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Fuzzy-Algebra Uncertainty Analysis for Abnormal-Environment Safety Assessment

J. Arlin Cooper

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550
for the United States Department of Energy
under Contract DE-AC04-94AL85000

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Distribution UC-706

Fuzzy-Algebra Uncertainty Analysis for Abnormal-Environment Safety Assessment

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Abstract

Many safety (risk) analyses depend on uncertain inputs and on mathematical models chosen from various alternatives, but give fixed results (implying no uncertainty). Conventional uncertainty analyses help, but are also based on assumptions and models, the accuracy of which may be difficult to assure. Some of the models and assumptions that on cursory examination seem reasonable can be misleading. As a result, quantitative assessments, even those accompanied by uncertainty measures, can give unwarranted impressions of accuracy. Since analysis results can be a major contributor to a safety-measure decision process, risk management depends on relating uncertainty to only the information available. The uncertainties due to abnormal environments are even more challenging than those in normal-environment safety assessments; and therefore require an even more cautious approach. A fuzzy algebra analysis is proposed in this report that has the potential to appropriately reflect the information available and portray uncertainties well, especially for abnormal environments.

Acknowledgment

Kathleen Diegert, Dick Smith, Roger Breeding, and Bob Easterling, Sandia National Laboratories; Doug Cooper, Naval Research Laboratory; Tim Ross, and Ron DeVries, University of New Mexico, have provided valuable assistance with technical aspects of this work.

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Introduction

Many safety (risk) analyses depend on uncertain inputs and on mathematical models chosen from various alternatives, but give fixed results (implying no uncertainty). While solving these types of problems may give insight to the analyst (an important benefit), there is a possibility that such results can give others an unwarranted impression of accuracy. There has been considerable noteworthy work [e.g., Refs. 1-6] on this problem, specifically emphasizing measures of uncertainty associated with analytical results and decisions. Conventional uncertainty analyses help, but are also based on assumptions and models, the accuracy of which may be difficult to assure. Some of the models and assumptions that on cursory examination seem reasonable can be misleading. As a result, quantitative assessments, even those accompanied by uncertainty measures, can give unwarranted impressions of accuracy. Since analysis results can be a major contributor to a safety-measure decision process, risk management depends on relating uncertainty to only the information available.

Safety analyses are frequently based on probabilities (e.g., probabilistic risk assessments). This approach almost always depends on models using logic structures (e.g., fault trees and event trees). It is appropriate to also consider "uncertain" inputs. The input uncertainty may be due to variability of potential input values, interpolation or extrapolation, measurement or human error, disagreements in interpretation, problem specification language vagueness or ambiguity, assumptions, simplifications or approximations, instrumentation resolution limits, sampling variability, etc.

One approach to describe input uncertainty is to use probability density functions. For most safety problems, this is an approximation, which introduces another contribution to uncertainty. The use of fault trees and event trees and combinations (one of many alternatives) implies properties that are often difficult to meet (discussed subsequently), which introduces additional uncertainty. Even the manner in which the results are presented can be varied, thereby varying the impression given to a decision-maker. The analyst is another important factor. It is tempting for analysts to focus so much on mathematical correctness that they may lose sight of some of the contributions to uncertainty. The literature on attempted verification of "confidence limits," for example, demonstrates that these are generally underestimated [Ref. 3, pp. 57-59]. Also, review of unexpected-accident histories reveals numerous situations for which assurance based on safety analysis was overly optimistic.

The uncertainties due to abnormal environments are even more challenging than "conventional" (normal-environment) safety assessments; and therefore require an even more cautious approach. Although uncertainty must be handled very carefully because of the above factors, safety analyses still afford the capability to contribute valuable information, since there is some semblance of natural order in almost all situations. The challenge is to do the best job possible of utilizing somewhat predictable phenomena, without being misleading about the uncertainty involved. This is the type of perspective that is most useful for the recipient of analytical results.

A fuzzy algebra analysis is proposed in this report that has the potential to appropriately reflect the information available and portray uncertainties well, especially for abnormal environments. Following a brief summary of uncertain-variable probabilistic operations and calculus, the application of fuzzy algebra is described, with emphasis on the differences (in concept and applicability) between the two approaches. The differences do not preclude transition from one to the other based on the amount of input knowledge available.

Frequentist and Bayesian Probability

Most treatments of uncertainty are probabilistic in nature. For example, an input might be modeled to have a probability density function (over ranges of probabilities):

$$f(x) = \frac{dF(x)}{dx} \quad , \tag{1}$$

where f(x) is a probability density function, and F(x) (the probability distribution function) is the probability that a variable has a value (for our purposes, a probability value) less than or equal to x. Integrating f(x) over all possibilities gives 1 (by convention, probability values are between 0 and 1):

$$\int_{-\infty}^{\infty} f(x)dx = 1 \quad . \tag{2}$$

Data for such a function might come from sufficient trials to convince oneself that the model was appropriate (frequentist approach), or they might be derived as a subjective expectation based on sparse data (Bayesian approach) [e.g., Ref. 7]. There are many mathematical models for probability density functions (and distribution functions), each of which has associated conditions that must be met (usually approximated) for validity.

An example probability density function, $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$, is shown below.

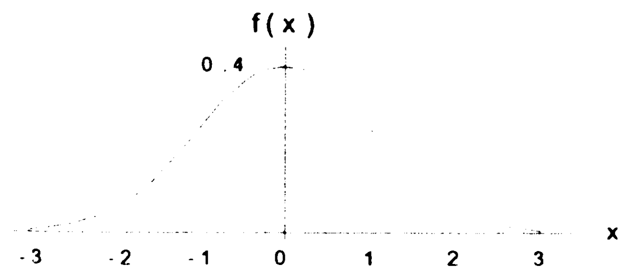


Figure 1. A Probability Density Function (Unit Gaussian)

The distributions of the results of logical and algebraic operations on random variables can be calculated using integral calculus[1] [Ref. 8, pp. 121, 129, 409]. For example, the density function of the sum of two independent random variables is (see Appendix 2):

$$f(y)=\int_{-\infty}^{\infty} f_1(x) f_2(y-x)dx \quad . \qquad (3)$$

The density function of the product of two such random variables (which are nonnegative, since they are probabilities) is (see Appendix 2):

$$f(y)=\int_{0}^{\infty} \frac{f_1(x) f_2(y/x)dx}{x} \quad . \qquad (4)$$

The complexity of the required analysis is inconvenient, but not overwhelming. The distribution of functions of random variables can also be computed in other ways. For example, Monte Carlo simulation is sometimes more efficient, and may be more applicable where the input distributions are estimated from data rather than assumed [Ref. 6].

Fuzzy Algebra

Probability density (or distribution) concepts provide a framework in which uncertain parameters can be described and operated on mathematically, transcending some of the limitations of fixed numbers. Fuzzy logic [e.g., Refs. 9-13] also extends capabilities beyond fixed numbers. Fuzzy logic first emphasized set membership [Ref. 9]. "Crisp" (conventional) set membership is fixed (an element is either a member of a set or it isn't). However, an entity can have some of the characteristics of more than one set description (e.g., a person's hair may be somewhat black and somewhat gray). Like probabilistic calculus, Fuzzy algebra also can be applied to introduce variability to fixed parameters. For example, an uncertain parameter can have some of the characteristics of more than one number (e.g., "approximately" five may indicate a range of real numbers including, but not limited to, five). Fuzzy models can therefore be applied to represent uncertainty of parameters in probability analysis [Ref. 14], and this has some similarity to strictly probabilistic descriptions. However, fuzzy algebra differs from probabilistic calculus both mathematically and in concept. It appears to be more appropriate for the uncertain inputs applicable to abnormal environments, particularly if probability distributions are unknown and must be assumed. Before exploring this, some mathematical background is helpful.

A fuzzy number (formally a convex and normal fuzzy set) can be represented mathematically [Ref. 10] as:

$$A^{\alpha}(x)=A^{\alpha}=[a_1^{\alpha},a_2^{\alpha}] \quad , \qquad (5)$$

[1]Transform techniques can be similarly used.

where the a_1 and a_2 values on x represent the lower and upper limits, respectively, of the variation possible for the number as a function of α, and α is a "level of presumption." The level of presumption represents a collection of subjective judgments[2] about the range specified. One must be more presumptuous in order to specify a narrower variable range (maximum level of presumption is presumption of minimum uncertainty). The "normal" restriction fixes the maximum level of presumption at 1 and the minimum to 0.[3] An example fuzzy number (over the real numbers) is shown below; this is called a triangular fuzzy number (TFN). A few other examples of fuzzy numbers are given in the Appendices.

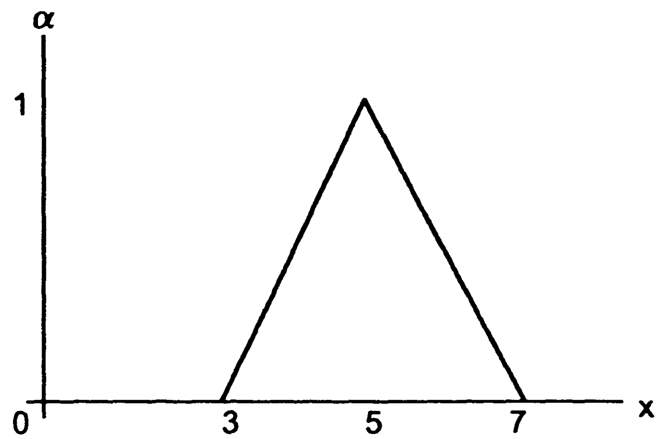


Figure 2. An Example of a Triangular "Fuzzy" Number (e.g., "Approximately" 5)

The graphical representations of fuzzy parameters should be viewed in a horizontal sense. That is, one should think of the function as representing the end points of intervals within which the parameter lies. If a particular value of presumption, a, is selected, a horizontal line can be drawn that intersects the ordinate at α=a. The two points where the line intersects the function represent the lower and upper bounds for the parameter at the specified presumption level.

Fuzzy addition is specified[4] as:

$$A^{\alpha} + B^{\alpha} = [a_1^{\alpha} + b_1^{\alpha}, a_2^{\alpha} + b_2^{\alpha}] \quad . \tag{6}$$

Fuzzy subtraction is:

$$A^{\alpha} - B^{\alpha} = [a_1^{\alpha} - b_2^{\alpha}, a_2^{\alpha} - b_1^{\alpha}] \quad . \tag{7}$$

---

[2]Preferably from "experts," preferably based on data (even if limited), and possibly weighted according to expertise.

[3]Note that this can sacrifice some uniformity, e.g., if various inputs are judged by various persons or groups.

[4]Fuzzy arithmetic can also be derived using the "extension principle" [Refs. 10, 14].

Addition and subtraction of TFNs yields a TFN (because these are linear operations).

Fuzzy multiplication is:

$$A^{\alpha} \times B^{\alpha} = [a_1^{\alpha} b_1^{\alpha}, a_2^{\alpha} b_2^{\alpha}] \quad . \tag{8}$$

Multiplication of TFNs does not yield a TFN, because multiplication is a nonlinear operation. This is demonstrated in Appendix 2, where algebraic expressions for the results of addition and multiplication are given. Note that the above fuzzy algebra operations only utilize ranges of values, and make no use of or assumptions about relationships between probability parameters, or of independence between probability parameters.[5] The applicability of the operations shown is useful for parameters for which relative probabilities and independence are not well known (a common situation). On the other hand, probabilistic operations are limited to parameters for which these characteristics are well known (a less common situation).

A useful aspect of fuzzy algebra is that accuracy (and uncertainty) of the results is not significantly affected by some common mathematical simplifications. For example, linearizing multiplicative nonlinearities (e.g., approximating the product of two TFNs by a TFN) is appropriate for most abnormal-environment assessments. Deviation from linearity increases with the number of multiplicative operands, but so does the difficulty of finding the independence necessary for multiplication to be valid. To emphasize this, we will portray these boundaries with linear dashed lines.

It is informative to consider an illustration of the applicability of fuzzy algebra, including its differences from probabilistic operations. Assume that four coins are to be flipped, but that they have been deformed. The expected probability of four "heads" is to be assessed, (e.g., as a threat to safety). If the coins could be subjected to frequentist experiments, or if their geometry could be accurately measured, we could use conventional techniques. If these data were limited (a typical abnormal environment problem), we could observe that over a large number of such situations, four heads would be expected about one-sixteenth of the time. However, fuzzy algebra assessment provides more appropriate information.

From the data given, we can only conclude that the probability of heads on any coin toss is bounded between 0 and 1. As fuzzy multiplication demonstrates, the probability of four heads is also bounded between 0 and 1. There is no way, based on the information available, to preclude the possibility of four heads.

Although probabilistic assessment is not appropriate for this problem, it could be used if sufficient information were available. Pursuing the example shows how results that assume too much information can be misleading. One might, for example, equate uniform distributions across the interval [0, 1] with no information about the probability of heads

[5]However, treatment of independence/dependence properties is not precluded.

for each coin. Although these density functions look similar to fuzzy descriptions, the meaning is quite different. The resultant probability density function, f(y) in Fig. 3, for four heads is $-(\ln y)^3/6$ (derived from Equations A2-8 and A2-9 in Appendix A2). This function is largely concentrated near zero, indicating large confidence that for any four coins, four heads is improbable, whereas no such concentration occurs when fuzzy no-knowledge descriptions are used. The concentration in probabilistic assessment arises from the assumptions of equal likelihood for probabilities and independence of coin deformation. These would be impossible to assure for general abnormal environments. The implication for abnormal-environment safety is that assuming more than is known can lead to unwarranted expectations.

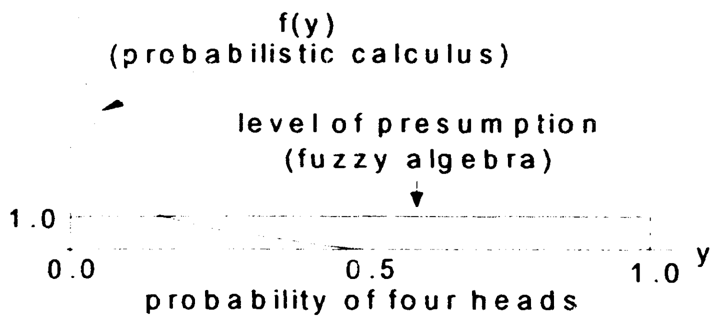


Figure 3. Probabilistic Assessment Example (Four Heads; Deformed Coins).

Pursuing the example, we can demonstrate that fuzzy algebra is not limited to trivial bounds. Suppose that "experts" have the opportunity to view the deformed coins. Their experience may allow them to make qualitative judgments about the possibility of heads for each coin. Furthermore, each expert may have varying levels of presumption in his or her own judgment, and the level of expertise may vary from expert to expert. The guidance provided for generating such descriptions is that the maximum level of presumption (one) corresponds to the smallest range of values the expert would judge plausible, and the lowest level of presumption (zero) corresponds to the largest range of values the expert would judge plausible. This allows for a characterization of the fuzzy input parameters by "level of presumption." The boundaries for abnormal-environment inputs and outputs are estimated linearly. Assume that the expert judgments have been consolidated (e.g., by a weighted combination) into the four graphs shown for the four coins in Fig. 4 (as an example). Multiplication of the fuzzy input variables (Eqn. 8, linearized) is appropriate only if the coins are independently deformed and independently tossed (a restriction addressed subsequently). This leads to the output shown in Figure 4.

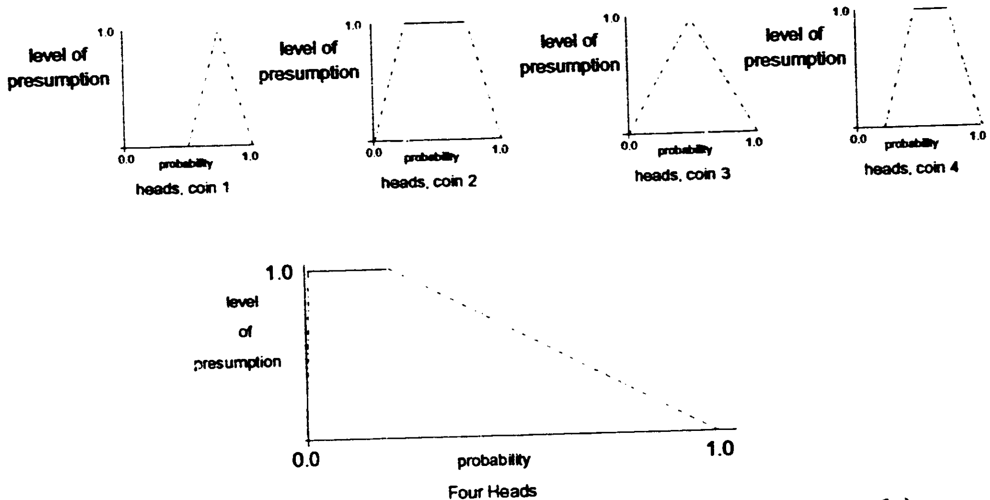


Figure 4. "Expert" Inputs and Fuzzy Algebra Output (Example)

The independence assumption above is convenient, but seldom assured. Questions would arise about the relationship of the deforming process from one coin to the next, for example. Since independence is not assured, expert judgment on dependence can also be useful. For example, the upper and the lower bounds on the range of values for four heads can vary, depending on the amount of dependence. The fuzzy result is bounded as indicated in Figure 5. Introducing another level of presumption from expert judgment (reflecting amount of dependence) would be necessary if the indicated range were to be narrowed.

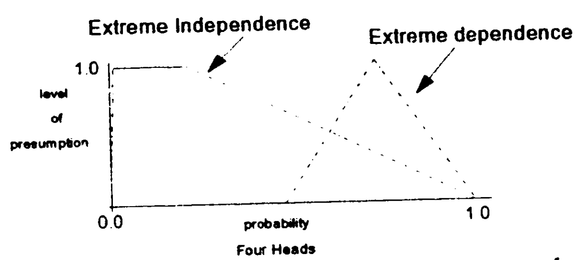


Figure 5. Effect of Dependence on Example

In abnormal-environment safety assessment, the input probabilities are often not known well, and the relations between possible probability values are not known well. The independence or dependence between inputs may also not be known well. However, we usually have access to expert judgment, along with limited data, which can be applied (to the appropriate extent) using fuzzy algebra.

The fuzzy algebra approach can transition toward the probabilistic approach as the amount of knowledge increases. It is also possible to combine probabilistic variables and fuzzy variables, as well as to combine probabilistic and fuzzy characteristics in the same variable [Ref. 10].

Fault Trees, Event Trees, and Combinations

An undesired outcome (loss of system safety) can frequently be described logically, leading to a "fault tree." The logical structure is an interconnected array of representations for logic functions (e.g., "and" and "or"). These are mostly combinational in nature (they do not handle sequences of events well). They are deductive descriptions of how an unwanted event might occur, once such an event has been postulated. An example of a fault that can be caused by occurrence A or by the combination of both occurrences B and C is shown below (assuming no other significant possibilities).

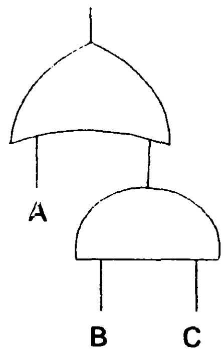


Figure 6. An Example (Simple) Fault Tree

The result of a sequence of occurrences can be described by a logical structure called an event tree (also scenario trees and event sequence trees). Event trees represent the sequence, each occurrence having two or more possible outcomes. They usually describe sequential occurrences, although they can be used for combinational factors. Event trees are inductive in nature, meaning that a starting occurrence is postulated, and it is necessary to consider the possibilities for subsequent occurrences. An example of an event tree for two coin tosses is shown below (assuming none remain on edge):

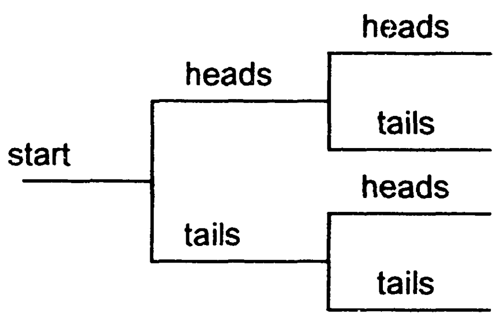


Figure 7. An Example (Simple) Event Tree

It is possible to glean most of the advantages of fault trees and event trees from combinations of the two, as discussed subsequently. First, it is important to examine the applicability of each.

The probability that two events in a fault tree or event tree both occur is:

$$P(AandB) = P(A)P(B|A) , \tag{9}$$

where the juxtaposition represents ordinary multiplication and the second term on the right hand side of the equation is the probability of B conditional on A having occurred. This either requires that the conditional probabilities be known, or that A and B be independent so that the probability of B occurring is not conditional on A. Similar extensions apply to any number of variables.

The "or" operation, which is an essential part of fault tree analysis[6] is:

$$P(AorB) = P(A) + P(B) - P(AandB) , \tag{10}$$

where the plus and minus signs indicate ordinary addition and subtraction. The third term on the right hand side of the equation is unnecessary if occurrences A and B are mutually exclusive. This term can be omitted as an approximation if the probabilities of occurrences A and B are very small. When the term must be included, occurrences A and B must be independent or the dependency must be known.

If A and B are uncertain variables, the independence consideration must be treated with great care. For example, although probability distributions can be added, subtracted, and multiplied; addition, subtraction, and multiplication of density functions cannot be combined to solve the "or" function above (for uncertain variables), because independence cannot be maintained for the operations. Therefore the solution is not nearly as straightforward for uncertain variables. For fuzzy uncertain variables, this constraint means that addition and then subtraction of their product must be done at each level of presumption, rather than first adding fuzzy parameters and then subtracting their product. The above approach also extends to any number of variables. Each "or" function must cover all possibilities conditional on being at that point in the fault tree.

The end result of a fault tree can be represented as a Boolean expression which can be converted to minimal (Boolean simplification applied) "sum of product" form ("or" of "ands"). The individual terms are called "cutsets." It should be noted that the overall result is a Boolean sum, not an algebraic sum of cutset values. These differ because of variables in common among the terms.

The branches at a node in an event tree must represent "partitions," meaning that they must be mutually exclusive occurrences and span all possibilities at that point in the event tree. The accumulation of probabilities along a path in the event tree can be made by multiplying probabilities if they are independent (or if the conditional dependence is known).

---

[6]Exhausting all possibilities for inputs is a particularly challenging problem.

The use of, and combination of, probabilities in fault trees and event trees also depends on consistent units. For example, the probability of a lightning strike to ground is not meaningful unless it is given as a probability that it occurs in a unit time interval or area. The probability of heads on a coin toss is a probability per toss. The units of the two examples above would not be generally compatible for fault tree or event tree combination.

There are many other logical constructs that can be used in fault trees and event trees (e.g., "inhibit," "not," "priority-and," "exclusive-or"). The use of each of these implies further limiting assumptions for the analysis.

The reason for pointing out these restrictions is that the recipient of the analysis results may be misled as to the degree of uncertainty added to the results by the modeling assumptions and potential inputs remaining unidentified. Lack of rigorous attention to these restrictions is a significant source of uncertainty for quantitative safety analysis results.

Fault tree and event tree structures are frequently combined. Accidents may be caused by a sequence of occurrences (matching an event tree structure). At each stage in the sequence, combinations of factors can lead to a safety problem (matching a fault tree structure). An example is given in Appendix 1.

Conclusions

Systematic treatment of uncertainty has been approached in a large number of ways, some of which were reviewed above or in the references. Each of these approaches has applicability in particular situations. Qualitative decision-making-assistance algorithms [e.g., Ref. 15] are becoming widely used because of quantitative analysis uncertainties. However, these are mostly heuristic-based. For abnormal environments, fuzzy algebra structured with an event-tree-fault-tree combination is a mathematically correct reflection of the input data. It therefore could offer advantages over other approaches. This is basically because fuzzy descriptions and the logical processing required are ideally suited to the knowledge base for most abnormal-environment situations.

Appendix 1. An Example

An example will be given involving the risk of a safety system failure following a particular type of accident. The probability of this type of situation can be expressed as:

$$P(failure) = P(accident)P(response|accident) \quad . \qquad \text{(A1-1)}$$

In general, we would trace through a particular type of accident, using an event tree for the accident, combined with a fault tree for the response to the accident. For simplicity of illustration, only one specific type of accident was chosen for this example. One specifically chosen response failure is evaluated (logical "and" of three independent contributors. All inputs are represented both as probabilistic and as fuzzy variables. The fuzzy values are chosen for illustration. There is no justification for the probabilistic values; these are chosen only for range comparison. The result sought is the probability of a safety failure per year for the particular type of accident. For illustration, all constraints necessary for fault tree/event tree modeling are assumed to be satisfied.

The diagram below shows a combination event tree/fault tree.

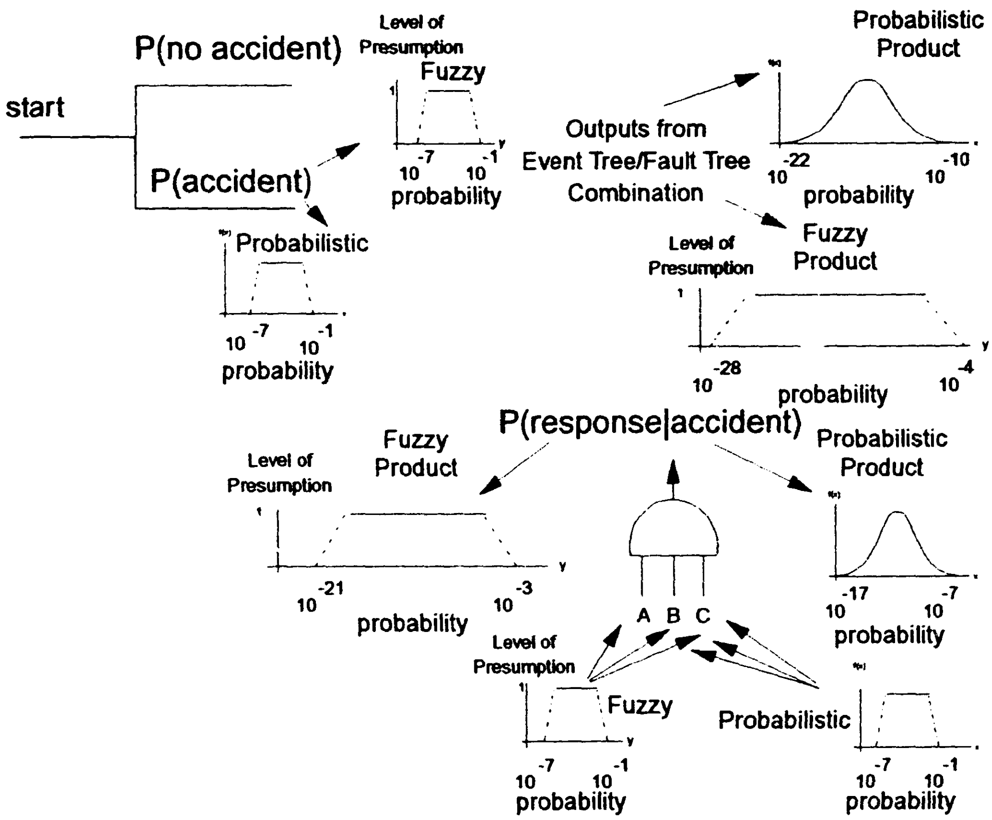


Figure A1.1. (Simple) Example of Combined Fault Tree/Event Tree

The event tree branch is for the occurrence of some accident per year. The abscissa scale for the inputs are shown logarithmically (proportional to the log to the base ten of the probability). The outputs of the event tree are variables for which the lower and upper limits and the spread (uncertainty) are directly affected by all event tree branches.

At this point, the response fault tree enters the computation. It shows that an undesired response can be due to a logical combination of effects described by uncertain variables (shown in fuzzy and probabilistic form). The fault tree output is also an uncertain variable (also shown in both forms).

The final output upper and lower limits (and spread) are affected by both the event tree output and the fault tree output. The combination of the event tree and the fault tree is through multiplication, representing an "and" operation, because the fault tree output is conditional on the event tree output.

Examples even as simple as this tend to illustrate how spread (uncertainty) grows as a result of the number of uncertain inputs. However, the spread indicated by probabilistic calculus is much smaller than the fuzzy bounds because the assumption of probability distributions implies extra knowledge. The general conclusions are that complex event tree/fault tree structures for describing abnormal environment response are almost certain to have substantial uncertainty, and the amount of uncertainty can be underestimated by using probability distributions that overassume knowledge.

Appendix 2. Mathematical Details

This appendix summarizes some of the mathematical details on logical combinations using probabilistic variable calculus and fuzzy variable algebra.

Probabilistic Calculus

1. Addition of Two Random Variables

This approach is extendible to any number of variables. Consider adding random variables X_1 and X_2 to obtain Y ($Y = X_1+X_2$). Let the variables be described by probability density functions (PDFs), f over x, and determine f(y) for y sum values from knowledge of $f_1(x)$ and $f_2(x)$. One solution (there are other approaches, e.g., through transforms) for independent variables[7] is:

$$f(y) = \int_{-\infty}^{\infty} f_1(x) f_2(y-x) dx \quad \text{(A2-1)}$$

The convolution integral results from the additive inverse linear contributions required to achieve a single sum from two contributors. As one operand increases, the other must decrease linearly.

Example 1. As an example, consider the addition of two random variables, each uniformly distributed between 0 and 1.

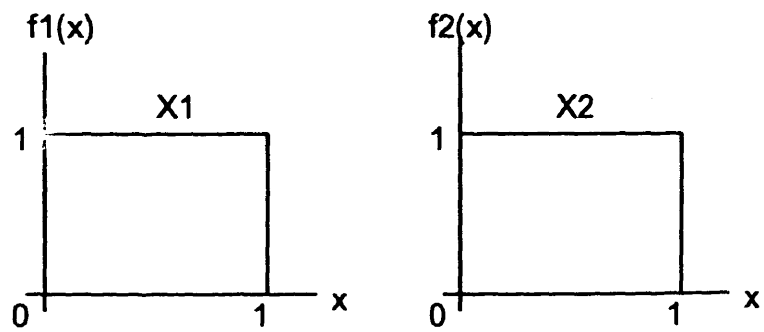


Figure A2.1. PDFs for Two Uniformly Distributed Random Variables

Solving equation A2-1:

$$0 \le y \le 1: f(y) = \int_0^y dx = y \quad \text{(A2-2)}$$

[7]Dependent functionality is similarly accounted for, but dependence is usually unknown. Probability values are nonnegative.

$$1 \le y \le 2: f(y) = \int_{y-1}^{1} dx = 2 - y \quad \text{(a2-3)}$$

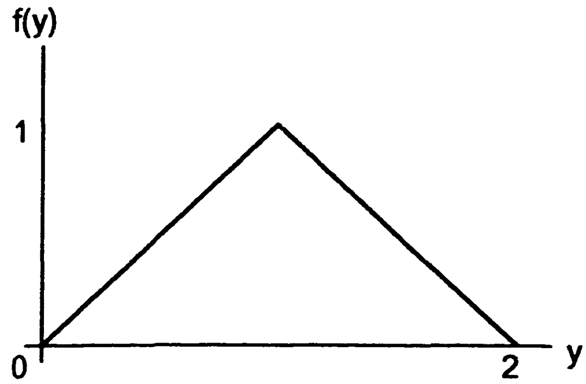


Figure A2.2. PDF for the Sum of Two Random Uniformly Distributed Variables

Example 2. For two triangular density functions ranging from 0 to 1:

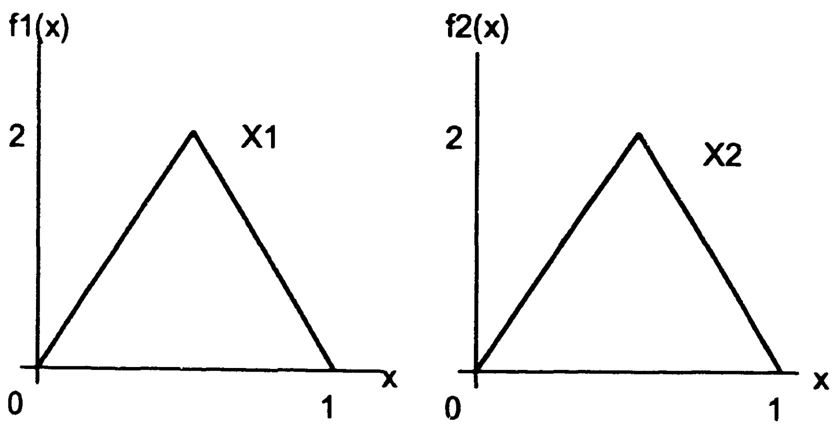


Figure A2.3. PDFs for Two Triangular-Distributed Random Variables

Solving equation A2-1:

$$0 \le y \le 1/2: f(y) = 16\int_{0}^{y} x(y-x)dx = 8y^3/3 \quad \text{(A2-4)}$$

$$1/2 \le y \le 1: f(y) = 16[\int_{0}^{y-1/2} x(1-y+x)dx + \int_{y-1/2}^{1/2} x(y-x)dx + \int_{1/2}^{y} (1-x)(y-x)dx]$$

$$= 8(-3y^3 + 6y^2 - 3y + 1/2)/3 \quad \text{(A2-5)}$$

The result is symmetrical in y about 1.

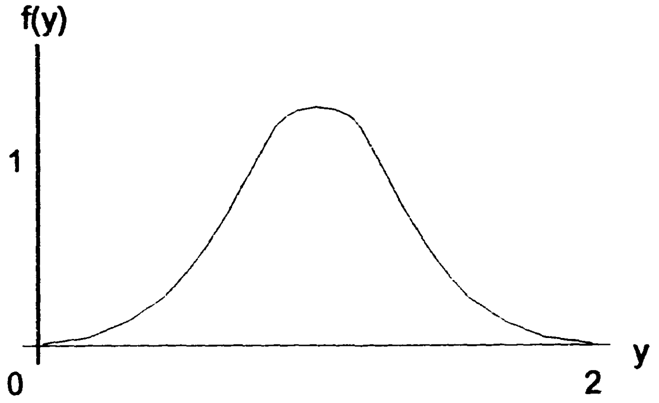


Figure A2.4. PDF for the Sum of Two Triangular-Distributed Random Variables

Example 3. For the addition of a random variable with a uniform density function to a random variable with a triangular density function:

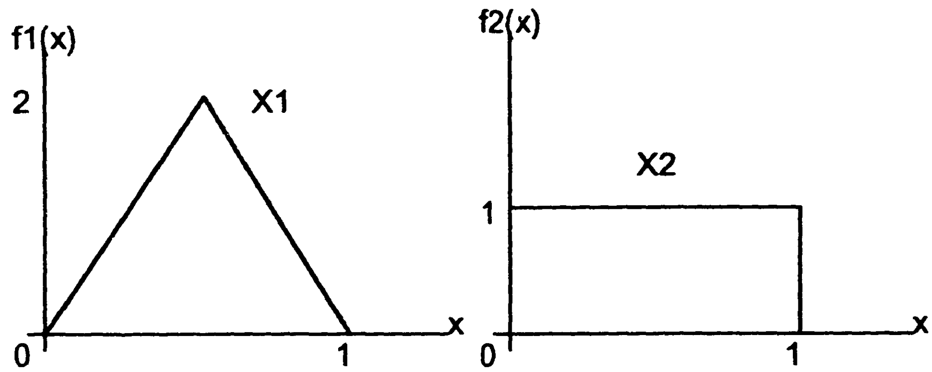


Figure A2.5. PDFs for Triangular- and Uniformly Distributed Random Variables

Solving equation A2-1:

$$0 \le y \le 1/2: f(y) = 4\int_0^y y dx = 2y^2 \tag{A2-6}$$

$$1/2 \le y \le 1: f(y) = 4[\int_0^{1/2} x dx + \int_{1/2}^{y} (1-x) dx] = -1 + 4y - 2y^2 \ . \tag{A2-7}$$

The result is symmetrical in y about 1.

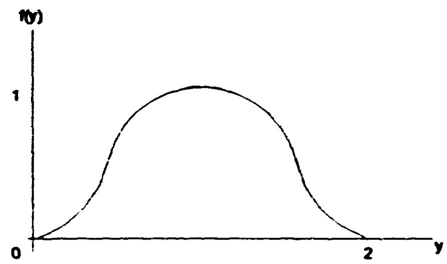


Figure A2.6. PDF for the Sum of Triangular- and Uniformly Distributed Random Variables

2. Multiplication of two random variables

As with addition, this is extendible to any number of variables. Consider multiplying X_1 and X_2 to obtain $Y(Y=X_1 x X_2)$. The solution (for nonnegative values and independent inputs) is:

$$f(y) = \int_0^\infty \frac{f_1(x) f_2(y/x)}{x} dx \ . \qquad \text{(A2-8)}$$

The structure is similar to addition, but the y/x functional dependence is to relate multiplicative inverses that generate each value, and the denominator is because derivatives in y are related to derivatives in x through the nonlinear inverse.

Example 4. Consider the multiplication of two random variables with uniform (0 to 1) PDFs (see Fig. A2.1).

Solving equation A2-8:

$$f(y) = \int_y^1 \frac{dx}{x} = -\ln y \ . \qquad \text{(A2-9)}$$

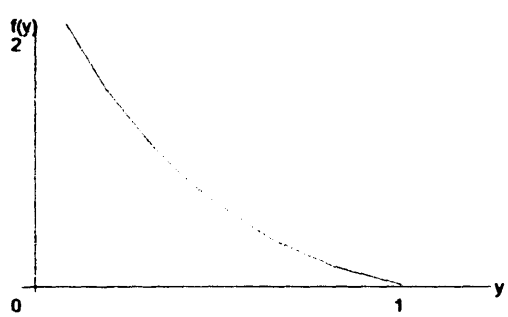


Figure A2.7. PDF for the Product of Two Uniformly Distributed Random Variables

Example 5. Consider two random variables, each of which has a triangular PDF (see Fig. A2.3).

Solving equation A2-8 requires meticulous care in accounting for the integration ranges. The ultimate result is:

$$0 \le y \le 0.25: f(y) = 16[\int_y^{2y}(1 - y/x)dx + \int_{2y}^{0.5}\frac{ydx}{x} + \int_{0.5}^{1}\frac{y(1-x)dx}{x^2}]$$

$$= 16[2y + 2y\ln 0.5 + y\ln y - 2y\ln(2y)] \qquad \text{(A2-10)}$$

$$0.25 \le y \le 0.5: f(y) = 16[\int_y^{0.5}(1 - y/x)dx + \int_{0.5}^{2y}\frac{(1-x)(1-y/x)dx}{x} + \int_{2y}^{1}\frac{y(1-x)dx}{x^2}]$$

$$= 16[2 - 6y - 2y\ln 0.5 - \ln 0.5 + y\ln y + 2y\ln(2y) + \ln(2y)] \qquad \text{(A2-11)}$$

$$0.5 \le y \le 1: f(y) = 16\int_y^1\frac{(1-x)(1-y/x)dx}{x} = 16(-2 + 2y - y\ln y - \ln y). \qquad \text{(A2-12)}$$

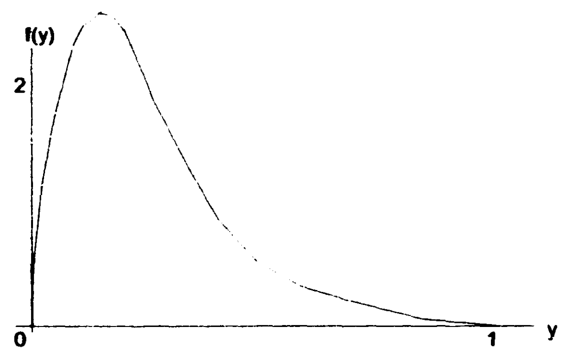


Figure A2.8. PDF for the Product of Two Triangular-Distributed Random Variables

Example 6. Consider multiplication of two random variables, one with a uniform PDF (0 to 1) and the other with a triangular PDF (0 to 1) (see Fig. A2-5).

Solving equation A2-8:

$$0 \le y \le 0.25: f(y) = 4[\int_y^{2y}dx + \int_{2y}^{0.5}dx + \int_{0.5}^{1}\frac{(1-x)dx}{x}]$$

$$= 4(-y - \ln 0.5) \qquad \text{(A2-13)}$$

$$0.25 \le y \le 0.5: f(y) = 4[\int_{y}^{0.5} dx + \int_{0.5}^{2y} \frac{(1-x)dx}{x} + \int_{2y}^{1} \frac{(1-x)dx}{x}]$$

$$= 4(-y - \ln 0.5) \qquad \text{(A2-14)}$$

$$0.5 \le y \le 1: f(y) = 4\int_{y}^{1} \frac{(1-x)dx}{x} = 4(-\ln y - 1 + y) \,. \qquad \text{(A2-15)}$$

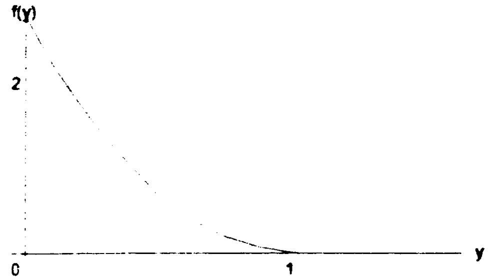


Figure A2.9. PDF for the Product of Triangular- and Uniformly Distributed Random Variables

Mean value computations depend on the calculation:

$$\mu = \int_{-\infty}^{\infty} yf(y)dy \,. \qquad \text{(A2-16)}$$

The results applied to examples 1-3 all give a mean value for y of 1; the results for examples 4-6 all give mean value 0.25.

Fuzzy Algebra

The presentation of fuzzy representations and of fuzzy algebra given in the report text is generally the most mathematically straightforward. However, additional fundamental understanding of processing can be given by an algebraic description. A fuzzy number can be described algebraically as follows:

$$A^{\alpha} \longrightarrow \alpha(x) \,. \qquad \text{(A2-17)}$$

This means that the relationship for level of presumption can be expressed as a function of x.

Given $\alpha(x)$, the sum of two variables is:

$$y(\alpha) = x_1(\alpha) + x_2(\alpha) \,. \qquad \text{(A2-18)}$$

From this, α can be determined as a function of y.

The product of two variables is:

$$y(\alpha) = x_1(\alpha)x_2(\alpha) \quad . \qquad \text{(A2-19)}$$

This can also be solved for α as a function of y.

An example algebraic description for a TFN is:

for $0 \le x \le 1/2$,

$$\alpha = 2x \ , \qquad \text{(A2-20)}$$

for $1/2 \le x \le 1$,

$$\alpha = 2(1-x) \quad . \qquad \text{(A2-21)}$$

Addition of the TFN to a similarly described TFN is solved as:

for $0 \le y \le 1$,

$$2x = y = \alpha \ , \qquad \text{(a2-22)}$$

for $1 \le y \le 2$,

$$2x = y = 2 - \alpha \ ,$$

$$\alpha = 2 - y \quad . \qquad \text{(A2-23)}$$

Multiplication of the TFN to a similarly described TFN is solved as:

for $0 \le y \le 1/4$,

$$x^2 = y = \alpha^2 / 4 \ ,$$

$$\alpha = 2\sqrt{y} \ , \qquad \text{(A2-24)}$$

for $1/4 \le y \le 1$,

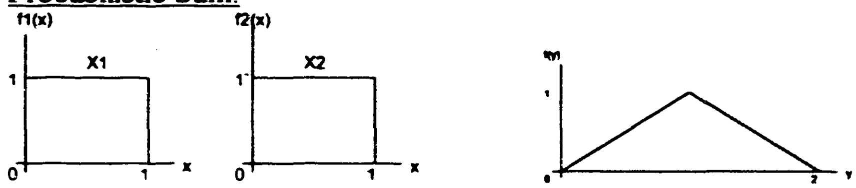
$$x^2 = y = (1 - \alpha/2)^2 \ ,$$

$$\alpha = 2 - 2\sqrt{y} \ . \qquad \text{(A2-25)}$$

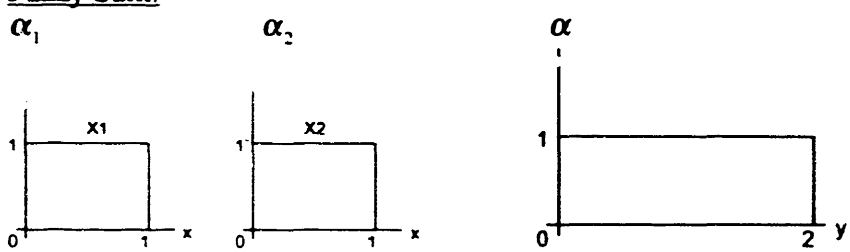
These techniques were used to solve six examples similar to those solved in the section on probabilistic calculus, but with the ordinate functionality in terms of level of presumption rather than probability density. For comparison, the six examples posed probabilistically and fuzzily are shown in the following table.

Comparison Table for Probabilistic Calculus and Fuzzy Algebra

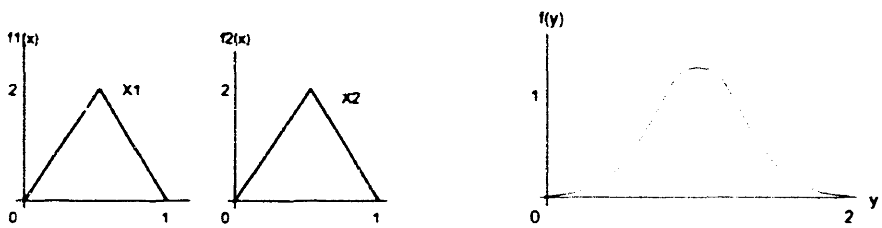
Probabilistic Sum:



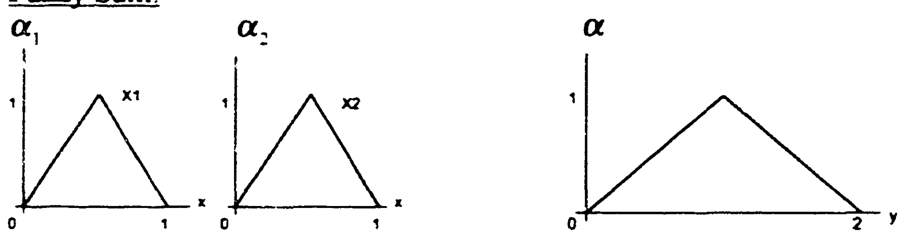
Fuzzy Sum:



Probabilistic Sum:



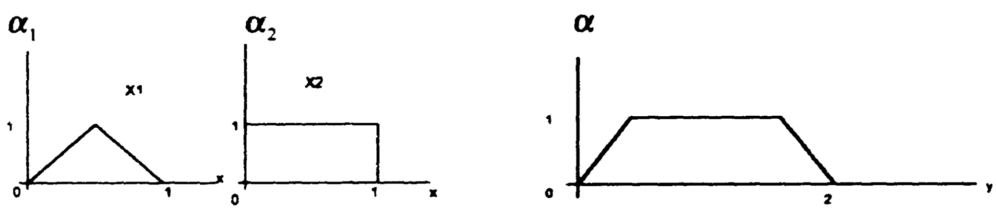
Fuzzy Sum:



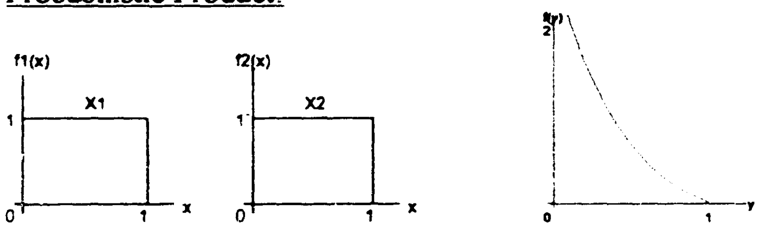
Comparison Table for Probabilistic Calculus and Fuzzy Algebra (Continued)

Probabilistic Sum:

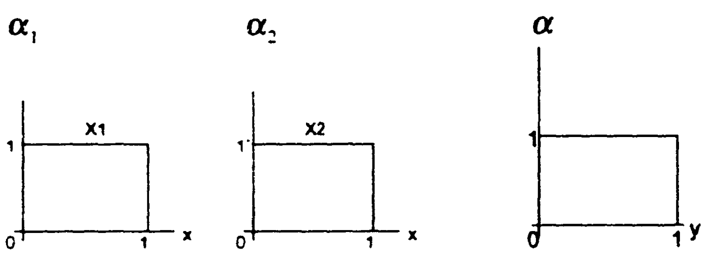
Fuzzy Sum:



Probabilistic Product:

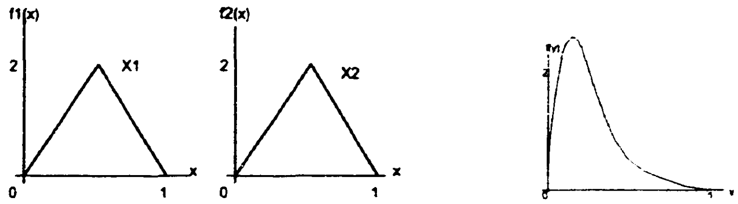


Fuzzy Product:

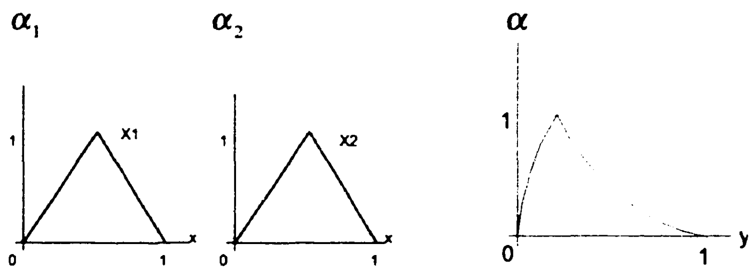


Comparison Table for Probabilistic Calculus and Fuzzy Algebra (Continued)

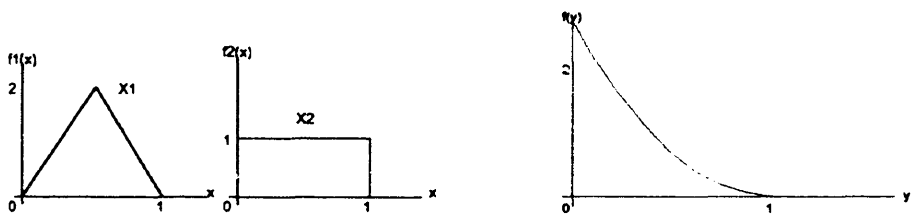
Probabilistic Product:



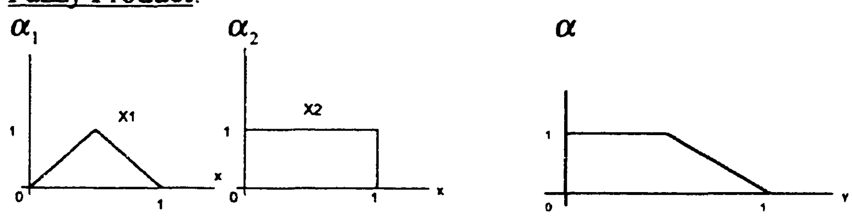
Fuzzy Product:



Probabilistic Product:



Fuzzy Product:



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