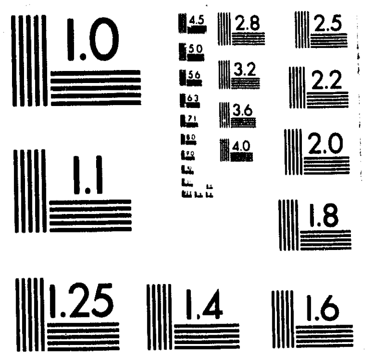

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Quarterly Progress Report

April 1, 1993 - June 30, 1993

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Introduction

In this quarter, we employed a kinetic theory to set up the boundary value problem for steady, fully developed, gravity-driven flows of identical, smooth, *highly inelastic* spheres down bumpy inclines. We treated the solid fraction, mean velocity, and components of the full second moment of fluctuation velocity as mean fields. In addition to the balance equations for mass and momentum, we treated the balance of the full second moment of fluctuation velocity as an equation that must be satisfied by the mean fields. However, in order to simplify the resulting boundary value problem, we retain fluxes of second moments in its isotropic piece only. The constitutive relations for the stresses and and collisional source of second moment depend explicitly on the second moment of fluctuation velocity, and the constitutive relation for the energy flux depends on gradients of granular temperature, solid fraction, and components of the second moment. The boundary conditions require that the flows are free of stress and energy flux at their tops, and that momentum and energy are balanced at the bumpy base.

In what follows, we provide the details of the boundary value problem. In the next quarter, we will develop a solution procedure, and employ it to obtain sample numerical solutions to the boundary value problem described here.

Balance Equations and Constitutive Relations

We are concerned here with steady, fully developed, gravity-driven flows of identical, smooth, highly inelastic spheres down bumpy inclines. The diameter of each sphere is σ, the mass density of each is ρ_p, and the coefficient of restitution between them is e. In what follows, e need not be close to unity. The vertical acceleration due to gravity is g, and the angle between the incline and the the horizontal is ϕ. We introduce an x_1-x_2-x_3 Cartesian coordinate system such that x_1 measures distance along the incline parallel to the flows, and x_2 measures distance above the incline perpendicular to the the flows. The flows are infinitely extended in the x_1- and x_3-directions.

The mean fields of interest in these granular flows are the solid fraction ν, the only non-zero velocity component u_1, the granular temperature T, and the components A_{11}, A_{22}, A_{33}, and A_{12}, of the *deviatoric* part of the second moment of particle fluctuation velocity. Their dimensionless counterparts ν, $u \equiv u_1/(\sigma g)^{1/2}$, $\tau \equiv T/\sigma g$, $a_{11} \equiv A_{11}/\sigma g$, $a_{22} \equiv A_{22}/\sigma g$, $a_{33} \equiv A_{33}/\sigma g$, and $a_{12} \equiv A_{12}/\sigma g$ depend on the dimensionless coordinate $y \equiv x_2/\sigma$ only. Their variations with y are governed by the x_1- and x_2-components of the balance of momentum, the balance of energy, and the x_1-x_1, x_2-x_2, x_3-x_3 and x_1-x_2 components of the balance of second moment.

Under these circumstances, the balance of mass is satisfied identically. If P_{ij} are the components of the pressure tensor, then in terms of their dimensionless counterparts $p_{ij} \equiv P_{ij}/\rho_p \sigma g$, the x_1- and x_2-components of the balance of momentum are,

$$p_{12}' = v\sin\phi \quad , \tag{1}$$

and

$$p_{22}' = -v\cos\phi \quad , \tag{2}$$

where primes denote differentiation with respect to y. The x_3-component of the balance of momentum demonstrates that p_{32} does not vary with y. The balance of energy is the isotropic part of the balance of the full second moment of fluctuation velocity. If Q_2 is the x_2-component of the energy flux, Γ is the rate of energy dissipation due to inelastic collisions, and their dimensionless counterparts are $q \equiv 2Q_2/\rho_p(\sigma g)^{3/2}$ and $\gamma \equiv -2\Gamma/\rho_p \sigma^{1/2} g^{3/2}$, then the balance of energy is,

$$q' = \gamma - 2p_{12}u' \quad . \tag{3}$$

The remaining equations are obtained from the deviatoric part of the balance of full second moment. In addition to the components P_{ij} of the pressure tensor, these equations involve the components Q_{ijk} and χ_{ij} of the flux and collisional source of the deviatoric part of the second moment. If the spatial gradients of Q_{ijk} are small compared to χ_{ij}, then, in terms of the dimensionless source components $\gamma_{ij} \equiv \chi_{ij}/\rho_p \sigma^{1/2} g^{3/2}$, the resulting approximate equations for a_{11}, a_{22}, and a_{12} are the x_1-x_1 deviatoric component of the balance of second moment,

$$\frac{4}{3} p_{12} u' = \gamma_{11} \quad ; \tag{4}$$

the x_2-x_2 deviatoric component of the balance of second moment,

$$\frac{-2}{3} p_{12} u' = \gamma_{22} \quad ; \tag{5}$$

and the x_1-x_2 component of the balance of second moment,

$$p_{22} u' = \gamma_{12} \quad . \tag{6}$$

The x_3-x_3 deviatoric component of the second moment equation determines a_{33}, and to within a minus sign is given by the sum of equations (4) and (5).

In what follows, we employ the constitutive theory derived by Richman and Martin [1993]. The constitutive relation for the shear stress p_{12} is given in terms of the solid fraction ν, the granular temperature τ, and the second moment component a_{12} by,

$$p_{12} = -2(1+e)\nu G\tau\left[\frac{2}{5\sqrt{\pi\tau}}u' - H\frac{a_{12}}{\tau}\right] \quad , \tag{7}$$

in which $G(\nu)$ is equal to $\nu(2-\nu)/2(1-\nu)^3$ and $H(G)$ is equal to $2[1+5/4(1+e)G]/5$. The normal pressure p_{11} is given in terms of ν, τ, and the deviatoric component a_{11} of second moment by,

$$p_{11} = 2(1+e)\nu G\tau\left[F + H\frac{a_{11}}{\tau}\right] \quad , \tag{8}$$

in which $F(G)$ is equal to $[1+1/2(1+e)G]$. Similarly, the remaining normal pressures p_{22} and p_{33} are given by,

$$p_{22} = 2(1+e)\nu G\tau\left[F + H\frac{a_{22}}{\tau}\right] \quad , \tag{9}$$

and

$$p_{33} = 2(1+e)\nu G\tau\left[F + H\frac{a_{33}}{\tau}\right] \quad . \tag{10}$$

Differences between the normal stresses result from corresponding differences between a_{11}, a_{22}, and a_{33}.

The energy flux q is related to gradients of τ, a_{22}, and ν according to the relation,

$$q = \frac{-4(1+e)\nu G\tau^{1/2}}{\pi^{1/2}}\left(\kappa\tau' + \lambda\tau\nu' + \eta\, a_{22}'\right) \quad , \tag{11}$$

in which the coefficients $\kappa(\nu, e)$, $\lambda(\nu, e)$, and $\eta(\nu, e)$ are given by,

$$\kappa = \left\{1 + \frac{9\pi(1+e)(2e-1)}{4(49-33e)}\left[1 + \frac{5}{3(1+e)^2(2e-1)G}\right]\left[1 + \frac{5}{6(1+e)G}\right]\right\} \quad , \tag{12}$$

$$\lambda = \frac{-9\pi e(1-e)}{4(49-33e)} \frac{d(\ln \nu G)}{d\nu} \left[1 + \frac{5}{6(1+e)G} \right] \quad , \tag{13}$$

and

$$\eta = \frac{2}{5} \left\{ 1 + \frac{25\pi(3e+1)(\beta+\alpha)}{24(3-e)(49-33e)} \left[1 + \frac{1}{(1+e)(\beta+\alpha)G} \right] \left[1 + \frac{5}{6(1+e)G} \right] \right.$$

$$\left. + \frac{5\pi\xi}{24(3-e)} \left[1 + \frac{1}{(1+e)\xi G} \right] \left[1 + \frac{5}{6(1+e)G} \right] \right\} \quad , \tag{14}$$

where $\beta \equiv (49-33e)[-6(1+e)/5+4(1+e)^2/3]/14(3e+1)$, $\alpha \equiv [-4/5-9(1+e)/5+2(1+e)^2/3]$, and $\xi \equiv [-4/5+6(1+e)/5+4(1+e)^2/21]$. If gradients of a_{22} are ignored and e is set equal to 1, then expression (11) reduces to the expression for the energy flux in assemblies of nearly elastic spheres obtained by Jenkins and Richman [1985].

The remaining constitutive quantity is the collisional source of second moment of fluctuation velocity. In it, we retain terms linear in a_{11}, a_{22}, a_{33}, a_{12}, and u'. In addition, we retain just those nonlinear terms that guarantee that, in the tensoral form of the balance of of second moment, the collisional contribution to the stress is multiplied only by the rate of strain. In this manner, the isotropic piece of the source of second moment is approximated by,

$$\gamma = \frac{-24\nu G(1-e^2)\tau^{3/2}}{\pi^{1/2}} \quad . \tag{15}$$

The corresponding result obtained by Jenkins and Richman [1985] may be obtained by replacing $(1-e^2)$ by $2(1-e)$ in expression (15). The deviatoric parts of the x_1-x_1 and x_2-x_2 components of the source of second moment are given in terms of ν, τ, a_{11}, a_{22}, a_{12}, p_{12}, and u' by the constitutive relations,

$$\gamma_{11} = \frac{-24\nu G(1+e)(3-e)\tau^{3/2}}{5\pi^{1/2}} \frac{a_{11}}{\tau} + (p_{12} - \nu a_{12})u' \quad , \tag{16}$$

and

$$\gamma_{22} = \frac{-24\nu G(1+e)(3-e)\tau^{3/2}}{5\pi^{1/2}} \frac{a_{22}}{\tau} - (p_{12} - \nu a_{12})u' \quad , \tag{17}$$

where p_{12} is given by equation (7). Similarly, the x_1-x_2 component of the source of second moment is,

$$\gamma_{12} = \frac{-24\nu G(1+e)\tau^{3/2}}{5}\left\{\frac{(3-e)}{\pi^{1/2}}\frac{a_{12}}{\tau} - \frac{(2-e)}{4}\frac{u'}{\tau^{1/2}}\right\} + [(p_{22} - p_{11}) - \nu(a_{22} - a_{11})]\frac{u'}{2} \quad , \qquad (18)$$

where p_{11} and p_{22} are given by equations (8) and (9). Constitutive relations (16), (17), and (18) have no counterparts in the theory of Jenkins and Richman [1985] for nearly elastic spheres.

In order to reduce the number of equations in the governing system, we employ constitutive relation (16) to eliminate γ_{11} from balance (4) to obtain,

$$\frac{a_{11}}{\tau} = \frac{-5\pi^{1/2}}{24\nu G(1+e)(3-e)\tau}\left[\frac{1}{3}p_{12} + \nu a_{12}\right]\frac{u'}{\tau^{1/2}} \quad , \qquad (19)$$

and constitutive relation (17) to eliminate γ_{22} from balance (5) to obtain,

$$\frac{a_{22}}{\tau} = \frac{-5\pi^{1/2}}{24\nu G(1+e)(3-e)\tau}\left[\frac{1}{3}p_{12} - \nu a_{12}\right]\frac{u'}{\tau^{1/2}} \quad . \qquad (20)$$

Equations (19) and (20) and constitutive relation (7) demonstrate that the deviatoric components a_{11} and a_{22} are sums of terms proportional to $(u')^2$ or to products of a_{12} and u'. These nonlinear terms were neglected by Jenkins and Richman [1985]. Consequently, they predicted that, for flows of nearly elastic spheres, the components a_{11}, a_{22}, and a_{33} all vanish. In that approximation, the constitutive equations (8), (9), and (10) simplify and guarantee that the normal pressures p_{11}, p_{22}, and p_{33} are all equal.

Finally, we employ constitutive relation (18) to eliminate γ_{12} from balance (6) to obtain,

$$\frac{a_{12}}{\tau} = \frac{-\pi^{1/2}(3e-1)}{12(3-e)}\left[1 + \frac{5}{2(1+e)(3e-1)G}\right]\frac{u'}{\tau^{1/2}} \quad , \qquad (21)$$

where we have neglected terms that are cubic in u', a_{12}, and products of u' and a_{12}. If equation (21) is employed to eliminate a_{12} from constitutive relation (7) and e is set equal to 1, then the resulting expression for the shear stress is identical to that obtained by Jenkins and Richman [1985].

Boundary Conditions

With appropriate conditions applied at the free surface and base of the incline, equations (1), (2), (3), (7), (9), (11), (15), (20), and (21) determine the variations with y of p_{12}, p_{22}, q, τ, γ, u', v, a_{12}, and a_{22}. Although the location of the free surface is not known, the stresses and the energy flux each vanish there; i.e.

$$p_{12} = 0 \quad \text{and} \quad p_{22} = 0 \quad , \tag{22}$$

and

$$q = 0 \quad . \tag{23}$$

Because the stresses both vanish at the top of the flow, ν may be eliminated between equations (1) and (2) to demonstrate that $p_{12}/p_{22}=-\tan\phi$.

If ν is equal to 0 and τ is not, then according to constitutive relation (9) the normal stress condition at the top of the flow is automatically satisfied. Near the top of the flow, therefore, ν is small, the normal stress may be approximated by

$$p_{22} = \nu(\tau + a_{22}) \quad , \tag{24}$$

and because the ratio p_{12}/p_{22} is everywhere equal to $-\tan\phi$, the shear stress may be approximated by

$$p_{12} = -\nu(\tau + a_{22}) \tan\phi \quad . \tag{25}$$

Furthermore, if equations (21) and (25) are employed to eliminate a_{12} and p_{12} from constitutive relation (7), then we find that near the top of the flow, u' is given approximately by,

$$u' = \frac{24(3-e)(1+e)(\tau+a_{22})\tan\phi}{5\pi^{1/2}\tau^{1/2}}\nu \quad . \tag{26}$$

With u' given by equation (26), the lowest order approximation of equation (7) dictates that,

$$a_{12} = -(\tau + a_{22}) \tan\phi \quad , \tag{27}$$

and with p_{12}, u', and a_{12} given by equations (25), (26), and (27), balance (20) yields,

$$1 + \frac{a_{22}}{\tau} = \frac{-3/2 + \sqrt{9/4 + 6\tan^2\phi}}{2\tan^2\phi} \quad . \tag{28}$$

For small values of ν and prescribed values of τ and ϕ, equation (28) fixes a_{22}, equations (24) and (25) fix p_{22} and p_{12}, and for prescribed values of e, equation (26) fixes u'. As ν approaches zero, so too do the stresses p_{22} and p_{12} and the velocity gradient u'. However, in the same limit the components a_{22} and a_{12} of second moment each approach *nonzero* limits that depend only on the inclination angle ϕ and the local value of τ.

Of interest also are the limiting behaviors of the gradients τ', ν', a_{12}' a_{22}' and u" as ν approaches zero. By differentiating approximations (24) and (28) with respect to y, for example, we find that

$$\nu' = \frac{-\nu\cos\phi}{f(\phi)\,\tau} - \frac{\nu\tau'}{\tau} \quad , \tag{29}$$

where $f(\phi)$ is given by the right-hand-side of equation (28), and

$$a_{22}' = [f(\phi) - 1]\,\tau' \quad . \tag{30}$$

If these are employed to eliminate ν' and a_{22}', then constitutive relation (11) for the energy flux demonstrates that τ', and therefore ν' and a_{22}', each approach zero with ν. Simple differentiation of approximations (26) and (27) with respect to y then demonstrates that both u" and a_{12}' approach zero in the same manner.

At the base of the incline (i.e. y=0), the rate **M** at which momentum is supplied to the flows by inelastic collisions between flow particles and the base must balance the traction vector at the base. Furthermore, the difference between the rate $-M_1u_1$ at which energy is supplied by slip work and the rate D at which it is absorbed by inelastic collisions between flow particles and the base must balance the energy flux at the base.

The transfer rates **M** and D depend on the geometry and dissipative nature of the incline. Here we focus on inclines that are flat surfaces to which identical, smooth, hemispherical particles of diameter d are randomly attached at an average distance s apart. In order to prevent flow particles from colliding with the flat part of the boundary, the maximum allowable value of s/d is $-1+(1+2\sigma/d)^{1/2}$. When a flow particle collides with a boundary particle the distance between their centers is $\delta\equiv(\sigma+d)/2$, and the energy dissipated is fixed by the coefficient of restitution e_w between them. A measure of the bumpiness of the boundaries is the angle $\theta\equiv\sin^{-1}(d+s)/(d+\sigma)$, which increases from 0 to $\pi/2$ as the boundaries evolve from perfectly flat to extremely bumpy.

We employ the general expressions for **M** and D obtained by Richman and Martin [1993] for assemblies of inelastic spheres that interact with bumpy boundaries described above. The expression for **M** involves an unknown factor that accounts for excluded volume and particle shielding at the boundary. If we first employ the balance between the x_2-components of **M** and the traction vector to write the unknown factor in terms of p_{22}, a_{22}, τ, and

θ, then the balance between the x_1-components of **M** and the traction vector determines the slip velocity u(0) according to,

$$\frac{u}{\tau^{1/2}} = \frac{-\pi^{1/2}}{2^{1/2} I}\left[1 + \frac{a_{22}}{\tau}\left(1 - \frac{3}{4}\sin^2\theta\right) \right]\frac{p_{12}}{p_{22}} + \frac{\delta}{\sigma}\frac{(2I - \sin^2\theta)}{2I}\frac{u'}{\tau^{1/2}} + \frac{\pi^{1/2}}{2^{3/2} I}\frac{a_{12}}{\tau} \quad , \qquad (31)$$

where $I(\theta) \equiv 2[2\csc^2\theta(1-\cos\theta)-\cos\theta]/3$. Furthermore, the energy flux at the boundary is determined by,

$$q = 2\left\{ -p_{12}u - \frac{2^{3/2}}{\pi^{1/2}}(1-e_w)\csc^2\theta(1-\cos\theta)\left[1 + \frac{a_{22}}{\tau}\left(1 - \frac{3}{4}\sin^2\theta\right) \right]^{-1}\tau^{1/2}p_{22} \right\} \quad . \qquad (32)$$

Conditions (22), (23), (31), and (32) are the five conditions needed to complete the set of equations (1), (2), (3), (7), (9), (11), (15), (20), and (21).

References

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