The results are plotted as a function of the slenderness ratio, H/R, for two values of the layer thickness ratio, H_2/H_1, and several values of the density ratio, ρ_2/ρ_1. These coefficients and the associated natural frequencies and modes of vibration are numbered in reverse order, starting with $n = 1$ for the highest frequency and terminating with $n = N = 2$ for the lowest frequency. The rationale for this numbering is that the modes corresponding to the lower natural frequencies are associated with a higher order of waviness (larger number of points of zero crossings) in the vertical direction. This matter is examined further later in this section. It is observed that both the frequency coefficients and the associated natural frequencies for the cylindrical tanks are larger than those for the corresponding rectangular tanks; however, the differences are not significant, and the general trends of the results for the two systems are quite similar. Incidentally, the corresponding plots for the second horizontal mode of vibration, $m = 2$, also exhibit the same general trends and are not shown.

The uppermost curves in Figs. 4.1 and 4.2 are for a homogeneous liquid with a depth H equal to the total depth of the layered system. It is noteworthy that both frequency coefficients for the layered system are smaller than that for the associated homogeneous system. The effect of the heavier bottom layer is to decrease the effective total depth of the layered system and, as would be expected from Eqs. (43) and (44), this reduction leads to a corresponding reduction in the values of the frequency coefficient and of the associated natural frequency.

The interrelationship of the natural frequencies of the layered and the homogeneous systems can more clearly be seen in Fig. 4.3, in which the frequencies ω_{11} and ω_{12} for the cylindrical systems examined in Fig. 4.2 are plotted normalized with respect to ω_1, the fundamental natural frequency for a homogeneous liquid of the same total depth. It is observed that the results, particularly those corresponding to the lower values of H/R, are substantially less than unity.

The dotted curves in Figs. 4.1 and 4.2, which refer to systems of $\rho_2/\rho_1 = 0$, also represent the frequency coefficients for homogeneous liquids with depths H_1 and H_2 when they are considered to act independently. For the systems with equal layer thicknesses considered in the left-hand plots, there is naturally a single such curve, whereas for the systems with unequal depths considered in the right-hand plots, there are two distinct curves. Note that the highest natural frequency of the layered system is higher than the higher of these curves, whereas the lowest frequency is lower than the lower curve. This result is consistent with the well known interrelationship of the natural frequencies of systems having one and two degrees of freedom. If a single-degree-of-freedom system with a natural frequency f_1 is augmented by the addition of another such system, it is well known that the natural frequencies of the resulting two-degree-of-freedom system lie on either side of f_1. Since the systems in the right-hand plots of Figs. 4.1 and 4.2 may be formed either from the lower layer by the addition of an upper layer, or from the upper layer by the insertion of a lower layer, their natural frequencies must lie on either side of the pair of dotted curves, and there will be no frequencies in the region between.

Further insight into the free-vibrational characteristics of the two-layered systems may be gained from the natural modes of vibration $\{\hat{D}_{mn}\}$ shown in Figs. 4.4 and 4.5. The results displayed in these figures are for cylindrical systems with values of $H/R = 0.5$ and 2, respectively. Two values of H_2/H_1 and several values of ρ_2/ρ_1 are considered in each case. The modes on the left correspond to the first or higher of the two natural frequencies and are normalized with respect to the free-surface displacement, whereas those on the right correspond to the second or lower natural frequency and are normalized with respect to the interfacial displacement. Note that the first or fundamental mode is associated with no zero crossings, while the second mode is associated with a single such crossing. These modes naturally satisfy both the orthogonality relation defined by Eq. (42) and the somewhat more involved relation defined by Eq. (38). For a multi-layered system, the n th vertical mode of vibration is associated with $n-1$ zero crossings.

It is noteworthy that, for the fundamental mode of vibration, the displacement amplitude at the interface of the two layers for the layered system is lower than that for the homogeneous system, the difference increasing with decreasing values of ρ_2/ρ_1. This result confirms the earlier statement to the effect that the larger density of the lower layer decreases the effective total depth of the layered system leading to a reduction in frequency.

The very low frequency values of the second natural modes may also be explained by the location of the sections of zero modal amplitudes (points of zero crossings) near the top. Since the vertical motion of the liquid is zero at these sections, the natural frequency of the system for this mode must be equal to that of a homogeneous liquid with a depth equal to the distance from the free surface to the section of zero amplitude. As an illustration, it is noted that for the cylindrical system with values of $H/R = 0.5$, $H_2/H_1 = 1$ and $\rho_2/\rho_1 = 0.5$ considered in Fig. 4.4, the section of zero crossing for the second mode of vibration is located at a distance $0.140\,H$ from the top. This leads to an effective depth-to-radius ratio for the homogeneous liquid of 0.070. If this ratio is substituted into Eq. (44), the value of the resulting frequency coefficient turns out to be $C_1 = 0.358$, which is precisely the value of C_{12} reported in Fig. 4.2.

In concluding this section on two-layered systems, it should be noted that the frequency coefficients for the systems with $H_2/H_1 = 2$ considered in the right-hand plots of Figs. 4.1 and 4.2 also apply to systems with $H_1/H_2 = 2$. This follows from Eq. (40), which shows that interchanging the dimensionless thicknesses α_1 and α_2 does not alter the equation. However, the natural modes are different in the two cases, as may well be appreciated from Eq. (41).

4.2.2 Three-Layered Systems. As an illustration of the free vibrational characteristics of systems with more than two layers, in Fig. 4.6 are shown the natural frequency coefficients for the fundamental horizontal mode of vibration, $m = 1$, of a cylindrical system with three layers of identical depths. The mass densities of the layers are presumed to increase from top to bottom in proportion to 1:2:3, and a range of H/R values is considered. Also shown are the natural modes of the system for the special case of $H/R = 1$, with each mode normalized to a unit maximum amplitude. The dashed curves in this figure represent the corresponding results for a homogeneous system with a depth equal to the total depth of the layered system.

It is observed that all three frequencies are lower than that of the associated homogeneous system, that the highest frequency is associated with a vertical mode of vibration which has no zero crossing and is similar to that of the associated homogeneous system, whereas the modes of the next two lower frequencies have one and two zero crossings, respectively.

In Figs. 4.4 through 4.6, the modal displacement ordinates for sections between the liquid interfaces have been evaluated by substituting Eq. (36) for the mode under consideration into Eq. (27), differentiating the resulting expression with respect to η_j and integrating the resulting modal velocity with respect to time.

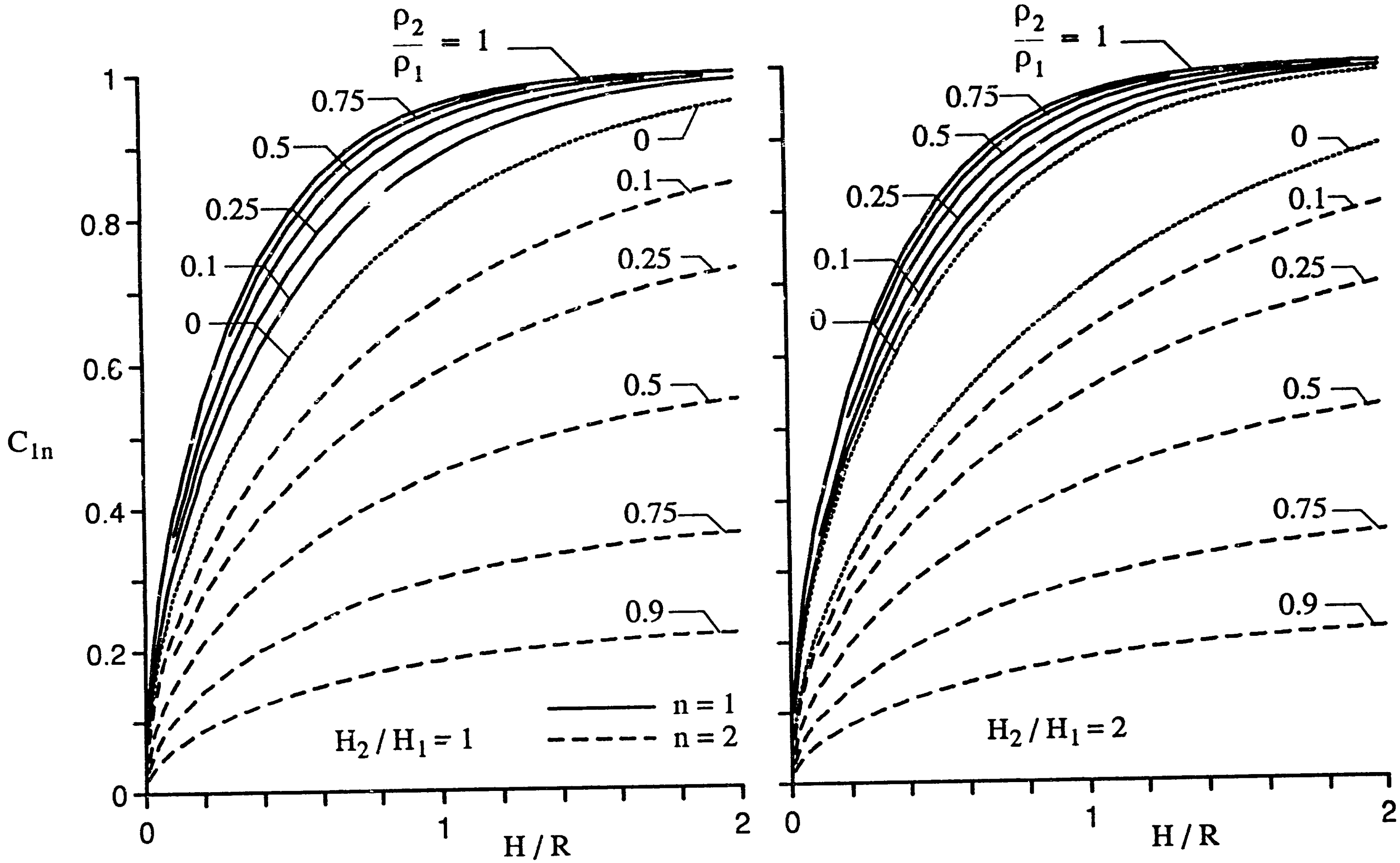

Figure 4.1 Frequency coefficients C_{1n} for fundamental horizontal mode of vibration of two-layered liquids in long rectangular tanks

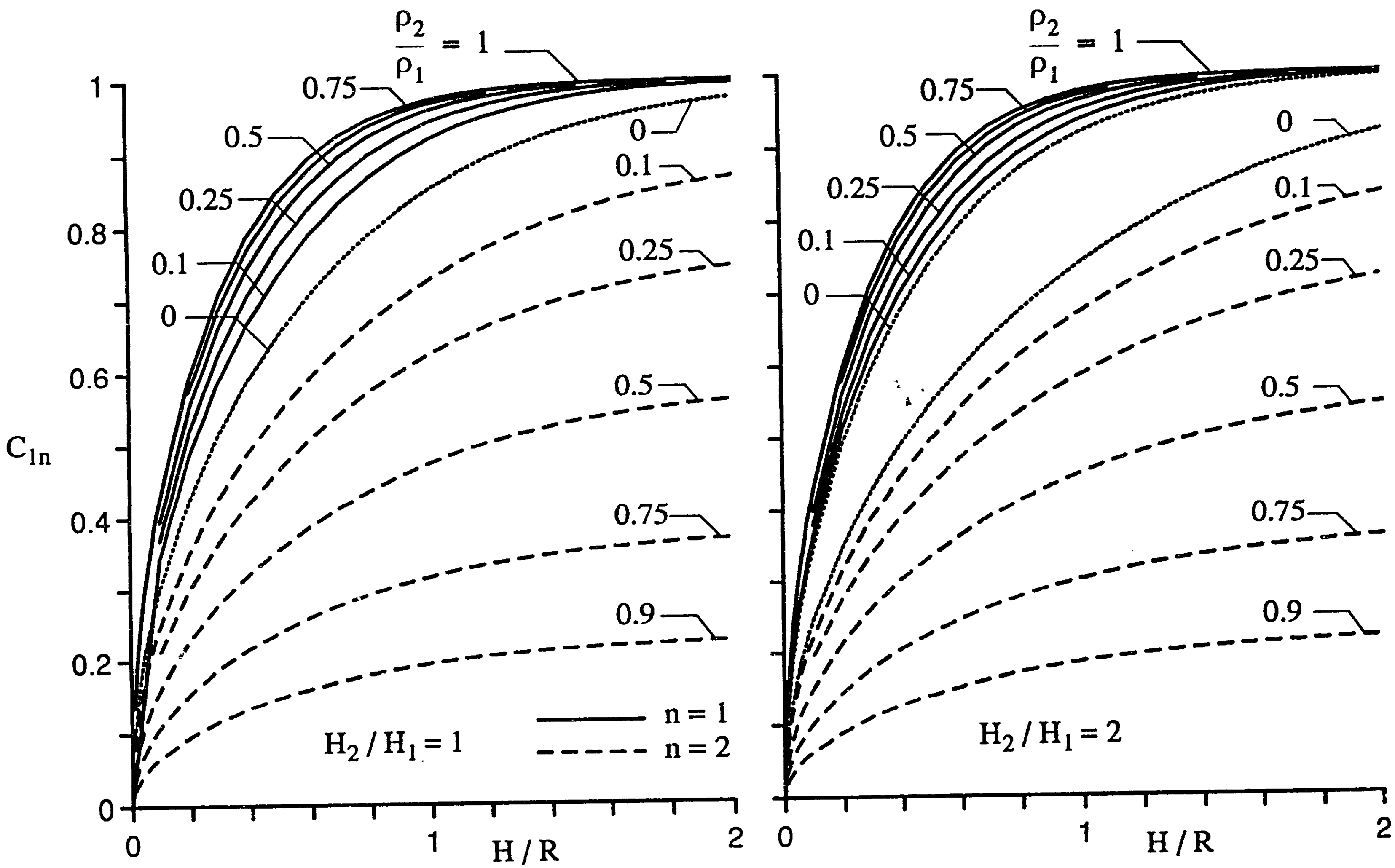

Figure 4.2 Frequency coefficients C_{1n} for fundamental horizontal mode of vibration of two-layered liquids in cylindrical tanks

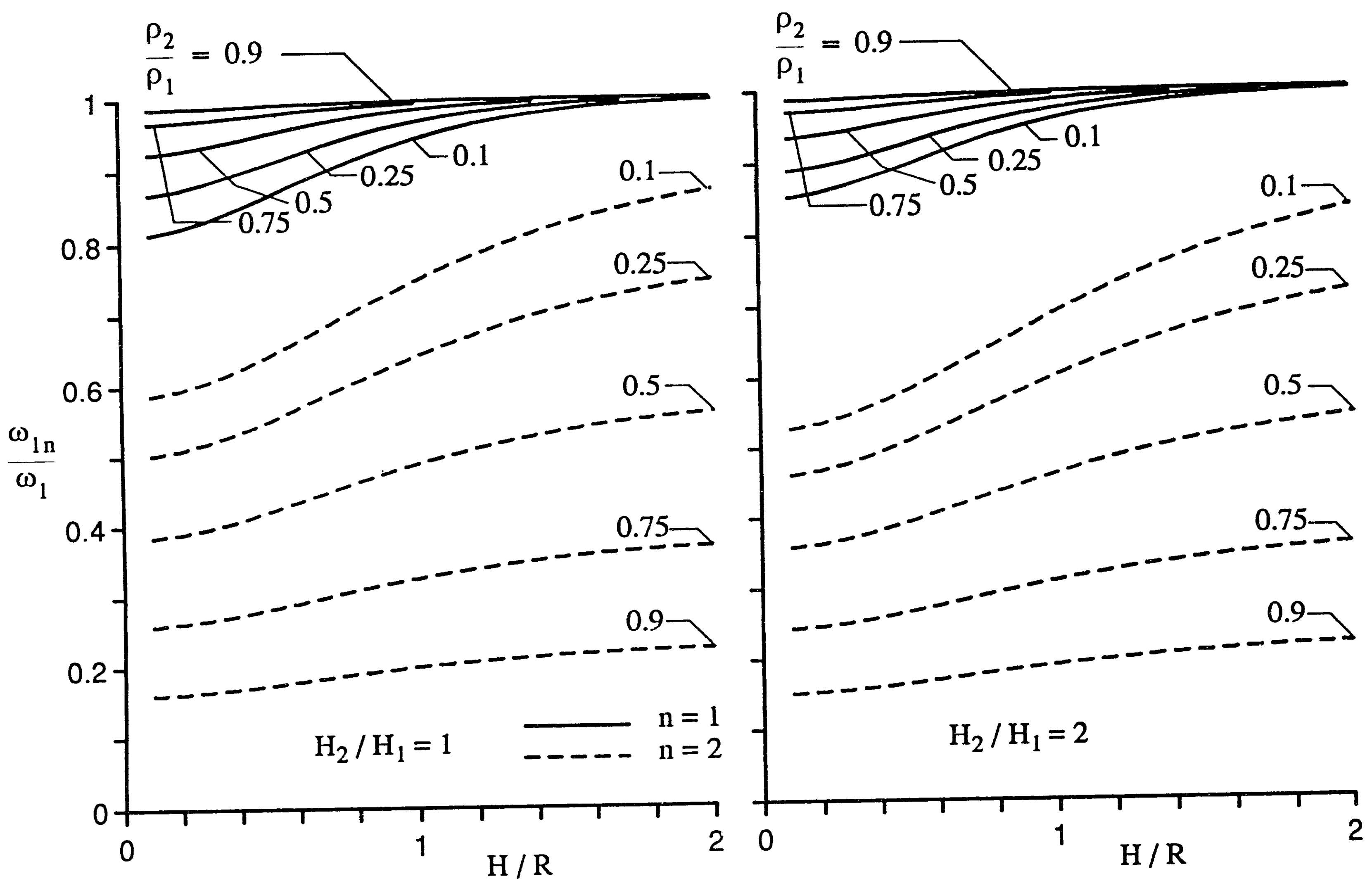

Figure 4.3 Ratio of natural frequencies of two-layered and homogeneous liquids in cylindrical tanks

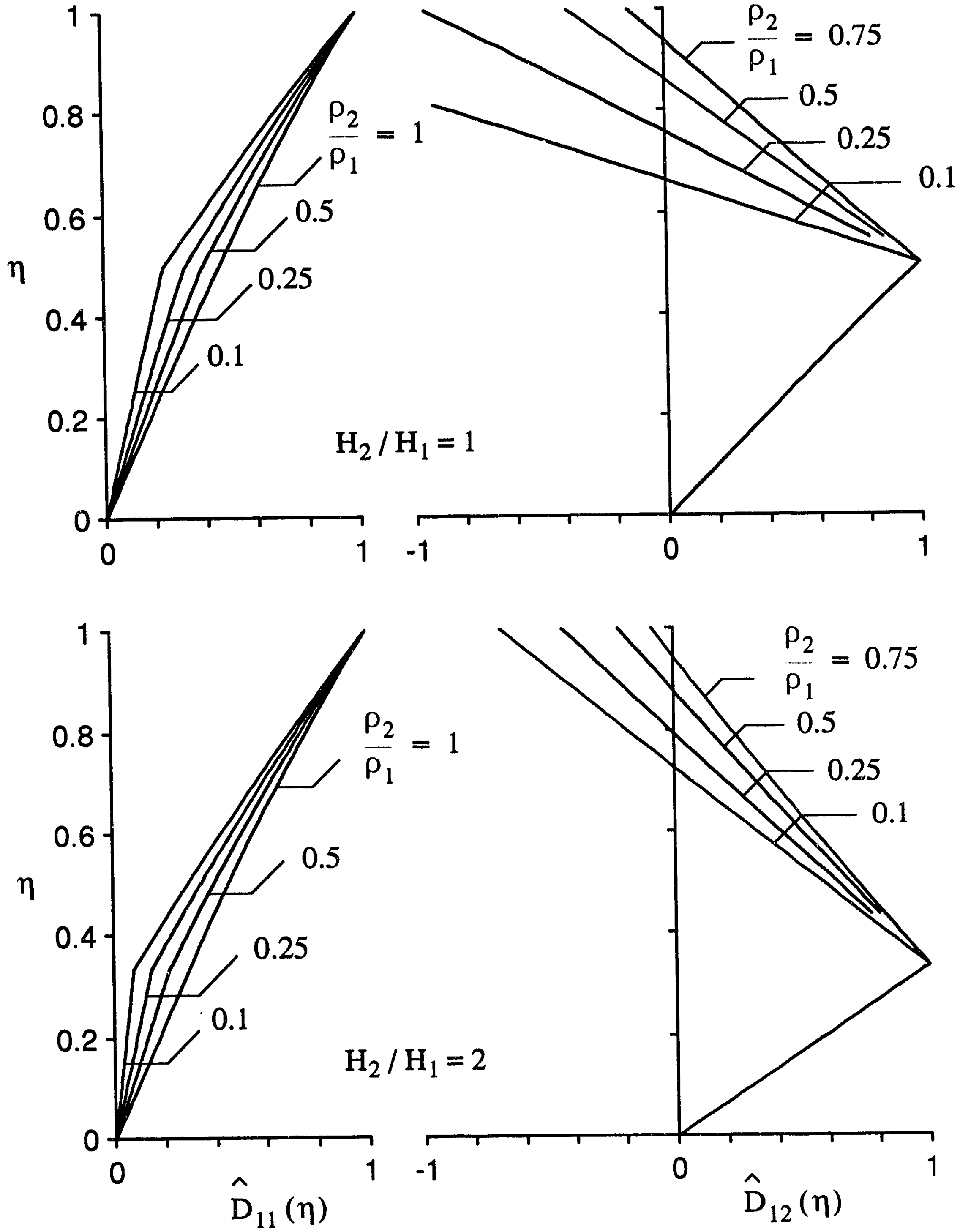

Figure 4.4 Vertical displacement configurations for fundamental horizontal mode of vibration of two-layered liquids in cylindrical tanks with H/R = 0.5

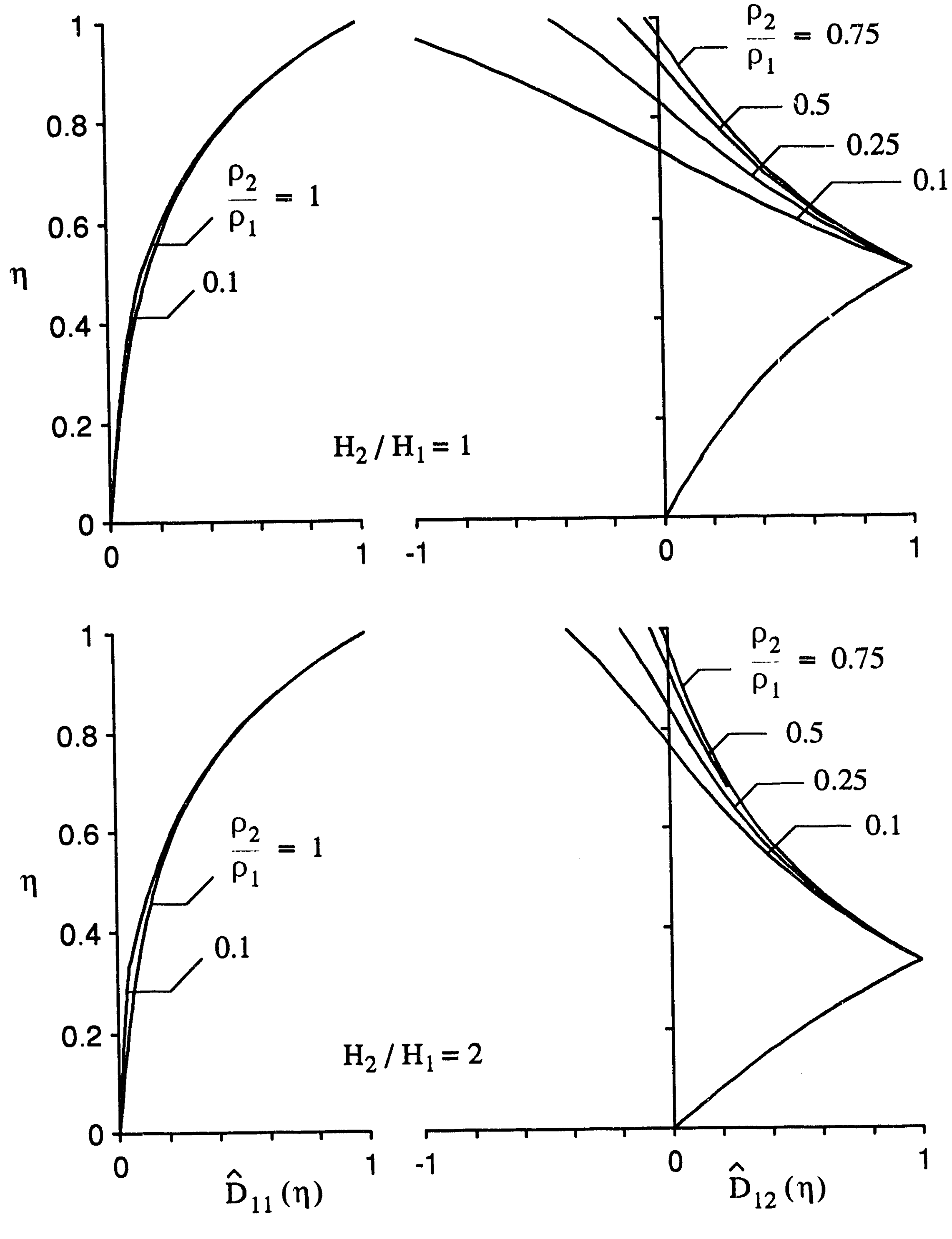

Figure 4.5 Vertical displacement configurations for fundamental horizontal mode of vibration of two-layered liquids in cylindrical tanks with $H/R = 2.0$

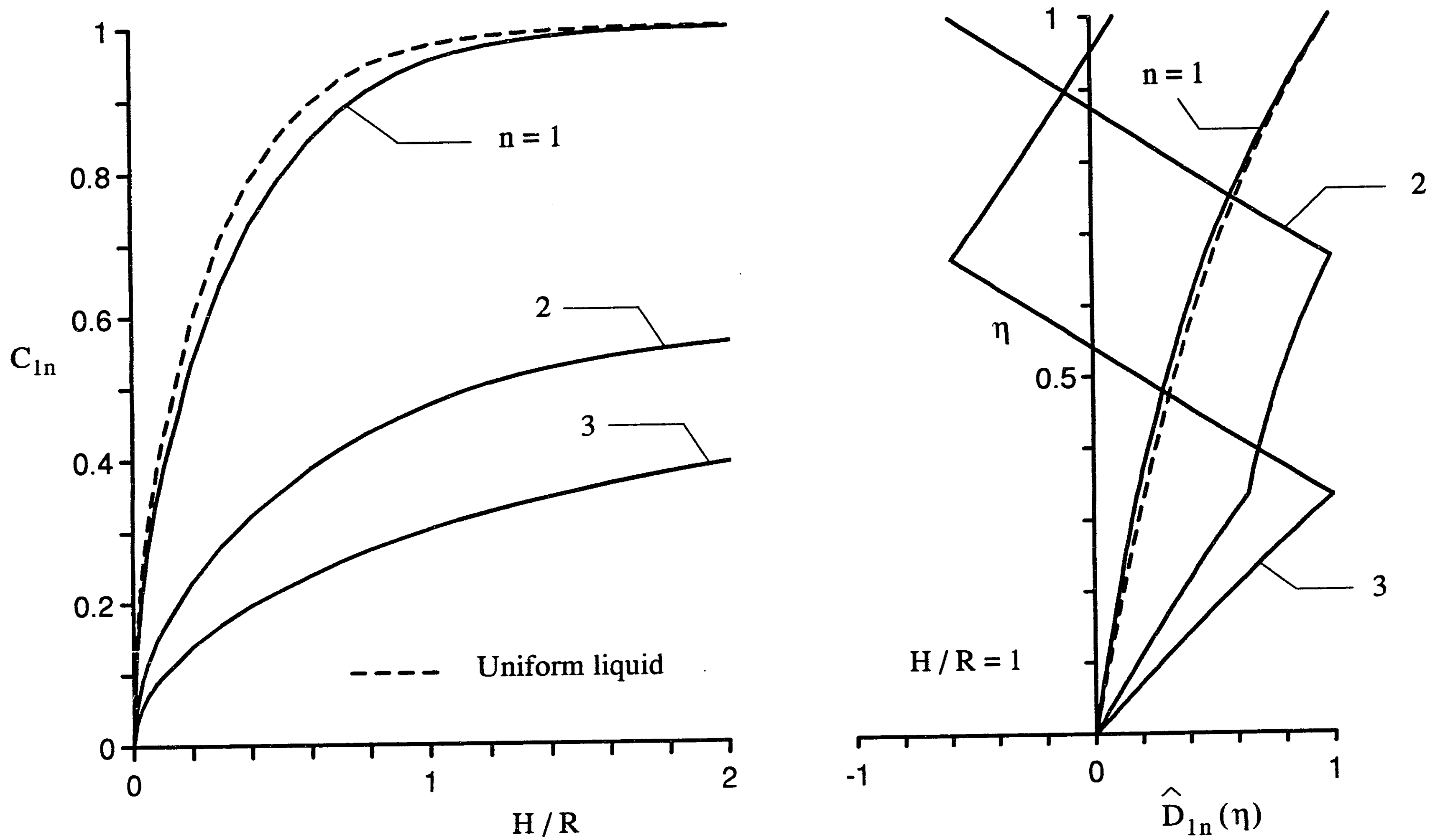

Figure 4.6 Frequency coefficients C_{1n} for fundamental horizontal mode of vibration of three-layered liquids in cylindrical tanks and associated vertical displacement configurations for systems with $H / R = 1$; $H_1 = H_2 = H_3$ and $\rho_3 : \rho_2 : \rho_1 = 1:2:3$

SECTION 5

FORCED VIBRATION

With the natural frequencies and modes of vibration of the system established, its response to an arbitrary lateral excitation may be obtained by the modal superposition method. In this approach, the vector $\{D_m(t)\}$ of the interfacial vertical displacements of the liquid along the tank wall is expressed as a linear combination of the characteristic vectors, $\{\hat{D}_{mn}\}$, as

$$\{D_m(t)\} = \sum_{n=1}^{N} \{\hat{D}_{mn}\}\, q_{mn}(t) \tag{46}$$

in which $q_{mn}(t)$ is a generalized time-dependent coordinate corresponding to the mth horizontal and nth vertical mode of vibration. On substituting Eq. (46) into Eq. (35), premultiplying the resulting expression by $\{\hat{D}_{mr}\}^T$, and making use of Eq. (37) and of the orthogonality properties of the natural modes defined by Eqs. (38) and (39), the resulting system of Eqs. is uncoupled, leading to

$$\ddot{q}_{mn}(t) + \omega_{mn}^2\, q_{mn}(t) = -\lambda_m\, \epsilon_m \frac{\{\hat{D}_{mn}\}^T\{c\}}{\{\hat{D}_{mn}\}^T\,[A]\,\{\hat{D}_{mn}\}}\, \ddot{x}_g(t) \tag{47}$$

It is convenient to replace the tri-diagonal matrix $[A]$ on the right-hand member of th.s expression by the diagonal matrix $[B]$. On making use of Eq. (37), Eq. (47) may be rewritten as

$$\ddot{q}_{mn}(t) + \omega_{mn}^2\, q_{mn}(t) = -\epsilon_m\, \omega_{mn}^2 \Gamma_{mn} R\, \frac{\ddot{x}_g(t)}{g} \tag{48}$$

in which Γ_{mn} is a dimensionless factor given by

$$\Gamma_{mn} = \frac{\{\hat{D}_{mn}\}^T\{c\}}{\{\hat{D}_{mn}\}^T\,[B]\,\{\hat{D}_{mn}\}} \tag{49}$$

The solution of Eq. (48) is then given by

$$q_{mn}(t) = \epsilon_m\, \Gamma_{mn}\, R\, \frac{A_{mn}(t)}{g} \tag{50}$$

in which $A_{mn}(t)$ represents the pseudoacceleration function defined by

$$A_{mn}(t) = -\omega_{mn} \int_0^t \ddot{x}_g(\tau)\; sin\, \omega_{mn}(t-\tau)\; d\tau \tag{51}$$

For a base-excited single-degree-of-freedom oscillator with a circular frequency ω_{mn}, the pseudoacceleration $A_{mn}(t)$ represents the product of the square of ω_{mn} and the deformation of the oscillator, $U_{mn}(t)$. The maximum value of $A_{mn}(t)$ is the quantity displayed on a pseudoacceleration response spectrum.

The interfacial displacements at arbitrary points are finally determined from Eq. (24) or (25) by making use of Eqs. (46) and (50). For the long rectangular system, they are given by

$$\{d(\xi,t)\} = R \sum_{m=1}^{\infty} \sum_{n=1}^{N} \{d_{mn}\} \frac{sin\lambda_m \xi}{sin\lambda_m} \frac{A_{mn}(t)}{g} \tag{52}$$

and for the cylindrical system, by

$$\{d(\xi,\theta,t)\} = R \sum_{m=1}^{\infty} \sum_{n=1}^{N} \{d_{mn}\} \frac{J_1(\lambda_m \xi)}{J_1(\lambda_m)} \frac{A_{mn}(t)}{g} \cos\theta \tag{53}$$

where

$$\{d_{mn}\} = \epsilon_m \Gamma_{mn} \{\hat{D}_{mn}\} \tag{54}$$

It must be recalled that the factors λ_m and the expressions for ϵ_m are different for the two systems. The same is also true of $\{\hat{D}_{mn}\}$, Γ_{mn}, $\{d_{mn}\}$, ω_{mn} and $A_{mn}(t)$.

For a single-layered system with a homogeneous liquid, the only interfacial displacements are those at the surface. In this case, $\{D_{mn}\}$ and $\{d_{mn}\}$ reduce to the scalars D_m and d_m; $\{d\}$ reduces to the surface displacement, $d(\xi,\theta,t)$; the matrix $[B]$ and vector $\{c\}$ become unit scalars; and the product $\Gamma_{mn}\{\hat{D}_{mn}\}$ in Eq. (54) reduces to unity. It can then be concluded from Eq. (54) that the factors ϵ_m, which are defined by Eq. (33) for the long rectangular system and by Eq. (34) for the cylindrical system, represent the displacement coefficients for the surface sloshing motion along the tank wall of the homogeneous liquid. The latter factors can be shown [Reference 1] to satisfy the relation

$$\sum_{m=1}^{\infty} \epsilon_m = 1 \tag{55}$$

Because of their special meaning, these factors were not absorbed into the Γ_{mn} factors but were retained as multipliers in the expressions for the layered systems as well.

The surface displacement of the uniform system may then be determined from the following specialized forms of Eqs. (52) and (53), of which the second has been reported previously, e.g., Reference 1:

$$d(\xi,t) = R \sum_{m=1}^{\infty} \epsilon_m \frac{sin\lambda_m \xi}{sin\lambda_m} \frac{A_m(t)}{g} \tag{56}$$

and

$$d(\xi, \theta, t) = R \sum_{m=1}^{\infty} \epsilon_m \frac{J_1(\lambda_m \xi)}{J_1(\lambda_m)} \frac{A_m(t)}{g} \cos\theta \tag{57}$$

In these expressions, $A_m(t)$ represents the instantaneous pseudoacceleration of a simple oscillator with a natural frequency equal to that of the mth sloshing mode of vibration of the actual system when it is subjected to the prescribed ground motion.

If the contribution of only the fundamental mode of vibration is considered, the surface displacement along the tank wall, d_w, for a homogeneous liquid in a cylindrical tank reduces to the well known expression

$$d_w(\theta, t) = 0.837\, R \frac{A_1(t)}{g} \cos\theta \tag{58}$$

On replacing $A_1(t)$ by $\omega_1^2 U_1(t)$, where $U_1(t)$ = the instantaneous deformation of the single-degree-of-freedom oscillator, and making use of Eq. (45), Eq. (58) can also be written as

$$d_w(\theta, t) = 1.54\, C_1^2\, U_1(t) \cos\theta \tag{59}$$

in which C_1 is the dimensionless frequency coefficient defined by Eq. (44).

For a multi-layered system, it can be shown that

$$\sum_{n=1}^{N} \{d_{mn}\} = \epsilon_m \{1\} \tag{60}$$

and by virtue of Eq. (55), it can further be concluded that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{N} \{d_{mn}\} = \{1\} \tag{61}$$

Equation (60) is proved in the following by examining the hydrodynamic pressure difference at the jth interface of the system, Δp_j. The instantaneous value of this difference is determined from Eq. (11) to be

$$\Delta p_j(\xi, t) = (\rho_j - \rho_{j+1})\, g\, d_j(\xi, t) \tag{62}$$

in which d_j is defined by Eq. (52) for the long rectangular system, and by Eq. (53) for the latter system. For the cylindrical system, Eq. (62) may thus be written as

$$\Delta p_j(\xi, \theta, t) = (\rho_j - \rho_{j+1})\, R \sum_{m=1}^{\infty} \sum_{n=1}^{N} d_{mn,j} \frac{J_1(\lambda_m \xi)}{J_1(\lambda_m)} A_{mn}(t) \cos\theta \tag{63}$$

Now, if the natural frequencies of the system are very high compared to the dominant frequency of the ground motion, all the pseudoacceleration functions $A_{mn}(t)$ will reduce to the ground acceleration $\ddot{x}_g(t)$, the liquid will respond as a rigid body, and the resulting expression for Δp_j will reduce to the pressure induced by the inertia of a rigid disk which has unit depth, radius r, mass density $\rho_j - \rho_{j+1}$ and is subjected to a horizontal motion with acceleration $\ddot{x}_g(t)$. The latter pressure is given by

$$\Delta p_j(\xi, \theta, t) = (\rho_j - \rho_{j+1})\, r\, \ddot{x}_g(t) \cos\theta \tag{64}$$

On equating the right-hand members of Eqs. (63) and (64) and cancelling the common terms, one obtains

$$\sum_{m=1}^{\infty} \sum_{n=1}^{N} d_{mn,j} \frac{J_1(\lambda_m \xi)}{J_1(\lambda_m)} = \xi \tag{65}$$

Finally, on multiplying through by $\xi\, J_1(\lambda_m \xi)\, d\xi$ and integrating from 0 to 1, by virtue of the orthogonality of the Bessel functions, the double summation on the left-hand side reduces to a single summation over n only, and the final expression reduces to the j th element of Eq. (60). The validity of Eq. (60) for the long rectangular system can be demonstrated in a similar manner working with trigonometric rather than Bessel functions.

5.1 Sloshing Displacement Coefficients

Of special interest in practice is the sloshing motion of the liquid at its free surface, as the maximum surface displacement is needed to define the freeboard that must be provided to prevent the liquid from overflowing or impacting the roof. This displacement is defined by the top elements of Eqs. (52) and (53).

In Table 5.1 are listed the surface values of the displacement coefficients $d_{mn,2}$ for two-layered, long rectangular and cylindrical tanks. Systems with two different slenderness ratios, H/R, two liquid thickness ratios, H_2/H_1, and several mass density ratios, ρ_2/ρ_1, are considered. Results for the first two horizontal modes of vibration, $m = 1$ and 2, and for each of the two vertical modes are presented. The following trends are worth noting:

1. The results for the two vertical modes of vibration are of opposite signs, and their numerical values increase with decreasing ρ_2/ρ_1; the increase is particularly large for the lower values of H/R, especially for $H_2/H_1 = 1$, for which the natural frequencies of the individual layers are equal. The larger displacement coefficients for the fundamental vertical mode of vibration of the layered systems are attributed to the fact that, in addition to being excited laterally, the upper layers of these systems are excited at their base by the rocking motion of the interfacial sloshing.

2. For a specified horizontal mode of vibration, the sum of the displacement coefficients for the two vertical modes is equal to that obtained for a homogeneous liquid of the same total depth. This is in agreement with Eq. (60).
3. The values of the coefficients for the second horizontal mode of vibration, $m = 2$, are significantly smaller than those for the fundamental mode, $m = 1$.
4. Provided a sufficiently large number of horizontal modes of vibration is considered, the algebraic sum of the coefficients is unity, in agreement with Eq. (61).
5. The results for the long rectangular and cylindrical systems are very similar.

In Table 5.2, the top values of the displacement coefficients, d_{mn}, for the two horizontal modes of vibration of the three-layered cylindrical system examined in the right part of Fig. 4.6 are compared with those obtained for a homogeneous liquid of the same total depth. As before, the larger numerical values are obtained for the layered system, and the reported values satisfy both Equations (60) and (61).

Notwithstanding the importance of the displacement coefficients, it must be realized that the relative contribution of the various modes of vibration to the total response depends also on the relative values of the pseudoaccelerations, $A_{mn}(t)$. The latter quantities depend, in turn, on the characteristics of the ground motion and the natural frequencies of the system itself. This matter is considered further in the following section.

5.2 Hydrodynamic Pressures

The main focus of this paper has been on the sloshing motion of the system. With the information presented, however, it is also possible to determine the magnitude and distribution of the hydrodynamic pressures induced by the ground shaking. The hydrodynamic pressure at any point in the j th layer may be evaluated from Eq. (5) making use of the expression for the velocity potential function ϕ_j defined by Eq. (12). The functions χ and ψ_j in the latter Eq. may be evaluated from Eqs. (15) and (26) for the long rectangular system and from Eqs. (16) and (27) for the cylindrical system. The final expressions, along with numerical solutions that elucidate the interrelationship of the hydrodynamic pressures for layered and homogeneous systems, will be presented in a later publication.

Table 5.1: Surface-displacement coefficients for two-layered long rectangular and cylindrical systems

H/R	ρ_2/ρ_1	Values of $d_{mn,2}$ for long rectangular systems				Values of $d_{mn,2}$ for cylindrical systems			
		$m = 1$		$m = 2$		$m = 1$		$m = 2$	
		$n = 1$	$n = 2$	$n = 1$	$n = 2$	$n = 1$	$n = 2$	$n = 1$	$n = 2$
					(a) $H_2/H_1 = 1$				
0.5	1.0	0.811		0.090		0.837		0.073	
	0.75	0.873	-0.062	0.096	-0.006	0.901	-0.064	0.077	-0.004
	0.50	0.978	-0.167	0.106	-0.016	1.009	-0.172	0.085	-0.012
	0.25	1.215	-0.405	0.132	-0.042	1.254	-0.417	0.105	-0.032
	0.10	1.687	-0.876	0.184	-0.094	1.741	-0.904	0.148	-0.075
	0.01	4.458	-3.647	0.494	-0.404	4.602	-3.765	0.399	-0.326
2.0	1.0	0.811		0.090		0.837		0.073	
	0.75	0.853	-0.042	0.090	0.000	0.873	-0.036	0.073	0.000
	0.50	0.932	-0.121	0.091	-0.001	0.942	-0.105	0.073	0.000
	0.25	1.138	-0.327	0.092	-0.002	1.130	-0.293	0.074	-0.001
	0.10	1.598	-0.787	0.097	-0.007	1.579	-0.742	0.076	-0.003
	0.01	4.415	-3.604	0.168	-0.078	4.507	-3.670	0.108	-0.035
					(b) $H_2/H_1 = 2$				
0.5	1.0	0.811		0.090		0.837		0.073	
	0.75	0.863	-0.052	0.094	-0.004	0.890	-0.053	0.076	-0.003
	0.50	0.944	-0.133	0.101	-0.011	0.973	-0.136	0.081	-0.008
	0.25	1.099	-0.289	0.115	-0.025	1.132	-0.295	0.091	-0.018
	0.10	1.303	-0.493	0.137	-0.047	1.342	-0.505	0.109	-0.036
	0.01	1.573	-0.762	0.173	-0.083	1.623	-0.786	0.139	-0.066
2.0	1.0	0.811		0.090		0.837		0.073	
	0.75	0.836	-0.025	0.090	0.000	0.857	-0.020	0.073	0.000
	0.50	0.881	-0.070	0.090	0.000	0.893	-0.056	0.073	0.000
	0.25	0.986	-0.176	0.091	-0.001	0.983	-0.146	0.073	0.000
	0.10	1.170	-0.360	0.092	-0.002	1.156	-0.319	0.074	-0.001
	0.01	1.533	-0.722	0.104	-0.014	1.560	-0.723	0.078	-0.005

Table 5.2: Surface-displacement coefficients for three-layered cylindrical system with $H/R = 1$ and $H_1 = H_2 = H_3$

$\rho_3/\rho_2/\rho_1$	Values of $d_{mn,3}$					
	$m = 1$			$m = 2$		
	$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$
1/1/1	0.837			0.073		
1/2/3	1.101	-0.295	0.031	0.085	-0.013	0.001

SECTION 6

NUMERICAL EXAMPLE

The surface sloshing displacements induced by the lateral component of an earthquake ground motion are evaluated for two-layered liquids in a cylindrical tank of 60-ft radius. The depths of the lower and upper layers are taken as 12 ft and 24 ft, respectively, and several different values are used for the relative mass densities of the two layers. System damping for each mode of vibration is considered to be of the viscous type, and is taken as 0.5 percent of the critical value.

The ground motion is specified by the response spectrum shown in Fig. 6.1. which refers to viscously damped single-degree-of-freedom systems with the designated amount of damping. The spectrum is displayed in a tripartite logarithmic format with the abscissa representing the natural frequency of the system, f, and the pseudoacceleration, A, plotted on the right-hand diagonal scale. The vertical scale represents the pseudovelocity of the system, V, and the left-hand diagonal scale represents the associated deformation, U. The three spectral quantities are interrelated by

$$A = 2\pi f V = 4\pi^2 f^2 U \tag{66}$$

The maximum values of A, V and U are $1.68\,g$, 61 in./sec and 31 in., respectively, and the maximum values of the acceleration, velocity and displacement of the ground are $0.33\,g$, 15.9 in./sec and 10.2 in., respectively. The response spectrum considered is the same as that used for the illustrative example in Reference 1.

The cyclic natural frequencies of the liquid for the first two horizontal and each of the two vertical modes of vibration are listed in Table 6.1 along with the corresponding values of the surface displacement coefficients, $d_{mn,2}$. Several values of ρ_2/ρ_1 in the range between unity and 0.10 are considered. Note that all frequency values fall in the left-hand, displacement-sensitive region of the response spectrum, and that, for all cases considered, the frequencies f_{11} and f_{21} fall within the segment for which the deformation U attains its maximum value. Note further that the largest displacement coefficients are associated with the fundamental mode of vibration, $m = n = 1$.

Table 6.2 lists the maximum values of the components of the surface displacements along the tank wall contributed by each of the four modes of vibration. The results are normalized with respect to the maximum ground displacement, $(x_g)_{max}$. Also listed are the corresponding values of the total displacement computed by taking the square root of the sum of the squares of the component terms. The following trends are worth noting:

1. The displacements of the layered liquid are larger than those of the homogeneous liquid of the same total depth. However, as would be expected from the information presented in the preceding sections, the increase is not particularly significant for values of $\rho_2/\rho_1 \geq 0.5$.

2. The fundamental mode of vibration, which corresponds to values of $m = n = 1$, is the dominant contributor to the response. The contribution of the remaining modes is minor due to the smallness either of the relevant displacement coefficients or of the associated pseudoaccelerations or both.

3. The response contributed by the mode corresponding to $m = 2$ and $n = 1$ is greater than that contributed by the mode corresponding to $m = 1$ and $n = 2$.

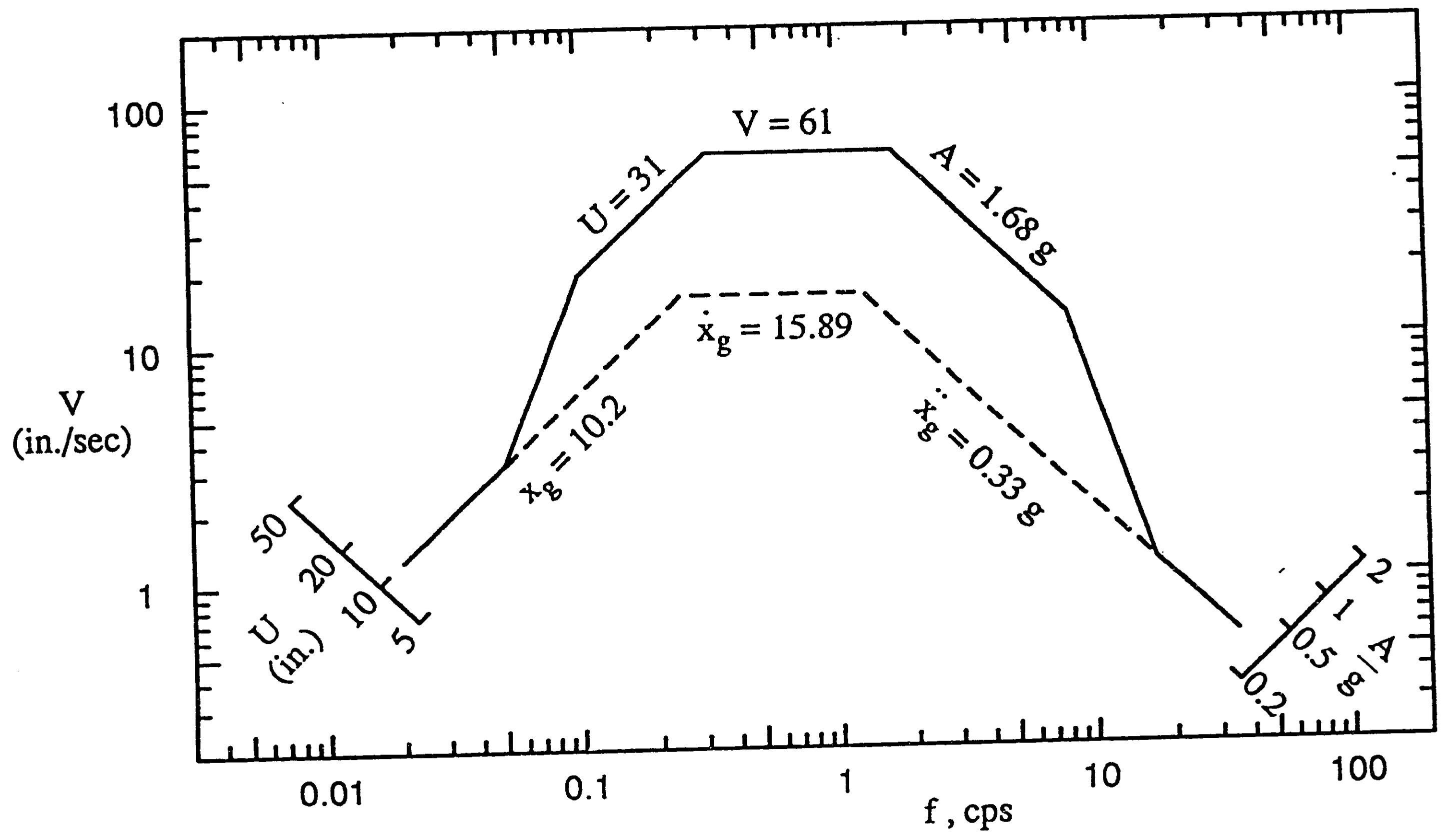

Figure 6.1 Design response spectrum considered in numerical example

Table 6.1: Natural frequencies and surface-displacement coefficients for system considered in numerical example

ρ_2/ρ_1	Natural frequency, f_{mn} in cps				Surface-displacement coefficients, $d_{mn,2}$			
	f_{11}	f_{12}	f_{21}	f_{22}	$d_{11,2}$	$d_{12,2}$	$d_{21,2}$	$d_{22,2}$
1.00	0.142		0.269		0.837		0.073	
0.75	0.140	0.039	0.269	0.094	0.888	-0.051	0.075	-0.002
0.50	0.137	0.058	0.268	0.142	0.970	-0.133	0.079	-0.006
0.25	0.133	0.075	0.268	0.188	1.127	-0.290	0.088	-0.015
0.10	0.129	0.086	0.267	0.217	1.337	-0.500	0.105	-0.032

Table 6.2: Maximum values of surface displacements of liquid along tank wall for system considered in numerical example

ρ_2/ρ_1	Values of $(d_w)_{max}/(x_g)_{max}$				
	Component contributed by				Total computed by RMS rule
	$m = 1$		$m = 2$		
	$n = 1$	$n = 2$	$n = 1$	$n = 2$	
1.00	3.775		1.181		3.956
0.75	3.893	0.006	1.213	0.004	4.078
0.50	4.071	0.047	1.269	0.028	4.265
0.25	4.459	0.263	1.414	0.119	4.687
0.10	4.975	0.706	1.673	0.337	5.307

SECTION 7

CONCLUSIONS

With the information presented herein, the free vibrational characteristics and the sloshing action of base-excited, layered liquids both in long rectangular and in cylindrical tanks may be evaluated readily and accurately. The comprehensive numerical data that have been presented provide valuable insights into the underlying response mechanisms and into the effects and relative importance of the numerous parameters involved. The principal conclusions of the study may be summarized as follows:

1. For a liquid with N homogeneous layers, there is an infinite number of horizontal natural modes of vibration, and corresponding to each such mode, there are N distinct vertical modes. The latter modes have from zero to $N-1$ points of zero crossings, and their frequencies are lower than the corresponding frequency of a uniform liquid of the same total depth.

2. For a specified horizontal mode of vibration, the natural frequencies of a two-layered system are, respectively, higher and lower than those computed considering the two liquid layers to act independently.

3. The natural modes of the layered liquid satisfy simple orthogonality relations that are identified in the text.

4. The maximum surface sloshing displacement of a base-excited layered system is generally greater than that induced in a homogeneous system of the same total depth. The increase is significant, however, only when the densities of individual layers differ substantially. The increased response is associated with the fact that, in addition to the lateral component of shaking, the base of the top layer is subjected to a rocking motion associated with the sloshing action of the interface.

5. For large-capacity tanks subjected to earthquake-ground motions, the mode of vibration corresponding to $m = n = 1$ is the dominant contributor to the surface sloshing displacements of the liquid. Furthermore, the contribution of the mode corresponding to $m = 2$ and $n = 1$ is generally greater than that of the mode corresponding to $m = 1$ and $n = 2$.

6. For the 2-layered system considered in the illustrative example, the maximum surface displacement along the tank wall was found to range from 3.96 times the maximum ground displacement when the densities of the two layers were considered to be equal, to 5.3 times the maximum ground displacement when the density of the top layer was taken as one-tenth that of the lower layer.

SECTION 8

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SECTION 9

NOTATION

A	maximum or spectral value of pseudoacceleration for a simple oscillator
$[A]$	tri-diagonal, symmetric matrix of size $N \times N$ with the elements of the j th row defined by Eqs. (29), (30) and (31)
$A_m(t)$	instantaneous pseudoacceleration for m th horizontal mode of vibration of homogeneous system
$A_{mn}(t)$	instantaneous pseudoacceleration for m th horizontal and n th vertical mode of vibration of layered system
$[B]$	diagonal matrix of size $N \times N$ with its j th element given by Eq. (32)
c_j	dimensionless factor given by Eq. (32)
C_m	dimensionless coefficient in expression for ω_m
C_{mn}	dimensionless coefficient in expression for ω_{mn}
d	instantaneous value of surface vertical displacement of homogeneous liquid
d_w	value of d along tank wall
$\{d\}$	instantaneous values of interfacial vertical displacements, measured from position of static equilibrium
d_j	j th element of $\{d\}$
$\{d_{mn}\}$	vector of displacement coefficients in expression for $\{d\}$, given by Eq. (54)
$\{D_m\}$	vector of interfacial vertical displacements of liquid along the tank wall when the system is vibrating in its m th horizontal mode of vibration
$D_{m,j}$	j th element of $\{D_m\}$
$\{\hat{D}_{mn}\}$	vector of amplitudes of interfacial vertical displacements for the m th horizontal and n th vertical natural mode of vibration
$\hat{D}_{mn,j}$	j th element of $\{\hat{D}_{mn}\}$
f	natural frequency, in cycles per second, of a simple oscillator
f_{mn}	$= \omega_{mn}/2\pi =$ the n th cyclic natural frequency of layered system for m th horizontal mode of vibration
g	acceleration due to gravity
H	total depth of liquid in tank
H_j	thickness of j th liquid layer
J_1	Bessel function of first kind and first order

N	number of superposed liquid layers of different densities
p_j	hydrodynamic pressure in j th liquid layer, given by Eq. (5)
$q_{mn}(t)$	time-dependent generalized coordinate corresponding to the m th horizontal and n th vertical mode of vibration
r	radial distance for cylindrical tank
R	radius of cylindrical tank and half-width of long rectangular tank
t	time
U	spectral value of deformation for simple oscillator
$U_m(t)$	instantaneous deformation of a simple oscillator the natural frequency of which is equal to the m th sloshing frequency of a homogeneous system
$U_{mn}(t)$	instantaneous deformation of a simple oscillator with a natural frequency equal to that of the m th horizontal and n th vertical mode of vibration of the layered system
v_j	instantaneous value of liquid velocity in j th layer, given by Eq. (4)
V	spectral value of pseudovelocity for simple oscillator
x	horizontal distance for long rectangular tank
$\ddot{x}_g(t)$	instantaneous value of free-field ground acceleration
z_j	vertical coordinate within the j th liquid layer
α_j	$= H_j/R =$ ratio of layer height to radius of tank
Γ_{mn}	dimensionless factor defined by Eq. (49)
ϵ_m	dimensionless factor defined by Eq. (33) for long rectangular tanks and by Eq. (34) for cylindrical tanks
η_j	$= z_j/R =$ normalized vertical distance coordinate for j th liquid layer
θ	circumferential angle
λ_m	m th root of J_1' for cylindrical tank, and factor defined by Eq. (20) for long rectangular tank
ξ	dimensionless horizontal distance coordinate, defined as x/R for long rectangular tank and as r/R for cylindrical tank
ρ_j	mass density of j th liquid layer
ϕ_j	velocity potential function for j th liquid layer
χ	velocity potential function associated with rigid body motion of tank walls
ψ_j	velocity potential function associated with relative motion of liquid and tank walls, given by Eqs. (19) and (21) for long rectangular and cylindrical tanks, respectively
ω_m	circular natural frequency of homogeneous system for m th sloshing mode of vibration

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