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TITLE: FROM THE SHELL MODEL TO THE INTERACTING BOSON MODEL

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FROM THE SHELL MODEL TO THE INTERACTING BOSON MODEL

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ABSTRACT

Starting from a general, microscopic fermion-pair-to-boson mapping of a complete Fermion space that preserves Hermitian conjugation, we show that the resulting infinite and non-convergent boson Hamilitonian can be factored into a finite (e.g., a 1+2-body fermion Hamiltonian is mapped to a 1+2-body boson Hamiltonian) image Hamilitonian times the norm operator, and it is the norm operator that is infinite and non-convergent. We then truncate to a collective boson space and we give conditions under which the exact boson images of finite fermion operators are also finite in the truncated basis.

1. Introduction

The phenomenological Interacting Boson Model (IBM) for nuclei successfully describes many states and transition amplitudes of heavy nuclei using only s- and d- (angular momentum $J = 0, 2$) bosons, which represent coherent nucleon pairs, a severe trunication of the large number of fermion degrees of freedom in heavy nuclei. On the other hand, a rigorous microscopic formalism which determines the IBM Hamiltonian from the shell model Hamiltonian is lacking.[2]

For this reason we have revisited boson mappings, which have a long history[3], from a new perspective[4,5].

2. Fermion Space to Boson Space

Consider a fermion space with 2Ω single-particle states, and a fermion Hamiltonian $\hat{H}$. The general problem is to solve the fermion eigenvalue equation

$$\hat{H}\,|\Psi_p\rangle = E_p\,|\Psi_p\rangle\,, \tag{1}$$

find transition amplitudes between eigenstates, and so on. To do this we require a many-body basis. Often the basis set for many-fermion wavefunctions are Slater determinants, antisymmeterized products of single-fermion wavefunctions which we can write as products of the Fock creation operators $a_j^\dagger, j = 1, \cdots, 2\Omega$ on the vacuum $a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle$ for n fermions. These states span the antisymmetric irreducible representation of the unitary group in 2Ω dimensions, SU(2Ω). But for an even number of fermions one can just as well construct states from $N = n/2$ fermion pair creation operators,

$$|\psi_\beta\rangle = \prod_{m=1}^{N} \hat{A}_{\beta_m}^\dagger |0\rangle , \tag{2}$$

with

$$\hat{A}_\beta^\dagger \equiv \frac{1}{\sqrt{2}} \sum_{ij} \left(\mathbf{A}_\beta^\dagger\right)_{ij} a_i^\dagger a_j^\dagger. \tag{3}$$

We always choose the $\Omega(2\Omega - 1)$ matrices $\mathbf{A}_\beta$ to be antisymmetric to preserve the underlying fermion statistics, and we choose for the normalization the trace, tr $\left(\mathbf{A}_\alpha \mathbf{A}_\beta^\dagger\right) = \delta_{\alpha\beta}$. For this paper we represent generic one- and two-body operators by

$$\hat{T} \equiv \sum_{ij} T_{ij} a_i^\dagger a_j, \quad \hat{V} \equiv \sum_{\mu\nu} \langle\mu| V |\nu\rangle \hat{A}_\mu^\dagger \hat{A}_\nu. \tag{4}$$

We begin with the straightforward mapping to boson states

$$|\psi_\beta\rangle \to |\phi_\beta) = \prod_{m=1}^{N} b_{\beta_m}^\dagger |0) , \tag{5}$$

where the $b^\dagger$ are boson creation operators. In conjunction with this simple mapping of states we construct boson *representations* that follow the philosophy of the Marumori expansion[6] and preserve matrix elements of the fermion operators, for example, by introducing boson representations $\hat{\mathcal{T}}_B$, $\hat{\mathcal{V}}_B$, and $\hat{\mathcal{H}}_B$ such that $(\phi_\alpha| \hat{\mathcal{T}}_B |\phi_\beta) = \langle\psi_\alpha| \hat{T} |\psi_\beta\rangle$, $(\phi_\alpha| \hat{\mathcal{V}}_B |\phi_\beta) = \langle\psi_\alpha| \hat{V} |\psi_\beta\rangle$, and $(\phi_\alpha| \hat{\mathcal{H}}_B |\phi_\beta) = \langle\psi_\alpha| \hat{H} |\psi_\beta\rangle$. In addition, because wavefunctions of the form (2) do not form an orthonormal set, we construct the norm operator $\hat{\mathcal{N}}_B$ with the property $(\phi_\alpha| \hat{\mathcal{N}}_B |\phi_\beta) = \langle\psi_\alpha |\psi_\beta\rangle$. The construction of these operators[5] follow directly from the matrix elements which are found by generalizing the vector coherent state method[7]. This procedure differs from the the usual Marumori expansion in that the latter does not have an explicit norm operator. With this mapping the fermion eigenvalue equation (1) becomes a generalized (because of the norm) boson eigvenvalue equation

$$\hat{\mathcal{H}}_B |\Phi_p) = E_p \hat{\mathcal{N}}_B |\Phi_p) . \tag{6}$$

Because we have defined our boson operators so as to preserve matrix elements, the original energy spectrum of (1) is found in (6). However, because the boson space is much larger than the original fermion space, (6) also has additional spurious states that do not correspond to physical fermion states. These by construction have zero eigenvalue and do not mix with the physical space.

When one constructs the norm operator one finds it can be conveniently expressed in terms of the kth order Casimir operators of the unitary group SU(2Ω), $\hat{C}_k = 2^k \operatorname{tr}(\mathbf{P})^k$, $\mathbf{P} = \sum_{\sigma\tau} b^\dagger_\sigma b_\tau \mathbf{A}_\sigma \mathbf{A}^\dagger_\tau$ ($\mathbf{P}$ is both a matrix and a boson operator; the trace is over the matrix indices and not the boson Fock space)[5]

$$\hat{\mathcal{N}}_B = :\exp\left(-\sum_{k=2}^{\infty} \frac{(-1)^k}{2k} \hat{C}_k\right): \tag{7}$$

where the colons ':' refer to normal-ordering of the boson operators.

Although the representations $\hat{\mathcal{T}}_B, \hat{\mathcal{V}}_B$ are complicated many-body operators, we can write them in compact form, a result we have not seen previously in the literature[5],

$$\hat{\mathcal{T}}_B = 2\sum_{\sigma,\tau} : \operatorname{tr}\left[\mathbf{A}_\sigma \mathbf{T} \mathbf{A}^\dagger_\tau \mathbf{G}\right] b^\dagger_\sigma b_\tau \hat{\mathcal{N}}_B :, \tag{8a}$$

$$\begin{aligned}\hat{\mathcal{V}}_B = \sum_{\mu,\nu} \langle \mu | V | \nu \rangle \sum_{\sigma,\tau} : \Big\{ & \operatorname{tr}\left[\mathbf{A}_\sigma \mathbf{A}^\dagger_\mu \mathbf{G}\right] \operatorname{tr}\left[\mathbf{A}_\nu \mathbf{A}^\dagger_\tau \mathbf{G}\right] \\ & + 2 \operatorname{tr}\left[\mathbf{A}_\sigma \mathbf{A}^\dagger_\mu (\mathbf{P}\mathbf{G} + \mathbf{G}\mathbf{P}) \mathbf{A}_\mu \mathbf{A}^\dagger_\tau \mathbf{G}\right]\Big\} b^\dagger_\sigma b_\tau \hat{\mathcal{N}}_B :,\end{aligned} \tag{8b}$$

where $\mathbf{T}$ is a matrix with matrix elements T_{ij} and $\mathbf{G} = (\mathbf{1} + 2\mathbf{P})^{-1}$. In general the boson representations given in (7) and (8) do not have good convergence properties, so that simple termination of the series in k-body terms is impossible and use of the generalized eigenvalue equation (6), as written, is problematic.

The explicit forms of (8) suggests, however, that these representations factor in a simple way: $\hat{\mathcal{T}}_B = \hat{\mathcal{N}}_B \hat{T}_B = \hat{T}_B \hat{\mathcal{N}}_B$ and $\hat{\mathcal{V}}_B = \hat{\mathcal{N}}_B \hat{V}_B = \hat{V}_B \hat{\mathcal{N}}_B$, where the factored operators $\hat{T}_B$, $\hat{V}_B$, which we term the boson images of $\hat{T}$, $\hat{V}$, have simple forms[5]. A one-body fermion operator has a one-body boson image,

$$\hat{T}_B = 2\sum_{\sigma\tau} \operatorname{tr}\left(\mathbf{A}_\sigma \mathbf{T} \mathbf{A}^\dagger_\tau\right) b^\dagger_\sigma b_\tau, \tag{9a}$$

while a two-body fermion operator has a one plus two-boson operator,

$$\hat{V}_B = \sum_{\mu\nu} \langle\mu| V |\nu\rangle \left[b^\dagger_\mu b_\nu + 2\sum_{\sigma\sigma'}\sum_{\tau\tau'} \mathrm{tr}\, \left(\mathbf{A}_\sigma \mathbf{A}^\dagger_\mu \mathbf{A}_{\sigma'} \mathbf{A}^\dagger_\tau \mathbf{A}_\nu \mathbf{A}^\dagger_{\tau'} \right) b^\dagger_\sigma b^\dagger_{\sigma'} b_\tau b_{\tau'} \right] \qquad (9b)$$

To prove these results we used the completeness relation

$$\sum_\alpha \left(\mathbf{A}^\dagger_\alpha\right)_{ij} \left(\mathbf{A}_\alpha\right)_{j'i'} = \frac{1}{2}\left(\delta_{i,i'}\delta_{j,j'} - \delta_{i,j'}\delta_{j,i'}\right). \qquad (10)$$

and the resulting identities

$$2\sum_\alpha \mathrm{tr}(\mathbf{Q}\mathbf{A}^\dagger_\alpha)\mathrm{tr}(\mathbf{A}_\alpha \mathbf{R}) = \mathrm{tr}(\mathbf{Q}\mathbf{R}) - \mathrm{tr}(\mathbf{Q}^T\mathbf{R}), \qquad (11a)$$

$$2\sum_\alpha \mathrm{tr}(\mathbf{Q}\mathbf{A}^\dagger_\alpha \mathbf{R}\mathbf{A}_\alpha) = \mathrm{tr}(\mathbf{Q})\mathrm{tr}(\mathbf{R}) - \mathrm{tr}(\mathbf{Q}^T\mathbf{R}). \qquad (11b)$$

where Q^T is the transpose.

Thus any boson representation of a Hamiltonian factorizes: $\hat{\mathcal{H}}_B = \hat{\mathcal{N}}_B \hat{H}_B$. Since the norm operator is a function of the SU(2Ω) Casimir operators it commutes with the boson images of fermion operators, $[\hat{H}_B, \hat{\mathcal{N}}_B] = [\hat{H}_B, \hat{\mathcal{N}}_B] = 0$, and one can simultaneously diagonalize both $\hat{H}_B$ and $\hat{\mathcal{N}}_B$. Then Eqn. (6) becomes

$$\hat{H}_B \left|\Phi_p\right) = E'_p \left|\Phi_p\right). \qquad (12)$$

where $E'_p = E_p$ for the physical states, but E'_p for the spurious states is no longer necessarily zero. The boson Hamiltonian $\hat{H}_B$ is Hermitian since $\hat{\mathcal{H}}_B$ is Hermitian and $\hat{H}_B$ commutes with $\hat{\mathcal{N}}_B$ and, if one starts with at most only two-body interactions between fermions, has at most two-body boson interactions. All physical eigenstates of the original fermion Hamiltonian will have counterparts in (12). It should be clear that transition amplitudes between physical eigenstates will be preserved. Spurious states will also exist but, since the norm operator $\hat{\mathcal{N}}_B$ commutes with the boson image Hamiltonian $\hat{H}_B$, the physical eigenstates and the spurious states will not admix. Identification of the spurious states is a serious though tractable problem, as $\hat{\mathcal{N}}_B$ annihilates such states; furthermore spurious states can be shifted up in the spectrum through the use of the Park operator[3], $\hat{M} =: \hat{C}_2 : +4N(N-1)$, which has zero eigenvalue for physical states and a positive definite spectrum for spurious states.

This result has been derived from the mapping of commutation relations of the fermion operators[8] but its equivalence to the Marumori mapping of matrix elements has not been demonstrated previously.

3. Truncation

A more critical question however is that of truncating the boson Fock space, by which we mean using states constructed from a restricted set of fermion pairs/bosons denoted by $\{\bar{\alpha}\}$; the operators in this space we denote by $\left[\hat{\mathcal{N}}_B\right]_T$, $\left[\hat{\mathcal{H}}_B\right]_T$, and so on, and are straightforward to construct: for example, the norm operator is

$$\left[\hat{\mathcal{N}}_B\right]_T =: \exp\left(\sum_{k=2}^{\infty} \frac{(-2)^{k-1}}{k} \text{tr } [\mathbf{P}]_T^k\right) : \tag{13}$$

where $[\mathbf{P}]_T = \sum_{\bar{\sigma},\bar{\tau}} b^\dagger_{\bar{\sigma}} b_{\bar{\tau}} \mathbf{A}_{\bar{\sigma}} \mathbf{A}^\dagger_{\bar{\tau}}$. These truncated representations still exactly preserve the matrix elements in the restricted fermion space: $(\phi_{\bar{\alpha}}| \left[\hat{\mathcal{N}}_B\right]_T |\phi_{\bar{\beta}}) = \langle\psi_{\bar{\alpha}} | \psi_{\bar{\beta}}\rangle$ and so on. This is true even when the truncated set of fermion pairs represented do not form a closed subalgebra, a fact apparently overlooked previously[9]. It should be evident that the truncated representations still do not mix physical and spurious states.

3.1 Factorization

Although the representations remain exact under truncation, the factorization into the truncated image does not persist in general: $\left[\hat{\mathcal{H}}_B\right]_T \neq \left[\hat{\mathcal{N}}_B\right]_T \left[\hat{H}_B\right]_T$. This was recognized by Marshalėk[10] and arises because the completeness relation (10) in general is only satisfied if the complete Fock space is used.

We ask under what conditions does $\left[\hat{\mathcal{H}}_B\right]_T$ have a finite factorization? That is,

$$[\hat{\mathcal{H}}_B]_T = [\hat{\mathcal{T}}_B]_T + [\hat{\mathcal{V}}_B]_T = [\hat{\mathcal{N}}_B]_T \bar{H}_B \tag{14a}$$

where $\bar{H}_B$ has only 1 + 2 body terms,

$$\bar{H} = \sum \langle\bar{\sigma}|t|\bar{\tau}\rangle\, b^\dagger_{\bar{\sigma}} b_{\bar{\tau}} + \sum \langle\bar{\sigma}\bar{\sigma}'|v|\bar{\tau}\bar{\tau}'\rangle\, b^\dagger_{\bar{\sigma}} b^\dagger_{\bar{\sigma}'} b_{\bar{\tau}} b_{\bar{\tau}'} \tag{14b}$$

By using the fact that

$$\left[\hat{\mathcal{N}}_B\right]_T b^\dagger_{\bar{\sigma}} = \sum_{\alpha} : \text{tr }(\Lambda_\alpha \mathbf{A}^\dagger_{\bar{\sigma}} \bar{\mathbf{G}}) b^\dagger_\alpha \left[\hat{\mathcal{N}}_B\right]_T :$$

where $\bar{\mathbf{G}} = (1 + 2\bar{\mathbf{P}})^{-1}, \bar{\mathbf{P}} = \sum \mathbf{A}_{\bar{\alpha}} \mathbf{A}^\dagger_{\bar{\beta}} b^\dagger_{\bar{\alpha}} b_{\bar{\beta}}$, we write

$$\left[\hat{\mathcal{N}}_B\right]_T \bar{H}_B =: \left[\hat{\mathcal{N}}_B\right]_T \hat{h} := \left[\hat{\mathcal{H}}_B\right]_T \tag{15}$$

From the left-hand side

$$\hat{h} = \sum \langle \bar{\alpha}\,|t|\,\bar{\tau}\rangle\, \mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{A}^{\dagger}_{\bar{\alpha}}\bar{\mathbf{G}})b^{\dagger}_{\bar{\sigma}}b_{\bar{\tau}} + \langle \bar{\alpha}\bar{\alpha}'\,|v|\,\bar{\tau}\bar{\tau}'\rangle\, [\mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{A}^{\dagger}_{\bar{\alpha}}\bar{\mathbf{G}})\ \ \mathrm{tr}(\mathbf{A}_{\bar{\sigma}'}\mathbf{A}^{\dagger}_{\bar{\alpha}'}\bar{\mathbf{G}}) \tag{16a}$$

$$-2\ \ \mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{A}^{\dagger}_{\bar{\alpha}}\bar{\mathbf{G}}\mathbf{A}_{\bar{\sigma}'}\mathbf{A}^{\dagger}_{\bar{\alpha}'}\bar{\mathbf{G}})]\ \ b^{\dagger}_{\sigma}b^{\dagger}_{\sigma'}b_{\tau}b_{\tau'}.$$

From the right-hand side we get

$$\hat{h} = 2\sum \mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{T}\mathbf{A}^{\dagger}_{\bar{\tau}}\bar{\mathbf{G}})b^{\dagger}_{\bar{\sigma}}b_{\bar{\tau}} + \sum \langle \mu\,|V|\,\nu\rangle\, \{\mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{A}^{\dagger}_{\mu}\bar{\mathbf{G}})\mathrm{tr}(\mathbf{A}_{\nu}\mathbf{A}^{\dagger}_{\bar{\tau}}\bar{\mathbf{G}}) \tag{16b}$$

$$+2\ \mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{A}^{\dagger}_{\mu}(\bar{\mathbf{P}}\bar{\mathbf{G}} + \bar{\mathbf{G}}\bar{\mathbf{P}})\mathbf{A}_{\nu}\mathbf{A}^{\dagger}_{\bar{\tau}}\bar{\mathbf{G}})\}b^{\dagger}_{\bar{\sigma}}b_{\bar{\tau}}$$

Then the condition to factorize into an $1+2$-boson operator is that (16a) and (16b) are equal. Because $\bar{\mathbf{G}}$ is a many-body operator, this equation can not be satisfied in general.

If it were satisfied, then the one-boson term ($\bar{\mathbf{G}} = 1$) determines $\langle\bar{\sigma}|t|\bar{\tau}\rangle$,

$$\langle\bar{\sigma}|t|\bar{\tau}\rangle = 2\sum \mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{T}\mathbf{A}^{\dagger}_{\bar{\tau}}) + \langle\bar{\sigma}|V|\bar{\tau}\rangle \tag{17}$$

The two-boson term determines $\left\langle\bar{\sigma}\bar{\sigma}'|v|\bar{\beta}\bar{\beta}'\right\rangle$. Defining

$$\langle\bar{\alpha}\bar{\alpha}'|w|\bar{\tau}\bar{\tau}'\rangle = 4\{\sum \langle\mu|V|\nu\rangle \left[2\ \mathrm{tr}(\mathbf{A}_{\bar{\alpha}}\mathbf{A}^{\dagger}_{\mu}\mathbf{A}_{\bar{\alpha}'}\mathbf{A}^{\dagger}_{\bar{\tau}}\mathbf{A}_{\nu}\mathbf{A}^{\dagger}_{\bar{\tau}'}) - \mathrm{tr}(\mathbf{A}_{\bar{\alpha}}\mathbf{A}^{\dagger}_{\bar{\tau}}\mathbf{A}_{\nu}\mathbf{A}^{\dagger}_{\bar{\tau}'})\delta_{\mu,\alpha'}\right] \tag{18}$$

$$-2\left[\ \mathrm{tr}(\mathbf{A}_{\bar{\alpha}}\mathbf{T}\mathbf{A}^{\dagger}_{\bar{\tau}}\mathbf{A}_{\bar{\alpha}'}\mathbf{A}^{\dagger}_{\bar{\tau}'}) - \mathrm{tr}(\mathbf{A}_{\bar{\sigma}}\mathbf{T}\mathbf{A}^{\dagger}_{\bar{\tau}})\mathrm{tr}(\mathbf{A}_{\bar{\alpha}}\mathbf{A}^{\dagger}_{\bar{\sigma}}\mathbf{A}_{\bar{\alpha}'}\mathbf{A}^{\dagger}_{\bar{\tau}'})\right]\},$$

then,

$$\langle\bar{\sigma}\bar{\sigma}'|v|\bar{\tau}\bar{\tau}'\rangle = \sum_{\bar{\alpha}\bar{\alpha}'} \left\langle\bar{\sigma}\bar{\sigma}'|\hat{\mathcal{N}}^{-1}_{\beta}|\bar{\alpha}\bar{\alpha}'\right\rangle \langle\bar{\alpha}\bar{\alpha}'|w|\bar{\tau}\bar{\tau}'\rangle \tag{19}$$

The norm in the two-boson space will more than likely have an inverse; if not, the truncation scheme is probably not a good one. In general, the image $\bar{H}_B$ will not be Hermitian.

Having determined the image Hamiltonian, the three, four,..., boson terms in (16) will impose conditions on the Hamiltonian for factorization which will not be satisfied in general, although in the next subsection we give an example of a truncation for which these conditions are satisfied. Of course, we could extend the ansatz (14b) and include three, four,..., boson interactions, and then determine an image. However, even this image most likely will not be Hermitian. In fact, the image will be Hermitian if, and only if, the truncated boson representation and the norm commute with one another,

$$\left[[\hat{\mathcal{N}}_B]_T, [\hat{\mathcal{H}}_B]_T\right] = 0 \tag{20}$$

If they do not commute, this does not mean spurious and physical states are mixed; in fact, they are not. Unlike the norm in the full space, the truncated norm may not have the same eigenvalue for all of the physical states and hence may not have the same eigenvectors as the Hamiltonian in the physical subspace.

In the next subsection we give an example of a truncation for which the image is one plus two-boson operators.

3.2 An Analytical Example

We can find a truncation scheme such that a factorization

$$\left[\hat{\mathcal{H}}_B\right]_T = \left[\hat{\mathcal{N}}_B\right]_T \bar{H}_B \tag{21}$$

does exist, with $\bar{H}_B$ at most two-body; then with a condition on the Hamiltonian can guarantee $\bar{H}_B$ is Hermitian and commutes with $\left[\hat{\mathcal{N}}_B\right]_T$. First, consider a partition of the single fermion states labeled by $i = (i_a, i_c)$, where the dimension of each subspace is $2\Omega_a$, $2\Omega_c$ so that $\Omega = 2\Omega_a\Omega_c$. We denote the amplitudes for the truncated space as $\mathbf{A}^\dagger_{\bar{\alpha}}$ and assume they can be factored, $(\mathbf{A}^\dagger_{\bar{\alpha}})_{ij} = (\mathbf{K}^\dagger)_{i_a j_a} \otimes (\bar{\mathbf{A}}^\dagger_{\bar{\alpha}})_{i_c j_c}$, with $\mathbf{K}^\dagger\mathbf{K} = \mathbf{K}\mathbf{K}^\dagger = \frac{1}{2\Omega_a}$ $\mathbf{K}^T = (-1)^p\mathbf{K}$, where $p = 0$ (symmetric) or $p = 1$ (antisymmetric).

Furthermore we assume the completeness relation (10), which was crucial for proving that $\hat{\mathcal{H}}_B = \hat{H}_B\hat{\mathcal{N}}_B$, is valid for the truncated space,

$$\sum_{\bar{\alpha}} (\bar{\mathbf{A}}^\dagger_{\bar{\alpha}})_{i_c j_c} (\bar{\mathbf{A}}_{\bar{\alpha}})_{j'_c i'_c} = \frac{1}{2}\left[\delta_{i_c, i'_c}\delta_{j_c, j'_c} - (-1)^p \delta_{i_c, j'_c}\delta_{i'_c, j_c}\right]. \tag{22}$$

A necessary condition is that the the set of operators, $\left\{\hat{A}_{\bar{\alpha}}, \hat{A}^\dagger_{\bar{\beta}}, \left[\hat{A}_{\bar{\alpha}}, \hat{A}^\dagger_{\bar{\beta}}\right]\right\}$ form a closed subalgebra, as in the example given below.

The norm operator in the truncated space then becomes

$$\left[\hat{\mathcal{N}}_B\right]_T = :\exp\sum_{k=2}\left(\frac{-1}{\Omega_a}\right)^{k-1}\frac{1}{k}\mathrm{tr}(\bar{\mathbf{P}}^k):, \tag{23}$$

where $\bar{\mathbf{P}} = \sum_{\bar{\sigma}\bar{\tau}} b^\dagger_{\bar{\sigma}} b_{\bar{\tau}} \bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}^\dagger_{\bar{\tau}}$ so that $[\mathbf{P}]_T = \left(\frac{1}{2\Omega_a}\right)\bar{\mathbf{P}}$. In this case the boson image of a one-body operator is the truncation of the boson image in the full space,

$$\left[\hat{\mathcal{T}}_B\right]_T = \left[\hat{\mathcal{N}}_B\right]_T \left[\hat{T}_B\right]_T \tag{24}$$

$$\left[\hat{T}_B\right]_T = 2\sum_{\bar{\sigma},\bar{\tau}} \mathrm{tr}\,\left(\mathbf{A}_{\bar{\sigma}}\mathbf{T}\mathbf{A}^{\dagger}_{\bar{\tau}}\right) b^{\dagger}_{\bar{\sigma}} b_{\bar{\tau}}. \tag{25}$$

The representation of a two-body interaction can be factored into a boson image times the truncated norm,

$$\left[\hat{\mathcal{V}}_B\right]_T = \left[\hat{\mathcal{N}}_B\right]_T \bar{V}_B; \tag{26}$$

however, $\bar{V}_B$, while finite (1+2-body), is not simply related to $[V_B]_T$ as is the case for one-body operators and in fact is not necessarily Hermitian.

The norm in the two-boson space is given by

$$\left\langle \bar{\tau}\bar{\tau}'|\hat{\mathcal{N}}_\beta|\bar{\beta}\bar{\beta}'\right\rangle = \{\delta_{\bar{\tau},\bar{\beta}'}\delta_{\bar{\tau}',\bar{\beta}'} + \delta_{\bar{\tau},\bar{\beta}'}\delta_{\bar{\tau}',\bar{\beta}'} - \frac{1}{\Omega_a}\left\langle \bar{\tau}\bar{\tau}'|M|\bar{\beta}\bar{\beta}'\right\rangle\}, \tag{27}$$

where

$$\left\langle \bar{\tau}\bar{\tau}'|M|\bar{\beta}\bar{\beta}'\right\rangle = \mathrm{tr}_c(\bar{\mathbf{A}}_{\bar{\tau}}\bar{\mathbf{A}}^{\dagger}_{\bar{\beta}}\bar{\mathbf{A}}_{\bar{\tau}'}\bar{\mathbf{A}}^{\dagger}_{\bar{\beta}'}) + \mathrm{tr}_c(\bar{\mathbf{A}}_{\bar{\tau}}\bar{\mathbf{A}}^{\dagger}_{\bar{\beta}'}\bar{\mathbf{A}}_{\bar{\tau}'}\bar{\mathbf{A}}^{\dagger}_{\bar{\beta}}). \tag{28}$$

Its inverse is

$$\left\langle \bar{\tau}\bar{\tau}'|\hat{\mathcal{N}}_\beta^{-1}|\bar{\beta}\bar{\beta}'\right\rangle = C\left\{\delta_{\bar{\tau},\bar{\beta}}\delta_{\bar{\tau}',\bar{\beta}'} + \delta_{\bar{\tau},\bar{\beta}'}\delta_{\bar{\tau}',\bar{\beta}'} + 2\frac{<\bar{\tau}\bar{\tau}'|M|\bar{\beta}\bar{\beta}'>}{(2\Omega_\sigma + (-1)^p)}\right\}, \tag{29}$$

where

$$C = \frac{\Omega_a(2\Omega_a + (-1)^p)}{4(2\Omega_a - (-1)^P)(\Omega_a + (-1)^p)}. \tag{30}$$

Using (18, 19) we find that the two-boson interaction is

$$\begin{aligned}\langle\bar{\sigma}\bar{\sigma}'|v|\bar{\tau}\bar{\tau}'\rangle = \frac{\sum\langle\mu|V|\nu\rangle}{\Omega_a(2\Omega_a - (-1)^p)(\Omega_a + (-1)^p)}&\mathrm{tr}_a\{\mathrm{tr}_c(\bar{\mathbf{A}}_{\bar{\sigma}}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}}\mathbf{A}_{\bar{\nu}}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}'})\mathrm{tr}_c(\bar{\mathbf{A}}_{\bar{\sigma}'}\mathbf{A}^{\dagger}_{\mu}) \\ +2\Omega_a[\mathrm{tr}_c(\bar{\mathbf{A}}_{\bar{\sigma}}\mathbf{A}^{\dagger}_{\mu}\bar{\mathbf{A}}_{\bar{\sigma}'}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}}\mathbf{A}_{\nu}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}'}) &- \mathrm{tr}_c(\bar{\mathbf{A}}_{\bar{\sigma}}\bar{\mathbf{A}}^{\dagger}_{\mu}\bar{\mathbf{A}}_{\bar{\sigma}'}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}}\mathbf{A}_{\nu}\mathbf{K}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}'})] \\ -\Omega_a(2\Omega_a + (-1)^p)&\mathrm{tr}_c(\mathbf{A}_{\nu}\mathbf{K}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}}\bar{\mathbf{A}}_{\bar{\sigma}'}\bar{\mathbf{A}}^{\dagger}_{\bar{\tau}'})\delta_{\bar{\sigma},\mu}\}\end{aligned} \tag{31}$$

which is not Hermitian in general.

Consider the additional condition

$$\begin{aligned}&\sum_{\mu,\nu}\langle\mu\,|V|\,\nu\rangle\sum_{i_a,j_a}(\mathbf{A}_{\nu})_{i_a i_c,j_a j_c}\left(\mathbf{A}^{\dagger}_{\mu}\right)_{j_a j_c',i_a i_c'} \\ &= N_a\sum_{\mu,\nu}\langle\mu\,|V|\,\nu\rangle\sum_{i_a,j_a}(\mathbf{A}_{\nu})_{i_a i_c,j_a j_c}\left(\mathbf{K}^{\dagger}\right)_{j_a,i_a}\sum_{i_a',j_a'}(\mathbf{K})_{i_a',j_a'}\left(\mathbf{A}^{\dagger}_{\mu}\right)_{j_a' j_c',i_a' i_c'}\end{aligned} \tag{32}$$

where the factor $N_a = \Omega_a(2\Omega_a + (-1)^p)$ is the number of pairs in the excluded subspace; while condition(32) looks complicated there are interactions that satisfy it, for example, two-body interactions constructed from one-body operators $\hat{V} = \hat{T}_{\bar{\alpha}\beta}\hat{T}_{\bar{\alpha}'\beta'}$ where $\hat{T}_{\bar{\alpha}\beta} = \left[A^\dagger_{\bar{\alpha}}, A_\beta\right]$. When (32) is satisfied then $\bar{V}_B$ is Hermitian and although $\bar{V}_B \neq [V_B]_T$ they are simply related:

$$\bar{V}_B = \sum_{\bar{\sigma},\bar{\tau}} \langle \bar{\sigma}\,|V|\,\bar{\tau}\rangle\, b^\dagger_{\bar{\sigma}} b_{\bar{\tau}} + 2 f_{\Omega_a} \sum_{\mu,\nu} \langle \mu\,|V|\,\nu\rangle \sum_{\bar{\sigma}\bar{\sigma}',\bar{\tau}\bar{\tau}'} \mathrm{tr}\left(\mathbf{A}_{\bar{\sigma}}\mathbf{A}^\dagger_\mu \mathbf{A}_{\bar{\sigma}'}\mathbf{A}^\dagger_{\bar{\tau}}\mathbf{A}_\nu \mathbf{A}^\dagger_{\bar{\tau}'}\right) b^\dagger_{\bar{\sigma}} b^\dagger_{\bar{\sigma}'} b_{\bar{\tau}} b_{\bar{\tau}'} \tag{33}$$

with $f_{\Omega_a} = 4\Omega_a^2/N_a$ renormalizing the two-boson part of $[V_B]_T$ by a factor which ranges from unity (full space) to 2 for a very small subspace.

Not all interactions satisfy (32); for example, the pairing interaction never does except in the full space. For the pairing interaction $\langle \mu\,|V|\,\nu\rangle = \delta_{\mu,0}\delta_{\nu,0}$, and $\mathbf{A}_0\mathbf{A}_0^\dagger = \frac{1}{2\Omega}$, the image (31) becomes (remembering $\Omega = 2\Omega_a\Omega_c$)

$$\langle \bar{\sigma}\bar{\sigma}'|v_p|\bar{\tau}\bar{\tau}'\rangle = \langle 0\,|V|\,0\rangle \left\{ \hat{\mathrm{N}}_0[1 - \frac{2}{\Omega}\hat{\mathrm{N}} + \frac{1}{\Omega} + \frac{\hat{\mathrm{N}}_0}{\Omega}] \right.$$

$$\left. + \sum_{\bar{\tau}\bar{\tau}'\neq 0,\bar{\sigma}} \mathrm{tr}\,(\bar{\mathbf{A}}_{\bar{\sigma}}\bar{\mathbf{A}}^\dagger_{\bar{\tau}}\bar{\mathbf{A}}_0\bar{\mathbf{A}}^\dagger_{\bar{\tau}'}) b^\dagger_{\bar{\sigma}} b^\dagger_0 b_{\bar{\tau}} b_{\bar{\tau}'} \right\}, \tag{34}$$

where $\hat{\mathrm{N}}$ is the total number of bosons, $\hat{\mathrm{N}} = \sum_{\bar{\tau}} b^\dagger_{\bar{\tau}} b_{\bar{\tau}}$, and $\hat{\mathrm{N}}_0 = b^\dagger_0 b_0$. The second term in the above is not Hermitian and can be transformed away by a similarity transformation, leaving the first term as a finite Hermitian image which gives the correct eigenvalues for all N.

The SO(8) and Sp(6) models [11] belong to a class of models which have a subspace for which (22) is valid. In these models the shell model orbitals have a definite angular momentum $\vec{\mathbf{j}}$ and are partitioned into a pseudo orbital angular momentum $\vec{\mathbf{k}}$ and pseudospin $\vec{\mathbf{i}}$, $\vec{\mathbf{j}} = \vec{\mathbf{k}} + \vec{\mathbf{i}}$. The amplitudes are then given as products of Clebsch-Gordon coefficients, $\left(A^\dagger_\alpha\right)_{ij} = \frac{(1+(-1)^{K+I})}{2}(k\,m_i, k\,m_j|K_\alpha\,M_\alpha)\,(i\,\mu_i, i\,\mu_j|I_\alpha\,\mu_\alpha)$, where K and I are the total pseudo orbital angular momentum and pseudospin respectively of the pair of nucleons. For the SO(8) model $\mathbf{i} = \frac{3}{2}$ and one considers the subspace of pairs with $K = 0$ $(p = 0)$, $(\bar{A}^\dagger_{\bar{\alpha}})_{ij} = \frac{(1+(-1)^I)}{2}(i\,\mu_i, i\,\mu_j|I_\alpha\,\mu_\alpha)$; in the Sp(6) model $\mathbf{k} = 1$ and one considers the subspace with $I = 0$ $(p = 1)$, $(\bar{A}^\dagger_{\bar{\alpha}})_{ij} = \frac{(1+(-1)^K)}{2}(k\,m_i, k\,m_j|K_\alpha\,M_\alpha)$. For these subspaces the completeness relation (22) holds, and the multipole-multipole interactions of satisfy (32)and yield finite, Hermitian images as defined in (33). Again, the pairing interaction can be made Hermitian and kept finite by a similarity transformation.

3.3 Hermitian Image

Although the truncated Hamiltonian will in general factorize into a non-Hermitian image times the truncated norm, we can make a similarity transformation $\mathcal{S}$ such that

$$\mathcal{S}[\hat{\mathcal{H}}_{\rm B}]_{\rm T}\mathcal{S}^{-1} = \bar{\rm H}\mathcal{S}[\hat{\mathcal{N}}_{\rm B}]_{\rm T}\mathcal{S}^{-1}, \tag{35}$$

where $\bar{H}$ is Hermitian, as was done for the pairing interaction in the last section. To see this write

$$\mathcal{S} = \hat{\mathcal{U}}[\widetilde{\mathcal{N}}_{\beta}]_{\rm T}^{-\frac{1}{2}}. \tag{36}$$

where $\hat{\mathcal{U}}$ is a unitary operator. Because the norm may be a singular operator, $[\mathcal{N}_B]_T^{-\frac{1}{2}}$ is calculated from the norm only in the physical subspace, with the zero eigenvalues which annihilate the spurious states retained. Then $\bar{H}_B$ does not mix physical and spurious states. Then clearly

$$\mathcal{S}\hat{\mathcal{N}}_{\rm B}\mathcal{S}^{\dagger} = {\rm Q} \tag{37}$$

where Q is unity in the physical space and zero in the spurious space and the image is given by

$$\bar{H} = \mathcal{S}[\hat{\mathcal{H}}_{\rm B}]_{\rm T}\mathcal{S}^{\dagger} \tag{38}$$

Clearly $\bar{H}$ is Hermitian and doesn't mix spurious states, but will not be finite in general. However from (36) we see there is freedom in the choice of $\mathcal{S}$ and we would like to choose $\mathcal{S}$ such that $\bar{H}$ is rapidly convergent. We are investigating possible choices and testing their convergence.

4. Summary

We have reviewed the map of fermion pairs into bosons so that the physical states (those with non-zero norm) of the resulting boson Hamiltonian have the same spectrum as the original fermion Hamiltonian. We have shown that this boson Hamiltonian, while infinite and not convergent, can be factored into a finite boson image times the norm operator, and this boson image is finite while the norm operator is infinite and not convergent. In fact the boson image Hamiltonian is a one plus two boson operator if the original fermion Hamiltonian is one plus two fermion operator.

Since a truncated space in terms of a few collective pairs is the more relevant space physically (for example, the IBM), we have derived the truncated boson Hamiltonian which gives the same spectrum as the truncated fermion Hamiltonian, and does not mix physical and spurious states, where the number of the latter will have decreased if the truncation is a good one. We derived sufficient conditions for this Hamiltonian to factor into a finite image times the norm in the truncated space. We gave an example of a truncation in which these conditions are satisfied for a one plus two boson operator image. However we showed that this image is Hermitian only for certain fermion Hamiltonians. In general the truncated boson Hamiltonian will factor into an infinite Hermitian image times the truncated norm after the similarity transformation. Nevertheless there may be enough freedom in the choice of the similarity transformation to determine a boson image which is rapidly convergent. This investigation is in progress.

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