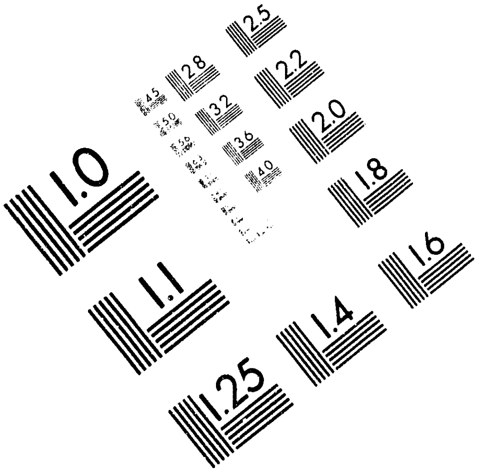

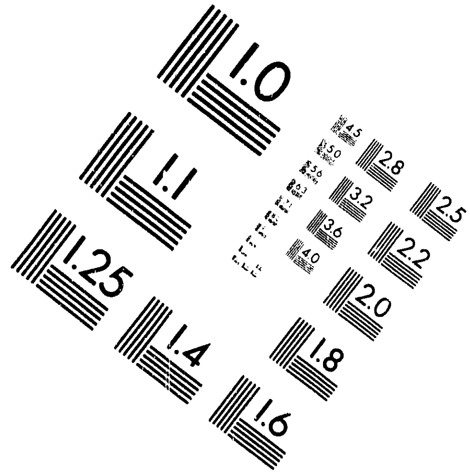

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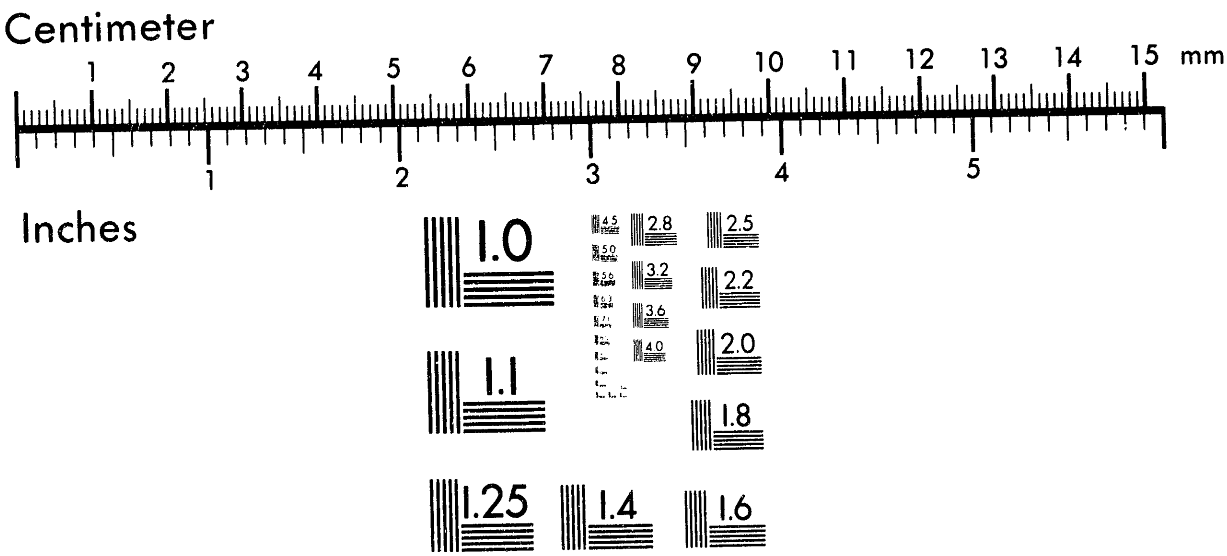

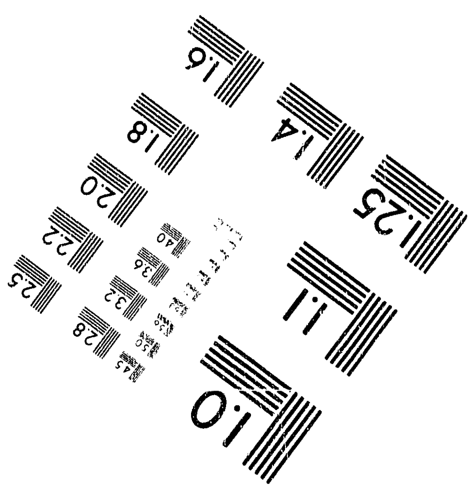

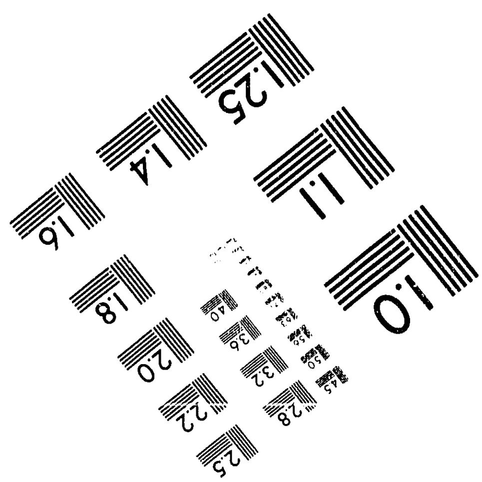

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THE FRACTIONAL QUANTUM HALL EFFECT AND THE ROTATION GROUP

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INTRODUCTION

When a magnetic field B is applied perpendicular (z- direction) to a two dimensional surface with a current in the x-direction, j_x, an electric field current is induced in the y- direction, E_y. This effect was discovered a century ago by Hall[1-3]. The Hall (transverse) resistivity is given by

$$\rho_{yx} = \frac{E_y}{j_x} = \frac{B}{\eta ec}, \tag{1}$$

where η is the electron density, e the electric charge, and c the velocity of light. Generally this linear dependence of the Hall transverse resistivity is observed. However, at low temperatures there is a plateau in the Hall resistivity when the magnetic field is the inverse of an integer times the electron density. Furthermore the longitudinal resistivity $\rho_{xx} = E_x/j_x$ goes through a minimum at these values of the magnetic field. (See for example Fig. 1.3 in Reference 2) These phenomena can be explained by taking into account the quantum motion of the electrons. The electrons move in quantized degenerate single particle levels called Landau levels. When these levels are filled the many - electron wave function becomes localized and the Hall resistivity remains constant producing the plateaus. Furthermore the longitudinal resistivity becomes small because there is a large energy gap for exciting electrons due to the closed shells. These closed shells are shown to occur when the magnetic field is the inverse of an integer times the density. The quantization of the Hall resistance for a filled Landau level is now recognized as a consequence of gauge invariance[4].

However, plateaus in the Hall resistivity and minima in the longitudinal resistivity have also been observed at low temperatures and high magnetic field when the magnetic field is a ratio of integers times the density. This fractional quantum Hall effect (FQHE) is thought to be the result of correlations between electrons induced by their mutual interactions. In this paper we explore the consequences of these correlations numerically and analytically. Following Haldane[5], we map the problem onto the surface of a sphere, so that the number of states in the first Landau level is finite. In the following three sections we review this geometry, give the matrix elements of the Coulomb interaction, and compare exact wave functions with the Laughlin/Haldane ground state ansatz[6,5] for fractional fillings 1/m, m an odd integer. While this ansatz accounts for many of the J=0 ground states found in numerical diagonalizations, it does not explain all such states. To make progress on this problem we study four-particle states in Section 5. A complete characterization of the most general four-particle J=0 wave function is given, including an algebraic enumeration of the number of such states. (The J=0 states are the analogs of translationally invariant states in the plane.) This construction provides one example of a 2/3-filled state, the particle-hole conjugate of the 3-particle Laughlin/Haldane m=3 state. This result suggests a generalization that may give the particle-hole conjugates of all Laughlin/Haldane states of arbitrary N: the construction depends on an equivalence of single-particle spinors of rank N/2 and ones formed by coupling the elementary spinors of N different particles. We further generalize this result to produce a wave function ansatz that may account for other fractional fillings of physical interest. In the last section we summarize our results and their implications.

THE QUANTUM HALL EFFECT IN A SPHERICAL GEOMETRY

Consider a two-dimensional electron gas of N particles each with mass M on a sphere of radius R with a magnetic monopole at the center. The total magnetic flux, Φ, is required to be an integral multiple of the elementary flux, $\Phi_0 = \frac{hc}{e}$, where e is the magnetic charge, as required by Dirac's monopole quantization condition,

$$\Phi = 4\pi R^2 B = 2S\Phi_0. \tag{2}$$

Therefore, $R = \sqrt{S}d$, where d is the magnetic length, $d = \sqrt{\frac{\hbar c}{eB}}$. Using the fact that the electron density is $\eta = \frac{N}{4\pi R^2}$, the classical Hall resistivity (1) becomes

$$\rho_{yx} = \nu^{-1}\frac{h}{e^2}, \tag{3a}$$

where

$$\nu = \frac{N}{2S}\,. \tag{3b}$$

Haldane[5] has studied the quantum mechanics of this problem. The single-particle Hamiltonian is

$$H = \frac{\vec{\Lambda}\cdot\vec{\Lambda}}{2MR^2}, \tag{4a}$$

where

$$\vec{\Lambda} = \vec{L} + [e\vec{r}\times\vec{A}] \tag{4b}$$

is the dynamical angular momentum, $\vec{L}$ the orbital angular momentum, and $\vec{A}$ is the gauge field associated with the magnetic field,

$$\vec{\nabla}\times\vec{A} = B\hat{\Omega}, \tag{5}$$

and $\hat{\Omega}$ is the unit radial vector to the sphere,

$$\hat{\Omega} = \frac{\vec{r}}{R} = \{\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\}. \tag{6}$$

The eigenvalues of this Hamiltonian are

$$E_n(S) = \frac{\hbar^2[(j(j+1)+(2j+1)S]}{2M\,R^2}, j = 0, 1, ... \tag{7}$$

and the eigenvectors are Wigner D-functions with angular momentum $\ell = S+j, j = 0, 1\ldots$, and body- fixed angular momentum projection equal to S, since only angular momenta $\geq S$ are allowed,

$$\Psi^{(S)}_{\ell,\mu}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}}D^{(\ell)}_{\mu,S}(\phi,\theta,0). \tag{8}$$

Hence in the spherical geometry the Landau levels are spherical shell model states with single particle angular momentum ℓ and projection m, and energy proportional to $(\ell(\ell+1)-S^2)$, with the lowest Landau level having $\ell = S$ (one-half the total magnetic flux through the surface) , but there is no upper bound on the allowed angular momenta. Since $m = \ell, \ell-1, ... -\ell$, each level has a $2\ell+1$ degeneracy. For $\ell = S$, the eigenfunctions (8) can be written in terms of powers of the spinor wave functions on the sphere,

$$u_{\frac{1}{2}}(\theta,\phi) = \cos\left(\frac{\theta}{2}\right)e^{\frac{i\phi}{2}},$$
$$u_{-\frac{1}{2}}(\theta,\phi) = \sin\left(\frac{\theta}{2}\right)e^{\frac{-i\phi}{2}}, \tag{9}$$

as

$$\Psi^{(S)}_{S,\mu}(\theta,\phi) = (-1)^{S-\mu}\sqrt{\frac{(2S+1)!}{4\pi(S+\mu)!(S-\mu)!}}(u_{\frac{1}{2}}(\theta,\phi))^{S+\mu}(u_{-\frac{1}{2}}(\theta,\phi))^{S-\mu}. \tag{10}$$

In the limit $R\to\infty$, we can map the sphere onto the plane, $x = rcos(\phi), y = rsin(\phi)$, by the transformation

$$r = \sqrt{x^2+y^2} = 2Rcot\left(\frac{\theta}{2}\right). \tag{11}$$

Then the three dimensional rotation group maps into the two- dimensional Euclidean group,

$$L_x \overset{R\to\infty}{\longrightarrow} R\,P_y, \tag{12a}$$
$$L_y \overset{R\to\infty}{\longrightarrow} -R\,P_x, \tag{12b}$$
$$L_z \overset{R\to\infty}{\longrightarrow} L_z, \tag{12c}$$

where P_i is the linear momentum in direction i, and the spherical Landau levels (8) map into the planar Landau levels.

The many-body wave functions for N electrons are determined by diagonalizing the Coulomb interaction between pairs of electrons in this shell model basis. This interaction will be rotationally invariant and will hence conserve the total angular momentum of the N electrons. The states with total angular momentum zero will map onto translationally invariant states in the plane and are the ones we shall focus on in this paper.

THE COULOMB INTERACTION

Using the mapping defined in (11), the Coulomb interaction between two electrons on a plane maps into the form

$$V = \frac{e^2}{|\vec{r_1} - \vec{r_2}|} \overset{R\to\infty}{\longrightarrow} \frac{e^2}{2R\sin(\Theta/2)} \equiv v, \tag{13}$$

on the sphere, where e is the electric charge and Θ is the angle between the two electrons on the sphere. The antisymmetric matrix elements of this interaction with respect to single-electron wave functions given in (8) for the two electrons coupled to angular momentum J are

$$< [\ell_1'\ell_2']J|v|[\ell_1\ell_2]J >_a =$$

$$(-1)^{\ell_1+\ell_1'+J}\frac{1}{2R}\frac{\hat{\ell}_1'\hat{\ell}_2'\hat{\ell}_1\hat{\ell}_2}{\hat{J}}\Bigg\{\sum_L \begin{pmatrix} \ell_1' & \ell_1 & L \\ S & -S & 0\end{pmatrix}\begin{pmatrix} \ell_2' & \ell_2 & L \\ S & -S & 0\end{pmatrix}\begin{Bmatrix} \ell_1 & \ell_1' & L \\ \ell_2' & \ell_2 & J\end{Bmatrix} - (-1)^{\ell_1+\ell_2+J}\left[\ell_1 \leftrightarrow \ell_2\right]\Bigg\}, \tag{14}$$

where $\hat{\ell} = \sqrt{2\ell+1}$ and $\begin{pmatrix} \ell & \ell & L \\ S & -S & 0\end{pmatrix}$ is a Wigner coefficient and $\begin{Bmatrix} \ell & \ell & L \\ \ell & \ell & J\end{Bmatrix}$ is a 6-j symbol. For a large magnetic field the single particle energy in (7) is large and admixtures between different Landau levels becomes small. We neglect this admixing by the Coulomb interaction and only consider the interaction in the lowest orbit, $\ell = S$. We can then do the sum in (14) analytically using techniques used in for other sum rules for 3-j symbols derived recently[7]. The result is

$$< [SS]J|v|[SS]J >_a = \frac{e^2((-1)^{2S-J}-1)(2S+1)^2}{2R}\sum_L \begin{Bmatrix} S & S & L \\ S & S & J\end{Bmatrix}\begin{pmatrix} S & S & L \\ S & -S & 0\end{pmatrix}^2$$

$$= \frac{e^2(1-(-1)^{2S-J})}{4R}\frac{\left[(2S+1)!\right]^2\Gamma(2S-J+\frac{1}{2})\Gamma(2S+J+\frac{3}{2})}{\left[\Gamma(2S+\frac{3}{2})\right]^2(2S-J)!(2S+J+1)!}. \tag{15}$$

For S large this becomes

$$< [SS]J|v|[SS]J >_a \approx \frac{e^2(1-(-1)^{2S-J})}{2d}\sqrt{\frac{S}{[(2S)^2-(J+\frac{1}{2})^2]}}. \tag{16}$$

HALDANE ANSATZ AND SHELL MODEL DIAGONALIZATION

Following Laughlin[6], Haldane[5] proposed that the lowest antisymmetric angular momentum zero state for N electrons is of the form

$$\chi_{N,m}^{(2S)} = \left\{ \prod_{i<j=1}^{N} u(i) \cdot u(j) \right\}^{m}, m \; odd, \tag{17}$$

For each electron there are $m(N-1)$ powers of the spinor wave function and hence, from Eq. (10), $2S = m(N-1)$ for each electron. For $N \to \infty$, and using (3b), these wave functions lead to fractional fillings $\nu \to \frac{1}{m}$, m *odd*.

Since each pair of electrons has m powers coupled to angular momentum zero, the maximum pairwise angular momentum is $J_{i \neq j} \leq 2S - m$. As we have seen in the last section the Coulomb interaction is very repulsive for pairs coupled to the highest angular momenta, there is good reason to believe that this ansatz may be a good one.

Table 1. Ground-state wave function overlaps $| < \psi_{\mathrm{HALDANE}} | \psi_{\mathrm{EXACT}} > |$ for the Haldane/Laughlin and exact wave functions for fractional fillings $1/m$, m an odd integer. Corresponding correlation energies (that is, the contributions to the energy from the electron-electron interaction) are also given (units e^2/d).

S	N	m	$\vert < \psi_H \vert \psi_E > \vert$	E^{corr}_{EXACT}	$E^{corr}_{HALDANE}$
3	3	3	1.000	1.087	1.087
9/2	4	3	0.998	1.871	1.872
6	5	3	0.999	2.805	2.806
15/2	6	3	0.996	3.871	3.873
9	7	3	0.996	5.060	5.062
21/2	8	3	0.995	6.363	6.365
12	9	3	0.994	7.771	7.774
5	3	5	1.000	0.817	0.817
15/2	4	5	0.984	1.410	1.413
10	5	5	0.997	2.127	2.128
25/2	6	5	0.949	2.938	2.943
7	3	7	1.000	0.681	0.681
21/2	4	7	0.974	1.177	1.179
14	5	7	0.997	1.780	1.781
9	3	9	1.000	0.595	0.595
27/2	4	9	0.974	1.030	1.032

Using a diagonalization program based on the Lanczos algorithm, we computed exact wave functions for arbitrary fillings through $2S = 22$, and for fillings $1/m$ through $2S = 27$. In Tables 1 and 2 we compare the exact Haldane and Laughlin wave functions for various fractional fillings and find good agreement for the overlaps and ground-state energies, and reasonable agreement for the two-body density matrices. In particular, Table 2 shows that the probability of finding pairs of electrons with $J_{i=j} > 2S - m$ in the exact wave function is very small. However plateaus in the Hall resistivity for fractional fillings other than $\frac{1}{m}$, m odd, have been observed. Also, not all of the J=0 ground states found in our numerical calculations correspond to odd-m fillings. We shall attempt to generalize this ansatz to wave functions with more general fractional fillings by studying few-electron systems in detail in the next section.

THE FOUR PARTICLE SYSTEM

In order to understand the more general fractional fillings, we examine few-electron systems. The two and three electron systems have only one angular momentum zero state and they are given by the Laughlin/Haldane wave function (17). The four electron system is the first for which more than one angular momentum zero state can exist for a given S and we study that system in detail in this section.

There are two orthonormal wave functions for four spin 1/2 particles with total angular momentum zero; one in which the particles 1 and 2 (and 3 and 4) are coupled to intermediate angular momentum zero, the other with them coupled to angular momentum one:

$$\Phi_J(12;34) = \left[[u(1)u(2)]^{(J)}[u(3)u(4)]^{(J)}\right]^{(0)}, J = 0,1 \tag{18}$$

Under the symmetric group on four particles, these two states form the basis for the irreducible representation (IR) $[2^2]$, the "mixed" symmetry representation. The remaining states with four spinor particles states coupled to angular momentum two and one form the basis for the symmetric, [4], and other mixed symmetry state, [3,1], respectively.

We are interested in generating all possible antisymmetric states with angular momentum zero. It can be shown that all such states can be produced by taking powers of the mixed symmetry states in (18) and there is no need to consider the [4] and [3,1] representations. We delete the proof in this paper for lack of space.

The product of the mixed IR with itself gives,

$$[2^2] \times [2^2] = [4] + [2^2] + [1^4], \tag{19}$$

while the product of the symmetric and antisymmetric representation with any other representation gives:

$$\begin{aligned} [4] \times [4] &= [4], \\ [4] \times [22] &= [22], \\ [4] \times [1^4] &= [1^4], \\ [1^4] \times [22] &= [22], \\ [1^4] \times [1^4] &= [4]. \end{aligned} \tag{20}$$

Table 2. The two-body ground-state density matrix ϕ_{J_0} for exact and Haldane wave functions, defined by

$$< 0^+_{gs} \| \frac{1}{2} \sum_{ij} 0_{ij} \| 0^+_{gs} > = \sum_{J_0} \phi_{J_0} < (S^2) J_0 \| 0_{12} \| (S^2) J_0 >,$$

where 0_{12} is any scalar two-body operator (e.g., the residual Coulomb interaction).

(S,N,m)		J0=0	2	4	6	8	10	12	14	16	18	20	22	24
(9/2, 4, 3)	Exact	0.185	0.194	0.468	1.101	0.002								
	Haldane	0.254	0.212	0.398	1.131	0.0								
(15/2, 6, 3)	Exact	0.191	0.379	0.415	0.428	0.444	0.733	1.194	0.002					
	Haldane	0.154	0.332	0.429	0.497	0.451	0.659	1.228	0.0					
(21/2, 8, 3)	Exact	0.115	0.272	0.402	0.518	0.586	0.590	0.567	0.598	0.888	1.301	0.002		
	Haldane	0.129	0.290	0.398	0.488	0.561	0.609	0.623	0.597	0.818	1.337	0.0		
(15/2, 4, 5)	Exact	0.025	0.026	0.062	0.191	0.514	0.623	0.013	0.0					
	Haldane	0.100	0.070	0.089	0.164	0.373	0.730	0.0	0.0					
(25/2, 6, 5)	Exact	0.164	0.305	0.274	0.185	0.114	0.089	0.111	0.207	0.407	0.655	0.576	0.016	0.0
	Haldane	0.073	0.156	0.191	0.205	0.207	0.199	0.182	0.219	0.312	0.495	0.721	0.0	0.0
(21/2, 4, 7)	Exact	0.003	0.003	0.007	0.024	0.079	0.234	0.475	0.370	0.020	0.0	0.0		
	Haldane	0.044	0.029	0.031	0.046	0.083	0.175	0.365	0.494	0.0	0.0	0.0		

The quadratic wave function for the symmetric representation [4] is, where we have dropped the arguments of the wave functions in (18) for convenience,

$$\Phi_S^{(2)} = \frac{1}{\sqrt{2}}(\Phi_1^2 + \Phi_0^2), \tag{21}$$

where the superscript (2) refers to the power in the single- particle spinors. The quadratic mixed symmetry states will be given by

$$\Phi_1^{(2)} = \frac{1}{\sqrt{2}}\left(\Phi_0^2 - \Phi_1^2\right), \Phi_0^{(2)} = \sqrt{2}\,\Phi_1\Phi_0, \tag{22}$$

The quadratic antisymmetric representation vanishes because the wave functions are the same in each representation.

We can take the product of the new mixed symmetry state (22) with the original one in (18) and repeat the process creating a trilinear symmetric and mixed symmetry state

$$\Phi_S^{(3)} = \frac{\Phi_1}{2}\left(3\Phi_0^2 - \Phi_1^2\right), \tag{23a}$$

$$\Phi_1^{(3)} = \frac{1}{2}\Phi_1\Phi_S^{(2)}, \quad \Phi_0^{(3)} = \frac{1}{2}\Phi_0\Phi_S^{(2)}, \tag{23b}$$

$$\Phi_A^{(3)} = \frac{\Phi_0}{2}\left(3\Phi_1^2 - \Phi_0^2\right). \tag{23c}$$

Besides generating the first antisymmetric wave function with the third power of the spinors, we also see that the third power mixed symmetry state is the linear mixed symmetry state times the quadratic symmetric state. From this fact and the rules given in (19) and (20), we conclude that all the possible antisymmetric states are given by

$$\Phi_{A;m,p,n}^{(2s)} = \left(\Phi_A^{(3)}\right)^m \left(\Phi_S^{(3)}\right)^{p_1} \left(\Phi_S^{(2)}\right)^{p_2}, \tag{24a}$$

where

$$2s = 3m + 3p_1 + 2p_2; \ m \ odd\ integer; \ p_i \ integer\ . \tag{24b}$$

The value of p_2 is restricted to 0, 1, or 2 because of the identity

$$(\Phi_S^{(3)})^2 = \frac{1}{\sqrt{2}}\left(\Phi_S^{(2)}\right)^3 - \left(\Phi_A^{(3)}\right)^2, \tag{25}$$

which follows from the fact that the square of an antisymmetric wave function is a symmetric wave function and hence can be written in terms of the symmetric functions $\Phi_S^{(3)}$ and $\Phi_S^{(2)}$ by virtue of (21), (23a) and (23c). Therefore all the antisymmetric four particle states with total angular momentum zero for particles with single-particle angular momentum S are given by the states in (24) which are linearly independent but not necessarily orthogonal. This implies that the number of antisymmetric four particle angular momentum zero is

$$N_0(S^4) = \left\{ \frac{2S + 3\frac{(1-(-1)^{2S})}{2}}{6} \right\}, \tag{26}$$

where $\{x\}$ is the largest integer not exceeding x. This is the first time such an enumeration has been given by an algebraic equation as opposed to a numerical counting procedure.

We can relate the wave functions in (24) to the Laughlin/Haldane wave function by writing Φ_J in terms of wave functions in which different pairs of particles have intermediate zero angular momentum coupling. For this purpose we re-introduce the argument of the wave function, and relate them to the wave functions in which different pairs of particles are coupled to intermediate angular momentum zero:

$$\Phi_1(12;34) = \frac{1}{\sqrt{3}}\left(\Phi_0(13;24) + \Phi_0(23;14)\right), \tag{27a}$$

$$\Phi_0(12;34) = \Phi_0(13;24) - \Phi_0(23;14). \tag{27b}$$

Inserting these results into (23c) we find

$$\Phi_A^{(3)} = 2\Phi_0(12;34)\Phi_0(13;24)\Phi_0(23;14) = \frac{1}{4}\prod_{i<j} u(i)\cdot u(j); \tag{28}$$

this is just the Laughlin/Haldane wave function for four particles and m=1. Therefore the wave functions in (24) with $p_1 = p_2 = 0$ are the general Laughlin/Haldane wave functions for N=4. By the same token

$$\Phi_S^{(3)} \sim \Big[(u(1)u(2)u(3)u(4))^{(2)} \otimes (u(1)u(2)u(3)u(4))^{(2)} \otimes (u(1)u(2)u(3)u(4))^{(2)}\Big]^{(0)} \tag{29a}$$

$$\Phi_S^{(2)} \sim \Big[(u(1)u(2)u(3)u(4))^{(2)} \otimes (u(1)u(2)u(3)u(4))^{(2)}\Big]^{(0)} \tag{29b}$$

That is, these wave functions have the spinors for each of the four electrons coupled symmetrically to maximum angular momentum and then each symmetric group coupled symmetrically (via the 3-j symbols $\begin{pmatrix} 2 & 2 & 2 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 2 & 0 \\ \mu_1 & \mu_2 & 0 \end{pmatrix}$) to total angular momentum zero. This equivalence between symmetric wavefunctions generated from the symmetric representation and the mixed representation is a reflection of the statement made at the beginning of this section that we need not consider both the linear symmetric and mixed representations in generating the high antisymmetric wave functions. The states with $p_i \neq 0$ are new, and one would anticipate that these would generalize to large-N states with new fractional fillings. In particlular, the S=3 N=4 state is unique and therefore must be the 2/3-filled particle-hole conjugate of the S=3 N=3 m=3 Laughlin/Haldane state.

A GENERALIZATION OF THE LAUGHLIN/HALDANE WAVE FUNCTION

These results led us to a number of conclusions that will summarize here and present in detail elsewhere[8].

1) One can decompose the Laughlin/Haldane wave function (Eq. (17)) into a product of antisymmetric and symmetric terms.

$$\chi_{N,m}^{(2S)} = \left\{ \prod_{i<j=1}^{N} u(i) \cdot u(j) \right\} \left\{ \prod_{i<j=1}^{N} u(i) \cdot u(j) \right\}^{m-1} \tag{30a}$$

$$= \left\{ \prod_{i<j=1}^{N} u(i) \cdot u(j) \right\} [\Psi^{(\bar{N}/2)}(1) \otimes \cdots \otimes \Psi^{(\bar{N}/2)}(N)]_{J=0} \tag{30b}$$

where $2S = m(N-1)$ for fractional fillings $1/m$. The single-particle wave functions $\Psi^{(\bar{N}/2)}(i)$, of rank $\bar{N}/2 = \frac{(m-1)}{2}(N-1)$, are defined in Eq. (10). The symmetric angular momentum coupling coefficient involved in the second bracket is a generalization of the 3-j symbol in which N tensors of rank $\bar{N}/2$ are coupled to zero. (We will relate $\bar{N}$ to the particle number of a different state below.) We have derived this coefficient and the recursion relations it satisfies.

(2) A manipulation of the Laughlin/Haldane state for S and $m = 1$, the closed shell, suggests that the particle-hole conjugate state (fractional filling $(m\text{-}1)/m$) with particle number $(2S+1) - N = (m-1)(N-1) = \bar{N}$ may have the form

$$\bar{\chi}_{\bar{N},m}^{(2S)} = \left\{ \prod_{i<j=1}^{\bar{N}} u(i) \cdot u(j) \right\} [(u(1) \cdots u(\bar{N}))_{\bar{N}/2} \otimes \cdots \otimes (u(1) \cdots u(\bar{N}))_{\bar{N}/2}]_{J=0} \tag{31}$$

The $(u(1) \cdots u(\bar{N}))_{\bar{N}/2}$ is an obviously symmetric quantity formed by maximally coupling the N distinct elementary spinors $u(i)$ to $\bar{N}/2$. N copies of these appear in the second bracket. Thus our introduction of $\bar{N}/2$ as the label on the spinors in Eq. (30b) anticipated its identification with the particle number of the $(m\text{-}1)/m$ filled state. The coupling $\otimes \cdots \otimes$ is identical to that in Eq. (30a).

If this ansatz is correct, there is a simple equivalence between the Laughlin/Haldane state and its $(m\text{-}1)/m$-filled particle-hole conjugate: the symmetric part of the latter is obtained from the symmetric part of the former by replacing the $\Psi^{(\bar{N}/2)}(i)$ by the conjugate form $(u(1) \ldots u(\bar{N}))_{\bar{N}/2}$.

We note in passing that a relation between such spinors is implicit in the Laughlin/Haldane wave function: the operator[8] which generates a Laughlin/Haldane state of $(N+1, m)$ from the state (N, m) involves a dot product between similar conjugate spinors. We used such a recursion relation to produce the Laughlin/Haldane wave functions from which Tables 1 and 2 were generated.

3) Returning to our four-particle results, we recall that the $m = 1, p_1 = 1, S = 3$ state, built from $\Phi_S^{(3)}$, has the form demanded by Eq. (31), since it is the 2/3-filled state conjugate to the $N = 3, m = 3, S = 3$ Laughlin/Haldane state (see Eq. (29a)). Likewise the $m = 1, p_2 = 1, S = 5/2$ state, built from $\Phi_S^{(2)}$, is the 4/5-filled state conjugate to the $N = 2, m = 5, S = 5/2$ Laughlin/Haldane state (see Eq. (29b)).

(4) However, in our numerical calculations for four-particle states, $J = 0$ ground states are found that correspond to neither Laughlin/Haldane states nor their particle-hole conjugates. For example, for integer S we find

$$(N=4, S=6): \qquad m=3, p_1=1, p_2=0$$

$$(N=4, S=9): \qquad m=5, p_1=1, p_2=0$$

The quantum number assignments are not unique, since multiple $J=0$ states exist, but only with this choice would these states and the $N=4, S=3$ state share a common structure. This suggests a generalization of Eq. (31)

$$\chi^{2S}_{\bar{N},m,\bar{m}} = \left\{ \prod_{i<j=1}^{\bar{N}} u(i)\cdot u(j) \right\}^{\bar{m}} \left[((u(1)\dots u(\bar{N}))_{\bar{N}/2} \otimes \dots \otimes (u(1)\dots u(\bar{N}))_{\bar{N}/2} \right]_{J=0} \qquad (32)$$

with m and $\bar{m}$ odd integers where, as in Eq. (31), the number of coupled spinors N appearing within the second bracket is given by $(m-1)(N-1)=\bar{N}$. It follows that

$$S = \frac{\bar{m}}{2}(\bar{N}-1) + \frac{\bar{N}}{2(m-1)} + \frac{1}{2} \qquad (33)$$

so that asymptotically,

$$\nu \equiv \frac{N}{2S} \rightarrow \frac{m-1}{\bar{m}(m-1)+1}\,. \qquad (34)$$

Thus the wave functions of Eq. (32) have fractional fillings of the form even/odd (2/7, 4/13, 2/11, etc.). Whether Eq. (32) is a useful generalization of our $(m\text{-}1)/m$-filled state (Eq. 31) is not yet known, since it has not been tested in numerical calculations. Numerical tests and additional generalizations of Eq. (32) will be presented elsewhere.[8]

SUMMARY AND CONCLUSIONS

We have reviewed the fractional quantum Hall effect in the spherical geometry, including the derivation of Coulomb matrix elements in the lowest spherical Landau level, and compared the Laughlin/Haldane ansatz for the correlated many-electron wave functions with exact results. While this ansatz describes fractional fillings of $1/m$, m odd, additional J=0 ground states appear in numerical calculations at unexpected fractional fillings. To make progress on this problem, we studied the four electron case, fully classifying all such wave functions as a function of S. A candidate wave function describing the particle-hole conjugates (fractional fillings $(m\text{-}1)/m$) of the Laughlin/Haldane states can be constructed by a simple spinor transformation affecting the symmetric part of the wave function. Two illustrations of this are contained in our four-particle results. The form of the particle-hole conjugate wave function suggests several possible generalizations, one of which we show produces fractional fillings of the form even/odd. Whether this generalization is useful is not yet known.

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