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1 of 1

Unstable Solitary-Wave Solutions of the Generalized Benjamin–Bona–Mahony Equation [1]

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Abstract. The evolution of solitary waves of the gBBM equation is investigated computationally. The experiments confirm previously derived theoretical stability estimates and, more importantly, yield insights into their behavior. For example, highly energetic unstable solitary waves when perturbed are shown to evolve into several stable solitary waves.

1. Introduction. The Benjamin–Bona–Mahony equation [1] is a model for unidirectional dispersive long waves with finite amplitude. It is a particular case of the generalized Benjamin–Bona–Mahony (gBBM) equation which is known to have solitary-wave solutions. In this paper a numerical method is used to approximate solutions to the initial value problem for the gBBM equation

$$u_t + u_x + (u^p)_x - u_{txx} = 0, \tag{1.1}$$

where p is a positive integer, with the initial condition

$$u(x,0) = u^0(x), \tag{1.2}$$

with the aim of investigating evolutions emanating from perturbations of unstable solitary-wave solutions for large p.

The equation (1.1) was studied in [4], where they analyzed the stability of the 1-parameter family of solitary-wave solutions, which may be written in the form

$$u_c(x,t) = \{A\,\mathrm{sech}[K(x-ct)]\}^{2/(p-1)}, \tag{1.3}$$

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where $c > 1$ is the speed and K and A are given by

$$K = \left(\frac{p-1}{2}\right)\sqrt{\frac{c-1}{c}} \qquad \text{and} \qquad A = \frac{(p+1)(c-1)}{2} . \tag{1.4}$$

One of the primary results of [4] is that for $p > 5$, the solitary-wave solution u_c is stable for $c > c_0(p)$, but unstable for $c \leq c_0(p)$ where c_0 is given in example 3.1 from [4]. In particular, we note that $c_0 = 1.301$ for $p = 8$ which corresponds to the examples reported on in Section 3 below.

2. Numerical Approximations. In order to accurately reflect the long-time behavior of instabilities and simulate solutions on the real line, the equation (1.1) is first scaled (in space) from an interval $[0, 2500]$ into $[0, 2\pi]$. The initial value problem (1.1) and (1.2) is then solved on a discrete grid of points belonging to $x \in [0, 2\pi]$ and $t > 0$ using periodic boundary conditions. The equation is projected into Fourier space using de-aliased Fourier pseudo-spectral collocation techniques [2]. The resulting finite-dimensional system of non-stiff ordinary differential equations for the Fourier coefficients is then integrated forward in t using a variable-order adaptive Adams-Bashforth multistep method. Fourier transforms are performed using the standard FFT package FFTPACK. The ODE solver used is Shampine and Gordon's DEABM package [3]. The time steps between calls to DEABM are fixed in size to enable better control of the dissipation of the scheme. The veracity of the scheme was tested by comparing numerical solutions to exact, stable solutions of the linear and nonlinear dispersive equations. Several norms are monitored for accuracy.

3. Numerical Simulations with Unstable Solitary Waves. As stated in Section 1 and proved in [4], the solitary-wave solutions u_c of (1.1) are unstable if $p > 5$ for values of c close enough to, but larger than 1. In this section, the results of some experiments are given for the case $p = 8$ in which u^0 is a perturbation of u_c with values of c taken very close to 1.

In the first numerical simulation, we use as initial data the slightly perturbed solitary wave

$$u^0(x) = \lambda \left\{A \operatorname{sech}[K(x - x_0)]\right\}^{2/(p-1)} \tag{3.1}$$

with $p = 8$, $\lambda = 1.05$, $x_0 = \pi$ and a speed of $c = 1.001$. The values of K and A are then given by (1.4). The resulting evolution is presented in Figure 1 where the profile of $u(x,t)$ is plotted at 4 different times. (Note that the solution travels around the interval due to the periodic boundary conditions.) In the first graph at $t = 0$, the initial condition is pictured. Next at $t = 1600$ followed by $t = 3200$, the effects of the perturbation in the unstable solution are clearly evident: the amplitude grows, symmetry is lost, and a dispersive tail forms. From $t = 3200$ to $t = 4800$, a stable soliton emerges and leaves the remaining part of the solution behind. The soliton has a measured speed of about $c = 1.71$ which corresponds to an amplitude of $A = 1.18$, in agreement with the final picture.

This evolution process occurs over a long time interval due to the scaling involved and the energy present in the initial condition. The energy is given by the square of the H^1-norm

$$E(v) = \|v\|_1^2 = \int_0^{2\pi} v^2 + (v')^2 \, dx \tag{3.2}$$

which is also an invariant for the equation (1.1). In order to increase the speed at which this process occurs and to obtain more interesting dynamics, it is helpful to increase the amount of energy in the initial condition. This may be done by decreasing the speed and using values of c closer to 1 since $E(u_c) \to \infty$ as $c \to 1$ from above.

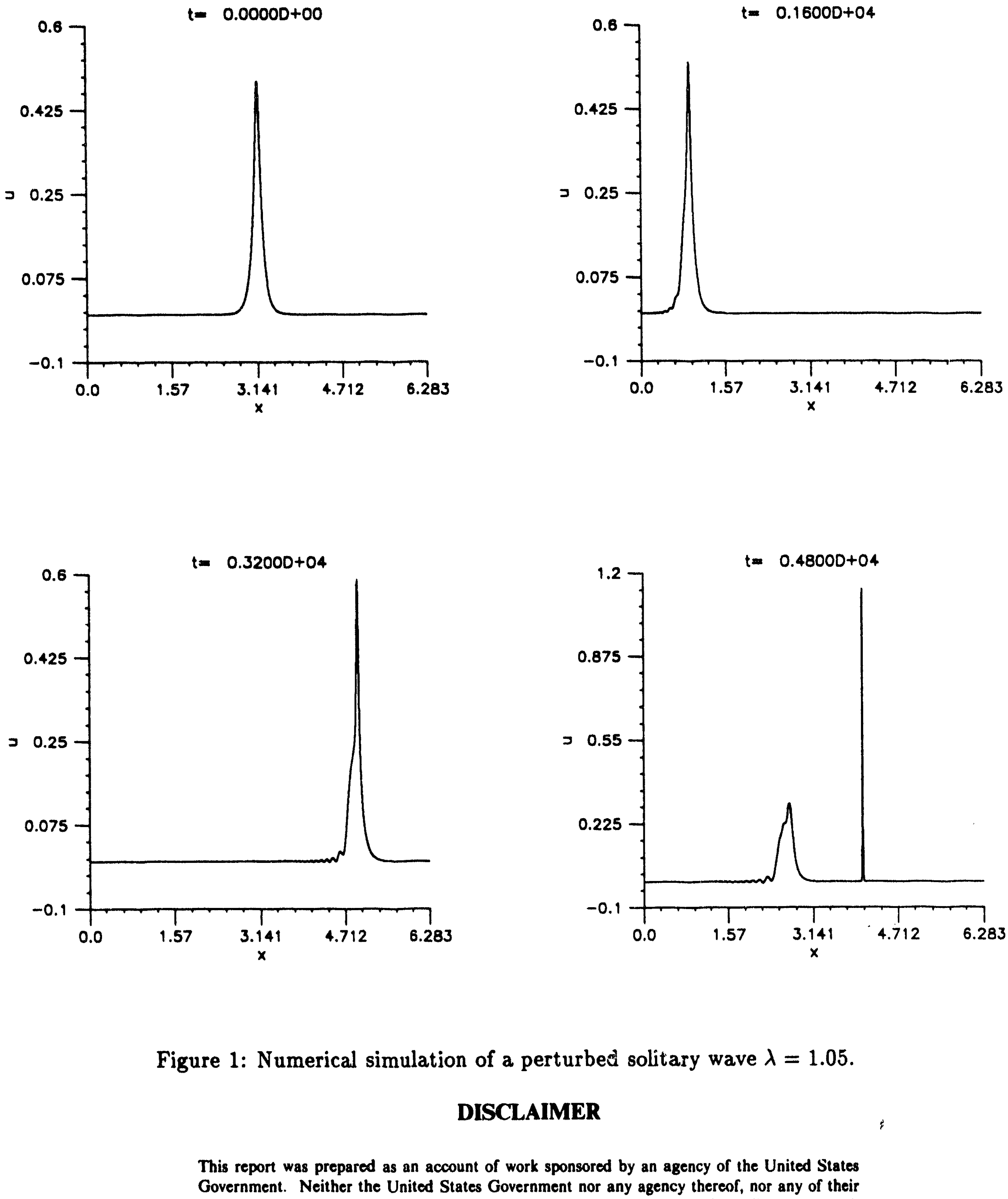

Figure 1: Numerical simulation of a perturbed solitary wave $\lambda = 1.05$.

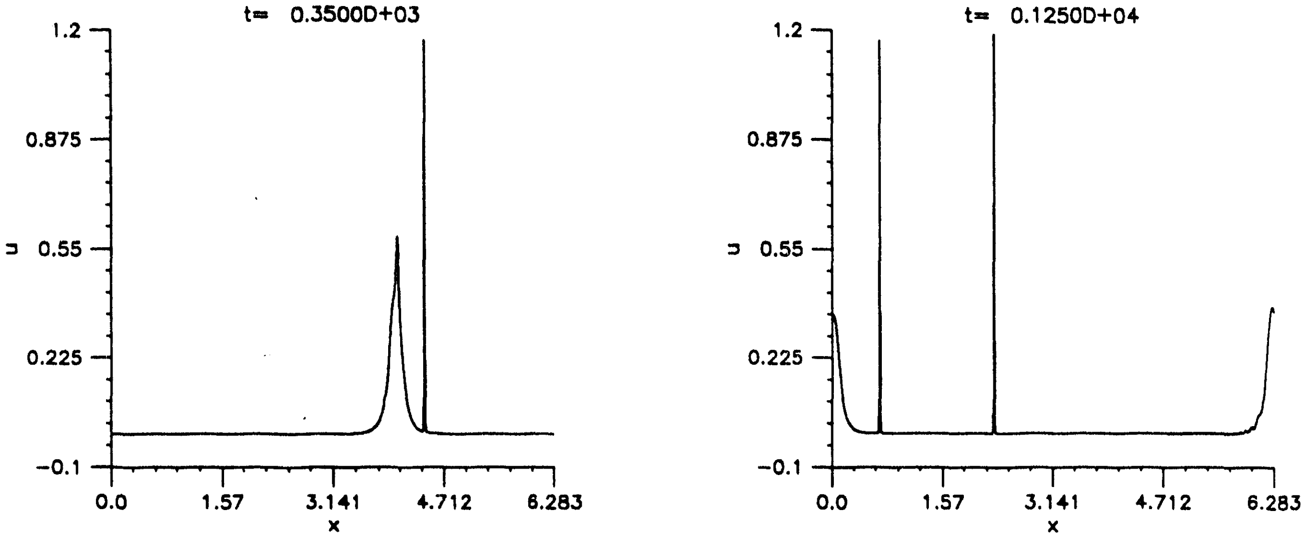

Figure 2: Numerical simulation of a perturbed solitary wave $\lambda = 1.40$.

Instead we increase the energy in the initial condition by means of increasing the value of the perturbation factor λ. In Figure 2, we show the evolution originating from the same initial condition as above, except now $\lambda = 1.40$. In this case the first solitary wave is already developed at $t = 350$, and leaves the remaining structure almost intact. Indeed, there is a sufficient amount of energy left in this structure to permit a second solitary wave to emerge at $t = 1250$. By either increasing the value of λ further or making c closer to 1, it is possible to get evolutions with several stable solitons present.

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