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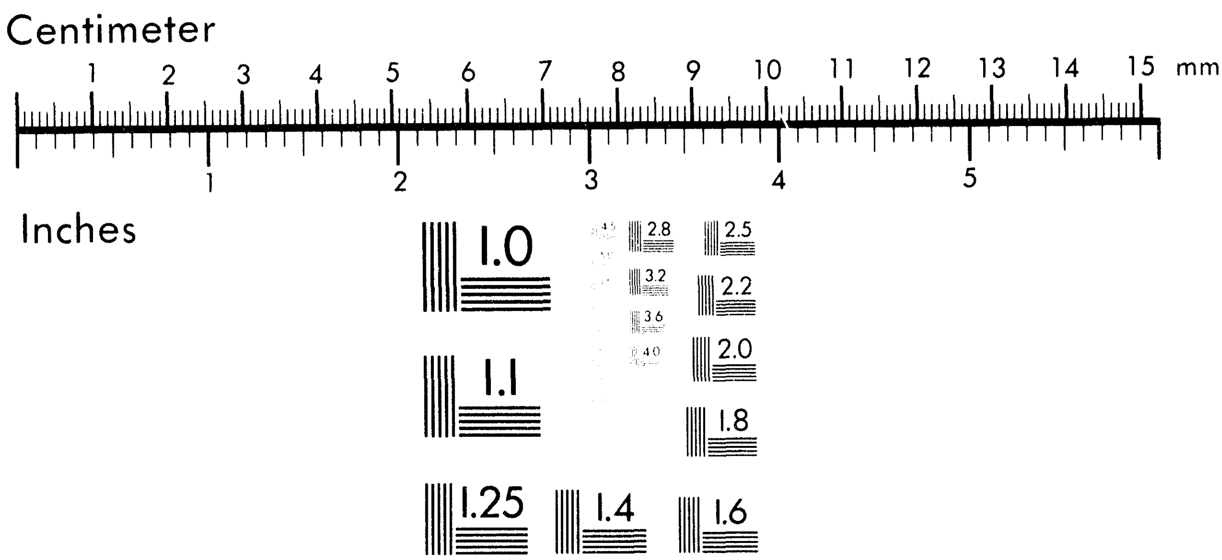

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NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS INVARIANT TO A ONE-PARAMETER FAMILY OF STRETCHING GROUPS*

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INTRODUCTION

Nonlinear partial differential equations (PDEs) in one dependent and two independent variables (call them c, z, and t) occur in many technological applications. Typical PDEs and the contexts in which they arise are the following: (1) $c_t = (c^n)_{zz}$, which occurs in plasma physics, hydrology, gas flow in porous media, and applied superconductivity; (2) $cc_t = c_{zz}$, which describes the expulsion of fluid from a long, slender, heated pipe; (3) $c_t = (c_z{}^{1/3})_z$, which describes heat transport in turbulent superfluid He-II; and (4) $c_{tt} = (c_{zz}/2) \int_0^1 c_z{}^2\, dz$, which describes the motion of a shock-loaded elastic membrane.

All of these equations are invariant to a *one-parameter family of one-parameter stretching groups* of the form

$$c' = \lambda^\alpha c, \quad t' = \lambda^\beta t, \quad z' = \lambda z, \qquad 0 < \lambda < \infty \tag{1}$$

where λ is the group parameter that labels the individual transformations of a group and α and β are the parameters that label groups of the family. The parameters α and β are connected by a linear relation

$$M\alpha + N\beta = L \tag{2}$$

where M, N, and L are numbers determined by the structure of the PDE.

Similarity solutions are solutions of the PDE that are invariant to *one group* of the family, say, that for which $\alpha = \alpha^*$ and $\beta = \beta^*$. Such solutions have the form

$$c = t^{\alpha^*/\beta^*} y(z/t^{1/\beta^*}) \tag{3}$$

where y is a function of the single variable $x = z/t^{1/\beta^*}$. When substituted into the PDE, (3) yields an ordinary differential equation (ODE) for the function $y(x)$. I call this ODE the *principal* ODE.

THE ASSOCIATED GROUP

The form of the principal ODE depends, naturally, on the form of the PDE, but it can be proved [1] that the principal ODE, whatever its form, is invariant to the stretching group

$$y' = \mu^{L/M} y, \; x' = \mu x, \qquad 0 < \mu < \infty \tag{4}$$

I call this group the *associated group*.

Often the principal ODE is of second order, as is the case for the illustrative PDEs given above. According to a theorem of Lie's [2], if we introduce as new variables an invariant of the associated group, e.g., $u = yx^{-L/M}$ or any function of u, and a first differential invariant, e.g., $v = \dot{y}x^{-L/M+1}$ ($\dot{y} \equiv dy/dx$) or any function of u and v, we reduce the second-order principal ODE in y and x to a first-order ODE in v and u. I call such a first-order ODE an *associated* ODE. Thus for problems of this kind, the computational task is the solution of a first-order associated ODE.

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ASYMPTOTIC BEHAVIOR

Before we turn to a concrete example of how this procedure works, certain additional generalities need to be described. There is a solution that is invariant not just to one group of the family, but to *all* groups of the family, namely, $c = Az^{L/M}t^{-N/M}$, where A is a constant that is determined by the structure of the PDE. When the ratios L/M and N/M are both negative and A is positive (as is the case for illustrative PDEs (2) and (3) above), this totally invariant solution obeys the partial boundary and initial conditions (BIC) $c(\infty,t) = 0$ and $c(z,0) = 0$. Often, technologically interesting solutions of the PDE obey the same BIC. For those that do, the totally invariant solution, under an additional conditions described below, gives their asymptotic behavior for large z [3].

The additional conditions are these: The solutions of the PDE must be ordered according to their boundary condition at $z = 0$. This means that if $c_1(0,t) \geq c_2(0,t)$ for $t > 0$, and if both c_1 and c_2 obey the BIC $c(\infty,t) = 0$ and $c(z,0) = 0$, then $c_1(z,t) \geq c_2(z,t)$ for all z and t. Furthermore, $c(0,t)$ must be bounded from below by a multiple of a power of t. The ordering condition is fulfilled for the example PDEs (1), (2) and (3) given above. For, these PDEs have the general form of conservation equations, namely, $S(c)c_t + q_z = 0$, where $S > 0$, and when $\partial q/\partial c_z \leq 0$, it can be shown [3] that solutions of such conservation equations for which $c(\infty,t) = 0$ and $c(z,0) = 0$ are ordered according to their boundary condition $c(0,t)$.

ILLUSTRATIVE EXAMPLE

To show how the procedure outlined above works let us consider illustrative PDE (3), for which $M = 2$, $N = -3$, $L = -4$, and $A = 4/3\sqrt{3}$. The principal ODE is

$$\beta d(\dot{y}^{1/3})/dx + x\dot{y} - \alpha y = 0 \tag{5}$$

Choosing as an invariant and a first differential invariant $p = u^{1/2} = xy^{1/2}$ and $q = v^{1/3} = x\dot{y}^{1/3}$, we find an associated ODE

$$dq/dp = 2p(\beta q - q^3 + \alpha p^2)/\beta(2p^2 + q^3) \tag{6}$$

Different choices of α and β correspond to different physical problems, i.e., to different BIC. Three problems of technological interest correspond to the following BIC: (i) $c(0,t) = 1$, $c(\infty,t) = 0$ and $c(z,0) = 0$ $(\alpha = 0, \beta = 4/3)$; (ii) $c_z(0,t) = -1$, $c(\infty,t) = 0$ and $c(z,0) = 0$ $(\alpha = 1, \beta = 2)$; (iii) $\int_{-\infty}^{\infty} c\, dz = 1$, $c(\infty,t) = 0$ and $c(z,0) = 0$ $(\alpha = -1, \beta = 2/3)$. Since the PDE describes heat transport, we can use the language of of that subject to describe these problems: (i) is the problem of an initially uniform semi-infinite tube of He-II the temperature of whose front face is raised and clamped at $t = 0$; hence, it is called the *clamped-temperature problem;* (ii) is the *clamped-flux problem* for the same semi-infinite tube; and (iii) is the problem of a tube infinite in both directions subjected to a sudden heat pulse per unit area at the plane $z = 0$ at time $t = 0$; it is called the *pulsed-source problem.*

Different values of α and β lead to different forms of the principal and associated ODEs (5) and (6). For the clamped-temperature problem (i) and the pulsed-source problem (iii), the principal ODE is analytically solvable:

$$\text{(i)} \quad c(z,t) = 1 - x/(x^2 + a^2)^{1/2}, \; a^2 = 8/3\sqrt{3}, \; x = z/t^{3/4} \tag{7a}$$

$$\text{(ii)} \quad c(z,t) = t^{-3/2}(4/3\sqrt{3})/(x^4 + b^4)^{1/2}, \; b = 2[\Gamma(1/4)]^2/3(3\pi)^{1/2} = 2.854535, \; x = z/t^{3/2} \tag{7b}$$

Both of the solutions (7a,b) are asymptotic to the totally invariant solution $c(z,t) = (4/3\sqrt{3})z^{-2}t^{3/2}$ when $x >> 1$, as expected. Solutions for different values of $c(0,t)$ or $\int_{-\infty}^{\infty} c\, dz$ than those given above can be obtained from the solutions (7a,b) by scaling $c(z,t)$ with the group (1) or $y(x)$ with the associated group (4). Scaling does not affect the asymptotic limit of the solutions since it is totally invariant to the family (1).

For the clamped-flux problem (ii), there is no simple solution to the principal ODE, which must be solved numerically subject to the *two-point* boundary conditions $\dot{y}(0) = -1$ and $y(\infty) = 0$. To avoid the labor of the shooting method, we turn for help to the associated ODE.

Corresponding to the solution $y(x)$ we seek, there is a curve in the (p,q)-plane which we now must identify. Shown in Fig. 1 is the fourth quadrant of the direction field of (6) for $\alpha = 1$ and $\beta = 2$. Only the fourth quadrant interests us since $p > 0$ and $q < 0$ (because $y > 0$ and $\dot{y} < 0$). The curves C_1 and C_2, the loci of zero and infinite slope dq/dp, respectively, divide the direction field into regions in which the slope dq/dp has one sign only. The intersections of these curves, the points O: (0,0) and P: $(2/3^{3/4}, -2/3^{1/2})$, are the singular points of (6).

The totally invariant solution $c = Az^{L/M}t^{-N/M}$ corresponds to the solution $y = Ax^{L/M}$ of the principal ODE, which is invariant to the associated group (5). For this solution, the invariant $u = A$ and the first differential invariant $v = (L/M)A$. Thus the totally invariant solution maps into a single point in the (p,q)-plane, namely, the point $(A^{1/2},[(L/M)A]^{1/3})$, which is the singular point P. (That the totally invariant solution always maps into a singular point in the (u,v)-plane follows from the fact that for the solution $y = Ax^{L/M}$ of the principal ODE, $du = dv = 0$ as x changes.) Thus the curve in the (p,q)-plane that corresponds to the solution $y(x)$ that we seek must pass through the singular point P. Furthermore, since P corresponds to the asymptotic behavior $y \sim Ax^{L/M}$ of the solution $y(x)$ that we seek, it corresponds to the limit $x = \infty$. When $x = 0$, on the other hand, $p = q = 0$, and the curve in the (p,q)-plane must also pass through the origin O. Only the separatrix S does so. It is the curve in the (p,q)-plane defined by the solution $y(x)$ that we seek.

Near the origin in the (p,q)-plane, the integral curves behave linearly, i.e., $p = -Bq$. Substituting the definitions of p and q, we find that $[y(0)]^{1/2} = -B\dot{y}^{1/3}(0)$, which means that $y(0) = B^2$ since $\dot{y}(0) = -1$. To find the value of B, we proceed as follows: Since the point P is a saddle point, two separatrices cross it. We can find their slopes by applying L'Hospital's rule to (6). The negative slope $m = -3^{1/4}(3+\sqrt{17}) = -1.562422$. Using this slope to get starting values $p = p_P - \varepsilon$, $q = q_P - m\varepsilon$ near P, we can integrate (6) numerically from P to O and find B = 0.912582.

Now we have values of both $y(0)$ ($= B^2$) and $\dot{y}(0)$ ($= -1$), so we can integrate (5) in the forward direction. Here a slight problem arises because integrating (5) in the forward direction carries us along the separatrix S from O towards P. Because the integral curves in the (p,q)-plane diverge away from O, integration in the direction from O to P is unstable: a small error (roundoff or truncation) throws us off the separatrix S and we eventually diverge to one side or the other. This instability is reflected in a corresponding instability as we attempt to integrate (5) in the forward direction. Nevertheless, as a practical matter, it is possible to advance to about $x \sim 1$ without undue errors; the computed behavior can then be joined to the known asymptotic behavior to achieve a reasonable estimate of $y(x)$.

Fortunately, a way exists to integrate (5) in the backward, stable direction. We proceed as follows: (i) we choose a point (p,q) on S close to P; (ii) guess a (large) value of x, say x_1; (iii) calculate y_1 and $\dot{y}_1$ from the chosen values of p and q; and (iv) use these values of as starting values for a backward integration from x_1 to 0. This procedure works for the following reason. Any image point of x_1, y_1, $\dot{y}_1$, say μx_1, $\mu^{-2}y_1$, $\mu^{-3}\dot{y}_1$, has the same values of p and q as the point x_1, y_1, $\dot{y}_1$ itself because p and q are functions of the group invariants u and v. Thus any value of x can be made to correspond to any p and q on the separatrix. In general, the backward integration will not give the curve for which $\dot{y}(0)$ has some specified value. But once the curve $y(x)$ has been calculated, it can be scaled with the associated group to a curve with any desired $\dot{y}(0)$.

SCALING WITH THE ASSOCIATED GROUP

Since all the curves $y(x)$ corresponding to different values of $\dot{y}(0)$ are images of one another under the associated group (4), all have the same value B of $[y(0)]^{1/2}/\dot{y}^{1/3}(0)$ because this quantity is invariant to transformations of the associated group. (Note that the point $x = 0$ transforms into the point $x' = 0$). From this it immediately follows that $c(0,t) = B^2\dot{y}^{2/3}(0)\, t^{1/2}$. This formula gives the dependence of the temperature on the front face $c(0,t)$ on the time t and the clamped flux $-\dot{y}^{1/3}(0)$, which are the only two parameters in the problem on which it can depend. We could have obtained this formula directly from knowledge of the associated group so that by group analysis alone we can obtain a formula for $c(0,t)$ correct up to a single undetermined constant. To find the value of the constant, however, we must integrate the associated ODE.

CONCLUDING REMARKS

The method outlined here does *not* depend on the PDE being linear. On the other hand, it does depend on the PDE being invariant to a *one-parameter family of one-parameter stretching groups*. This is a high degree of algebraic symmetry that is only found in the simplest equations. But many equations of technological interest have the high symmetry required and so can be dealt with by the method of this paper.

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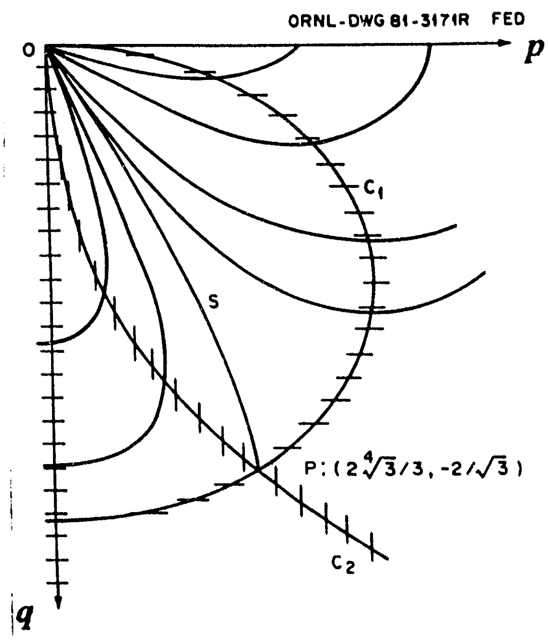

Fig. 1. The fourth quadrant of the direction field of (6) for $\alpha = 1$ and $\beta = 2$.

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