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LOS ALAMOS NATIONAL LABORATORY

THEORETICAL DIVISION, T–6 GROUP

FINAL REPORT

E. CARTAN MOMENT OF ROTATION IN CLASSICAL AND QUANTUM GRAVITY

Prepared by: Arkady Kheyfets, Ph. D.

Academic Rank: Associate Professor

Department and University: Department of Mathematics
North Carolina State University
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Abstract

The geometric construction of the E. Cartan moment of rotation associated to the spacetime curvature provides a geometric interpretation of the gravitational field sources and describes geometrically how the sources are "wired" to the field in standard geometrodynamics. E. Cartan moment of rotation yields an alternate way (as opposed to using variational principles) to obtain Einstein equations. The E. Cartan construction uses in an essential way the soldering structure of the frame bundle underlying the geometry of the gravitational field of general relativity. The geometry of Ashtekar's connection formulation of gravitation theory is based on a complex–valued self–dual connection that is defined not on the frame bundle of spacetime but, rather, on its complexification. We show how to transfer the construction of the E. Cartan moment of rotation to Ashtekar's theory of gravity and demonstrate that no spurious equations are produced via this procedure.

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I. Introduction.

The geometric construction of the moment of rotation associated to the spacetime curvature was introduced originally by E. Cartan. The meaning of this construction, as well as its importance for clear understanding of the geometric structure of general relativity was further investigated by C. W. Misner and J. A. Wheeler[2]. It was Wheeler[1] that first linked E. Cartan moment of rotation to the deeper foundational aspects of general relativity. Wheeler has also conjectured that such a relation can be extended to other basic field theories such as electrodynamics and Yang–Mills theory. His conjecture has been proven recently[3,4,5]. The difficulties of extending the E. Cartan construction to the field theories other than general relativity are related to the fact that this construction in general relativity uses, in an essential way, the soldering structure of the spacetime geometry. This structure appears to be absent in the geometry of electrodynamics and Yang–Mills theory. Nevertheless, the geometry of all basic field theories, when properly understood and reinterpreted, admits the E. Cartan construction. Furthermore, exactly as in general relativity, such an extension of the E. Cartan construction to other field theories describes geometrically how the sources of the fields are "wired" to the fields themselves and provides an alternate way (as opposed to the variational approach) to deduce the field equations in a way quite similar to that of obtaining Einstein equations in general relativity.

Ashtekar's theory of gravity introduces a new set of variables in the Hamiltonian (ADM) formulation of general relativity. Such a reformulation leads to a significant simplification[6,7,8,9] of both the evolution equations and constraints. It provides (1) a revolutionary new look at the whole structure of geometrodynamics, and (2) an opportunity of achieving considerable progress in many important problems of classical and quantum geometrodynamics. At this time Ashtekar's theory is viewed as a rapidly developing field of classical and quantum geometrodynamics.

Originally Ashtekar's theory was introduced in its Hamiltonian formulation. However, a Lagrangian formulation of Ashtekar's theory also exists[8,9]. This is fortunate because the results we obtain via an application of the E. Cartan construction can be compared much easier with the Lagrange formulation than with the Hamilton formulation.

The geometry of Ashtekar's formulation (sometimes also called the connection formulation) of general relativity is based on a complex valued self–dual connection that is, strictly speaking, not defined on the frame bundle of the real spacetime. The curvature of this connection is complex valued. The soldering form of Ashtekar's theory is, in general, complex valued (complex vector or spinor valued), and is defined on the complexified tangent spaces. In its variational formulation, the action of Ashtekar's theory is also complex valued. The usual worry in Ashtekar's theory is that the imaginary part of the curvature, and the imaginary part of the action in a variational approach, might lead to spurious equations; thus, making the theory different from general relativity even when the reality conditions are imposed. Fortunately, this does not occur when variational principles are used[6,9].

In this report we extend the E. Cartan moment of rotation construction to Ashtekar's theory and demonstrate that no spurious equation are produced via this procedure. Although the self–dual Riemann curvature is complex even in the case of real relativity, the E. Cartan moment of rotation turns out to be real in this case.

In Sec. II we describe the E. Cartan moment of rotation in standard general relativity. Here we introduce the terminology and our notations. Sec. III contains the translation of this construction in the language of the tetrad formulation, which is used frequently as a transitional step from standard general relativity to Ashtekar's theory[6]. Sec. IV considers the E. Cartan moment of rotation of the self–dual Riemann curvature and demonstrates that in real relativity it is real. The imaginary part of the E. Cartan moment of rotation vanishes due to the cyclic symmetries of the Riemann tensor. Sec. V provides a formulation of the results in the language of two–spinors. It contains the expression of the spinor components of the E. Cartan moment of rotation in terms of the unprimed spinor connection curvature.

II. E. Cartan Moment of Rotation in General Relativity.

The geometry underlying the spacetime formulation of general relativity can be defined on an orthonormal frame bundle with the Lorentz group as its structure group and spacetime as its base. The structure group is represented as the group of automorphisms of a four dimensional vector space – the space of values – isomorphic to the to the tangent space of spacetime at a point. The space of values at a point of spacetime can be considered as a copy of the tangent space to spacetime at this point.

The tangent space to spacetime at each point of spacetime is assumed to be endowed with a basis $\{\mathbf{e}_\mu\}_{\mu=0}^{3}$. In most cases we will assume that this basis is a coordinate basis ($\mathbf{e}_\mu = \partial/\partial x^\mu$) with the dual 1–form basis $\{dx^\mu\}_{\mu=0}^{3}$. The space of values, considered as a copy of the tangent space, is assumed to be spanned by the same basis $\{\mathbf{e}_\mu\}_{\mu=0}^{3}$. Spacetime has a pseudoriemannian metric $g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$ on it such that

$$g_{\mu\nu}(x) = \mathbf{e}_\mu(x) \cdot \mathbf{e}_\nu(x). \tag{1}$$

Although the tangent space and the space of values are described as two copies of the same space they should be treated quite differently in the description of the geometric structure of general relativity. Particularly, the Lie algebra of the structure group can be identified, as a vector space, with the space of bivectors (or 2–forms) of the space of values. All the geometric quantities (such as the connection form, the curvature form, etc.) are defined as linear maps from the tangent space (or its tensor products with itself and its dual) to the space of values (or its tensor products with itself and its dual).

The principal bundle of general relativity, being a frame bundle, admits the canonical soldering structure given by the canonical soldering form

$$\theta = \mathbf{e}_\mu \, \delta^\mu_\nu \, dx^\nu = \mathbf{e}_\mu \, dx^\mu. \tag{2}$$

In this formula $\mathbf{e}_\mu$ that stands to the left of the component expression δ^μ_ν is a vector in the space of values and dx^ν that stands to the right of the δ^μ_ν is a 1–form in the tangent space. From now on we are going to stick with the agreement that in all the expressions the vectors (multivectors, forms, etc.) standing to the left of the component expression are those of the space of values whereas the vectors (multivectors, forms, etc.) standing to the right of the component expression are those of the tangent space. It is clear that the canonical soldering form, in fact, is nothing but an isomorphism between the tangent space and the space of values.

The geometry itself is defined by a connection form on the principal bundle described above . Its pulldown to spacetime (a bivector valued 1–form) is

$$\Gamma = \mathbf{e}_\mu \wedge \mathbf{e}_\nu \, \Gamma^{\mu\nu}{}_\lambda \, dx^\lambda. \tag{3}$$

Here, just as in expression (2), the bivector $\mathbf{e}_\mu \wedge \mathbf{e}_\nu$ standing to the left of the $\Gamma^{\mu\nu}{}_\lambda$ is a bivector of the values space, and the form dx^λ standing to the right of $\Gamma^{\mu\nu}{}_\lambda$ acts on the vectors of the tangent space. This connection form determines the exterior covariant derivative. We use the symbol ∇ for its pulldown to spacetime. By pulldown to spacetime we mean the pullback of the form via a section of the frame bundle. When performing practical calculations at a point of spacetime it is often convenient to pick up a section horizontal at this point (such a choice of the section implies $\Gamma^{\mu\nu}{}_\lambda = 0$ at the point).

The curvature form of the geometry determined by (3) is defined as the exterior covariant derivative of the connection form. The pulldown of this curvature form to spacetime is expressed locally as

$$R = \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \, R^{\alpha\beta}{}_{\mu\nu} \, dx^\mu \wedge dx^\nu = \nabla\Gamma. \tag{4}$$

In general relativity physicists call R determined by the expression (4) the curvature of spacetime. This curvature satisfies the Bianchi identities

$$\nabla R = 0. \tag{5}$$

The connection in general relativity is restricted by the demand that it is compatible with the metric structure and the canonical soldering structure

$$\begin{aligned} \nabla g &= 0 \\ \nabla \theta &= 0. \end{aligned} \tag{6}$$

Such a connection is called the Levi–Civita connection.

In the standard formulation of general relativity[2] that we are considering here the space of values and the tangent space are closely related in the sense that the choice of the local coordinate system on spacetime fixes both the coordinate basis of the tangent space and the local section of the principal bundle (a basis of the space of values). However, they are treated very differently in operations involving derivatives which is of a crucial importance in considering the construction of E. Cartan moment of rotation and, particularly, when the operators of dualization and exterior covariant derivative are used in the same expression. Indeed, if we use the notations $*$ for the duality operator on the tangent space and $\star$ for the duality operator on the values space, then[2]

$$\begin{aligned} \nabla^* &\neq {}^*\nabla \\ \nabla^\star &= {}^\star\nabla. \end{aligned} \tag{7}$$

The E. Cartan moment of rotation is defined as the values space dual of the exterior product of the canonical soldering form and the curvature form

$$M = {}^{*}(\theta \wedge R)\,, \tag{8}$$

which can be rewritten using equations (2) and (4) in the following way

$$\begin{aligned} M &= {}^{*}\left(\mathbf{e}_\mu \wedge \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \, R^{\alpha\beta}{}_{\nu\lambda}\; dx^\mu \wedge dx^\nu \wedge dx^\lambda\right) \\ &= \mathbf{e}_\gamma\; \epsilon^{\gamma}{}_{\mu\alpha\beta}\, R^{\alpha\beta}{}_{\nu\lambda}\; dx^\mu \wedge dx^\nu \wedge dx^\lambda \\ &= \mathbf{e}_\gamma\; \epsilon^{\gamma}{}_{\mu\alpha\beta}\, R^{\alpha\beta}{}_{\nu\lambda}\; \epsilon^{\mu\nu\lambda\tau}\; d^3\Sigma_\tau \\ &= \mathbf{e}_\gamma\; G^{\gamma\tau}\; d^3\Sigma_\tau = \mathbf{e}_\alpha\; G^{\alpha\tau}\; \epsilon_{\tau\mu\nu\lambda}\; dx^\mu \wedge dx^\nu \wedge dx^\lambda = G, \end{aligned} \tag{9}$$

where G is the Einstein form (a vector valued 3–form) and $G^{\mu\tau}$ is the Einstein tensor.

Taking into account equations (7) (compatibility of the connection Γ with the canonical soldering structure), (5) (Bianchi identities), and the fact that the exterior covariant derivative operation Γ commutes with the internal space duality operator * we can write down the conservation equation for the Einstein form

$$\nabla G = \nabla\left[{}^{*}(\theta \wedge R)\right] = 0, \tag{10}$$

which suggests the standard form of the Einstein equation

$$G = T, \tag{11}$$

and provides a geometric interpretation of the source of gravitational field as the E. Cartan moment of rotation in the tetrad formulation of general relativity.

III. E. Cartan Moment of Rotation in Tetrad Formulation.

The tetrad formalism uses instead of the metric four linearly independent covariant vector fields e^I_μ related to the metric by the equation

$$g_{\mu\nu}(x) = e^i_\mu(x)\, e^j_\nu(x)\, \eta_{ij}, \tag{12}$$

where $\mu, \nu, \ldots$ are spacetime indices running from 0 to 3 , and $i, j, \ldots$ are internal indices also running from 0 to 3. The internal indices are raised and lowered with the Minkowski metric $\eta_{ij} = \mathrm{diag}[-1, 1, 1, 1]$.

The geometry underlying the tetrad formalism is defined on a $SO(3,1)$ principal bundle with the Lorentz group as its structure group and spacetime as its base. The structure group is represented as the group of automorphisms of a four dimensional vector space (internal space) with a basis $\{\mathbf{e}_i\}_{i=0}^{3}$. The tangent space to spacetime at each point of spacetime is assumed to be endowed with a basis $\{\mathbf{e}_\mu\}_{\mu=0}^{3}$. In most cases we will assume that this basis is a coordinate basis ($\mathbf{e}_\mu = \partial/\partial x^\mu$) with the dual 1–form basis $\{dx^\mu\}_{\mu=0}^{3}$. The described frame bundle has the canonical soldering structure determined by the canonical soldering form

$$\theta = \mathbf{e}_i \, e^i_\mu \, dx^\mu. \tag{13}$$

The form θ is a vector valued 1–form and defines a one to one map from the tangent space of spacetime at a point to the internal space.

The geometry itself is defined by a connection form. Its pulldown to spacetime is a bivector valued 1–form

$$\omega = \mathbf{e}_i \wedge \mathbf{e}_j \, \omega^{ij}{}_\mu \, dx^\mu. \tag{14}$$

The connection determined by (14) frequently called the spin connection[6] is defined to be compatible with the canonical soldering structure

$$\nabla_\omega \theta = 0, \tag{15}$$

where ∇_ω denotes the pulldown to spacetime of the exterior covariant derivative determined by the connection ω. In components, equation (15) takes the form

$$\partial_{[\mu} e^i_{\nu]} + \omega^i_{[\mu|j|} \, e^j_{\nu]} = 0. \tag{16}$$

The curvature bivector valued 2–form R (more precisely R is the pulldown of the curvature form to spacetime) is defined in a standard way as the covariant exterior derivative of the connection form ω

$$R = \mathbf{e}_i \wedge \mathbf{e}_j \, R^{ij}{}_{\mu\nu} \, dx^\mu \wedge dx^\nu = \nabla_\omega \omega, \tag{17}$$

so that it satisfies the Bianchi identities

$$\nabla_\omega R = 0. \tag{18}$$

The tetrad components $R^{ij}{}_{\mu\nu}$ of the curvature are related to the standard spacetime components $R^{\alpha\beta}{}_{\mu\nu}$ in a simple fashion

$$R^{ij}{}_{\mu\nu} = e^i_\alpha \, e^j_\beta \, R^{\alpha\beta}{}_{\mu\nu}. \tag{19}$$

The E. Cartan moment of rotation is defined as the internal space dual of the exterior product of the canonical soldering form and the curvature form

$$M_C = {}^\star (\theta \wedge R) \,, \tag{20}$$

which can be rewritten using equations (13) and (17) in the following way

$$\begin{aligned} M_C &= {}^\star \left(\mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k \; e^i_\mu \, R^{jk}{}_{\nu\lambda} \; dx^\mu \wedge dx^\nu \wedge dx^\lambda \right) \\ &= \mathbf{e}_m \; \epsilon^m{}_{ijk} \, e^i_\mu \, R^{jk}{}_{\nu\lambda} \; dx^\mu \wedge dx^\nu \wedge dx^\lambda \\ &= \mathbf{e}_m \; \epsilon^m{}_{ijk} \, e^i_\mu \, R^{jk}{}_{\nu\lambda} \; \epsilon^{\mu\nu\lambda\tau} \; d^3\Sigma_\tau \\ &= \mathbf{e}_m \; G^{m\tau} \; d^3\Sigma_\tau = \mathbf{e}_m \; G^{m\tau} \; \epsilon_{\tau\mu\nu\lambda} \; dx^\mu \wedge dx^\nu \wedge dx^\lambda = G, \end{aligned} \tag{21}$$

where G is the Einstein form (a vector valued 3–form) and $G^{m\tau} = e^m_\mu \, G^{\mu\tau}$ is the Einstein tensor expressed a half in internal components and a half in spacetime components.

Taking into account equations (15) (compatibility of the connection ω with the canonical soldering structure), (18) (Bianchi identities), and the fact that the exterior covariant derivative operation ∇_ω commutes with the internal space duality operator * we can write down the conservation equation for the Einstein form

$$\nabla_\omega G = \nabla_\omega [{}^*(\theta \wedge R)] = 0, \tag{22}$$

which suggests the standard form of the Einstein equation

$$G = T, \tag{23}$$

and provides a geometric interpretation of the source of gravitational field as the E. Cartan moment of rotation in the tetrad formulation of general relativity.

IV. E. Cartan Moment of Rotation in Ashtekar's Theory.

Geometry of Ashtekar's theory is quite different from that of general relativity in both spacetime and tetrad formulations. The principal bundle of Ashtekar's theory is an $SO(3,1;\mathbf{C})$ bundle over spacetime. Its structure group is the complex Lorentz group represented as the automorphisms group of a complex four dimensional vector space (internal space) obtained via a complexification of the internal space of the tetrad formalism.

Geometry on this principal bundle is determined by Ashtekar connection[6]

$$^+A = \mathbf{e}_i \wedge \mathbf{e}_j \; {}^+A^{ij}{}_\mu \, dx^\mu, \tag{24}$$

which is fixed to be the self–dual spin connection*

$$^+A^{ij}{}_\mu = \omega^{ij}{}_\mu + i\,\epsilon^{ij}{}_{mn}\,\omega^{mn}{}_\mu. \tag{25}$$

* We wish to point out the difference between the Rovelli's self–dual spin connection[6] and the self–dual connection used in some other references[9]. The difference is not important for our results. It would cause trivial changes in some expressions (factors 2 and $\frac{1}{2}$).

The curvature ${}^+F$ of the Ashtekar connection ${}^+A$ is defined in a standard fashion as the exterior covariant derivative $\nabla_{{}^+A}$ of ${}^+A$

$$ {}^+F = \mathbf{e}_i \wedge \mathbf{e}_j \, {}^+F^{ij}{}_{\mu\nu} \, dx^\mu \wedge dx^\nu = \nabla_{{}^+A} {}^+A. \tag{26} $$

The components ${}^+F^{ij}{}_{\mu\nu}$ are related to the components $R^{ij}\mu\nu$ as follows[6]

$$ {}^+F^{ij}{}_{\mu\nu} = R^{ij}\mu\nu + i\,\epsilon^{ij}{}_{mn}\, R^{mn}{}_{\mu\nu}. \tag{27} $$

We recall here that in both spacetime and tetrad formulations of general relativity the E. Cartan moment of rotation M_C is defined to be the left dual (the internal space dual in case of the tetrad formulation) of the exterior product of the canonical soldering form θ and the curvature form R.

The principal bundle of Ashtekar's theory is not a frame bundle in the strict sense of the word and, consequently, does not have the canonical soldering structure, very much like electrodynamics or Yang–Mills theory (although θ can be considered as an isomorphism between the complexified tangent space and the complex internal space). A more detailed analysis[3,5] has shown that in electrodynamics as well as in Yang–Mills fields it is possible to express the source of any of these fields in terms of E. Cartan moment of rotation, and that the soldering form taking a part in the E. Cartan construction in both cases is just the canonical soldering form of the frame bundle over spacetime. Following the same pattern, we form the E. Cartan moment of rotation in Ashtekar's theory using the expression (13) for the canonical soldering form in the tetrad basis

$$ M_C^A = {}^\star\left(\theta \wedge {}^+F\right). \tag{28} $$

From the definition of the Ashtekar connection (25), the compatibility of the connection with the spacetime soldering, commutativity of the exterior covariant differentiation with the duality operator on the internal space, and Bianchi identities

$$ \nabla_{{}^+A} {}^+F = 0 \tag{29} $$

it is easy to see that the E. Cartan moment of rotation (28) in Ashtekar's theory is conserved

$$\nabla_{+A} M_C^A = \nabla_{+A} \left[{}^* \left(\theta \wedge {}^+F \right) \right] = 0, \tag{30}$$

just as in both spacetime and tetrad formulations of general relativity. In expressions (29) and (30) ${}^+A$ stands for the pulldown of the Ashtekar connection to spacetime and ${}^+F$ for the pulldown of the Ashtekar curvature form to spacetime.

However, the curvature form of Ashtekar's theory contains, in addition to the standard tetrad curvature, an imaginary part proportional to the internal space dual of the tetrad curvature form. The usual worry in Ashtekar's theory[6,7,9] is that such an imaginary part might produce spurious equations and in this way lead to the theory different from general relativity. As it is well known[8,9] this does not occur when variational principles are used. To see whether it happens or not in our case we inspect closer the expression (28) for the E. Cartan moment of rotation via substituting expressions (13) for θ and (26), (27) for ${}^+F$. It is convenient to rewrite equations (26), (27) in the form

$${}^+F = R + i\,{}^*R. \tag{31}$$

Thus, the expression (28) for M_C^A yields

$$M_C^A = {}^*(\theta \wedge R) + i\,{}^*(\theta \wedge {}^*R)\,. \tag{32}$$

The first term of this expression yields the standard expression for the Einstein form (an internal vector valued 3–form) G as we have already shown in Sec. III of this report (cf. equation (21) and the related discussion). The second term can be calculated by means of straightforward performing all the operations of dualization and exterior multiplication listed in its expression, and using the standard identity for ϵ^{ijmn}

$$\epsilon^{mi}{}_{np}\,\epsilon_{jq}{}^{np} = -2(\delta_j^m \delta_q^i - \delta_q^m \delta_j^i). \tag{33}$$

The result is

$$
{}^\star(\theta \wedge {}^\star R) = \mathbf{e}_m \; e_{i\nu} \, R^{mi}{}_{\mu\lambda} \, dx^\nu \wedge dx^\mu \wedge dx^\lambda. \tag{34}
$$

Using the relations

$$
\begin{aligned}
&R^{mi}{}_{\mu\lambda} = e^m_\alpha \, e^i_\beta \, R^{\alpha\beta}{}_{\mu\lambda} \\
&e^i_\beta \, e_{i\nu} = g_{\beta\nu} \\
&e^m_\alpha \, \mathbf{e}_m = \mathbf{e}_\alpha
\end{aligned} \tag{35}
$$

between the tetrad and spacetime variables (34) can be reduced to

$$
\begin{aligned}
{}^\star(\theta \wedge {}^\star R) &= \mathbf{e}_m \; e^m_\alpha \, e^i_\beta \, e_{i\nu} \, R^{mi}{}_{\mu\lambda} \, dx^\nu \wedge dx^\mu \wedge dx^\lambda \\
&= \mathbf{e}_\alpha \; g_{\beta\nu} \; R^{\alpha\beta}{}_{\mu\lambda} \, dx^\nu \wedge dx^\mu \wedge dx^\lambda \\
&= \mathbf{e}_\alpha \; R^\alpha{}_{\nu\mu\lambda} \, dx^\nu \wedge dx^\mu \wedge dx^\lambda = 0.
\end{aligned} \tag{36}
$$

The last equality in (36) follows from the identity satisfied by the Riemann tensor

$$
R^\alpha{}_{\nu\mu\lambda} + R^\alpha{}_{\mu\lambda\nu} + R^\alpha{}_{\lambda\nu\mu} = 0. \tag{37}
$$

We can summarize our analysis by the statement that the E. Cartan moment of rotation in Ashtekar's theory is real and, just as in standard relativity, is equal to the Einstein form

$$
{}^\star\left(\theta \wedge {}^+F\right) = {}^\star(\theta \wedge R) = G. \tag{38}
$$

V. Two–Component Spinor Language.

As it has been noted above, in Ashtekar's theory of gravity we deal only with the self–dual part of of a frame bundle connection. Therefore, in this theory it is convenient to use a two–component spinor language[10], as it is done in most of the literature on the subject[7,8,9]. In most of this section we use common index notations that can be interpreted by a reader either as abstract indices or as the components of spinors in a basis. When the two–component spinor language is used the internal tetrad indices $i, j, \ldots$ are replaced by the pairs of unprimed and primed indices $AA', AB', \ldots$ referring to the internal two dimensional vector space (over complex numbers) of spinors and its complex conjugate. The internal space of spinors is assumed to be endowed with a fixed non–degenerate 2–form ε_{AB}, its inverse ε^{AB}, and the conjugates $\varepsilon_{A'B'}$ and $\varepsilon^{A'B'}$ which are used to raise and lower the spinor indices[10].

Just as it is the case for all other formulations of general relativity the tangent space to spacetime is soldered to the internal space. This time the soldering form $\sigma^{AA'\mu}$ is an isomorphism (over the field of complex numbers) between the complexified tangent space and the four dimensional space of $(1,1)$ spinors. The soldering form $\sigma^{AA'\mu}$ is a vector of the (complexified) tangent space and a $(1,1)$ spinor of the internal space. Using the tetrad field we can transform it into a 1–form on the complexified tetrad internal space $\sigma^{AA'}{}_i$ via the relation

$$\sigma^{AA'\mu} = e^{i\mu}\,\sigma^{AA'}{}_i. \tag{39}$$

In the case of real relativity $\sigma^{AA'\mu}$ is required to be such that $\overline{\sigma^{AA'\mu}} = \sigma^{AA'\mu}$. In such a case $\sigma^{AA'\mu}$ is related to the spacetime metric in a simple way

$$g^{\mu\nu} = \sigma^{AA'\mu}\,\sigma_{AA'}{}^{\nu}. \tag{40}$$

An $SL(2,C)$ connection $A_{BC\mu}$ and the related covariant derivative $\mathcal{D}_\mu$ is defined on the unprimed spinors bundle in such a way that

$$\mathcal{D}_\mu\,\varepsilon_{AB} = 0. \tag{41}$$

The action of the covariant derivative $\mathcal{D}_\mu$ on an unprimed spinor λ_B is expressed in terms of the connection 1–form $A_{BC\mu}$ by

$$\mathcal{D}_\mu\lambda_B = \partial_\mu\lambda_B + A_B{}^C{}_\mu\,\lambda_C. \tag{42}$$

It follows from (41) that $A_B{}^C{}_\mu$ is traceless, or equivalently $A_{BC\mu} = -A_{CB\mu}$. It is clear that $A_{BC\mu}$ has $4\times 3 = 12$ independent complex components. It is equivalent to a self–dual connection of Sec. IV via the one to one correspondence between ${}^+A^{ij}{}_\mu$ and $A_{BC\mu}\,\varepsilon_{B'C'}$ when the pair of indices BB' is identified with the tetrad index i. The identification is given by a fixed map $\sigma^{BB'}{}_i$ between the internal tetrad space and the internal space of $(1,1)$ spinors

$$\sigma^{BB'i}\,\sigma^{CC'j}\,A_{BC\mu}\,\varepsilon_{B'C'} = {}^+A^{ij}{}_\mu. \tag{43}$$

Likewise, the relation between the curvature $F_{BC\mu\nu}$ of the connection $A_{BC\mu}$ defined by

$$\frac{1}{2}F_{BC\mu\nu} = \partial_{[\mu}A_{|BC|\nu]} + A_B{}^P{}_{[\mu}\,A_{|PN|\nu]} \tag{44}$$

and the self–dual Riemann curvature ${}^+F^{ij}{}_{\mu\nu}$ is given by

$$\sigma^{BB'i}\,\sigma^{CC'j}\,F_{BC\mu\nu}\,\varepsilon_{B'C'} = {}^+F^{ij}{}_{\mu\nu}. \tag{45}$$

As we already know (cf. equation (38)) the E.Cartan moment of rotation does not change (and, in particular, remains to be real) when the Riemann curvature R in it is replaced with the self–dual curvature ${}^+F$

$$\begin{aligned} M = M^A_C &= {}^*(\theta\wedge R) = {}^*\left(\theta\wedge {}^+F\right) = G \\ &= \mathbf{e}_i\, G^{i\mu}\, d^3\Sigma_\mu = \mathbf{e}_\nu\, G^{\nu\mu}\, d^3\Sigma_\mu, \end{aligned} \tag{46}$$

where $G^{i\mu}$ and $G^{\nu\mu}$ are the tetrad and spacetime components of the Einstein tensor respectively. Equivalently, $G^{i\mu}$ and $G^{\nu\mu}$ can be called the tetrad and spacetime components of the E. Cartan moment of rotation. It follows from the expression (46) that the spacetime components of the E. Cartan moment of rotation can be expressed in terms of the self–dual Riemann curvature as follows

$$G^{\mu\nu} = {}^+F^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}\,{}^+F, \tag{47}$$

where ${}^{+}F^{\mu\nu}$ and ${}^{+}F$ are the Ricci part and the scalar part of the self–dual Riemann curvature

$$
\begin{aligned}
{}^{+}F^{\mu\nu} &= {}^{+}F^{\alpha\mu}{}_{\alpha}{}^{\nu} \\
{}^{+}F &= {}^{+}F^{\alpha}{}_{\alpha}.
\end{aligned}
\tag{48}
$$

In expressions (47) and (48) we have switched from the internal space indices of the Sec. IV to all spacetime indices. To avoid an ambiguity we should point out that the self–duality in these expressions means the self–duality with respect to the first pair of indices of ${}^{+}F^{\mu\nu\alpha\beta}$.

The expressions for the spinor components $G^{AB'\mu}$ of the E. Cartan moment of rotation can be obtained easily based on the equation (47)

$$
G^{AB'\mu} = \sigma^{AB'}{}_{i}\, G^{i\mu} = \sigma^{AB'}{}_{\nu}\, G^{\nu\mu} = \sigma^{AB'}{}_{\nu}\left({}^{+}F^{\nu\mu} - \frac{1}{2}\,{}^{+}F\right) \tag{49}
$$

via a substitution into it the expressions

$$
\begin{aligned}
{}^{+}F^{\alpha\beta}{}_{\mu\nu} &= \sigma^{BB'\alpha}\,\sigma^{CC'\beta}\,F_{BC\mu\nu}\,\varepsilon_{B'C'} \\
{}^{+}F^{\alpha}{}_{\nu} &= \sigma^{BB'\mu}\,\sigma^{CC'\beta}\,F_{BC\mu\nu}\,\varepsilon_{B'C'} \\
{}^{+}F &= \sigma^{BB'\mu}\,\sigma^{CC'\nu}\,F_{BC\mu\nu}\,\varepsilon_{B'C'}
\end{aligned}
\tag{50}
$$

together with the expression (40) for $g^{\mu\nu}$.

The resulting expression for the spinor components $G^{B'}_{C}{}^{\mu}$ of the E. Cartan moment of rotation is

$$
G^{B'}_{C}{}^{\mu} = \sigma^{BB'\nu}\,F_{BC}{}^{\mu}{}_{\nu} - \frac{1}{2}\,\sigma^{B'}_{C}{}^{\mu}\left(\sigma^{MM'\alpha}\,\sigma^{N}_{M'}{}^{\beta}\,F_{MN\alpha\beta}\right). \tag{51}
$$

This expression is not manifestly real. However, in the case of real relativity it is, indeed real, which is clear from the considerations of the Sec. IV and reality of the $\sigma^{AA'\mu}$. Comparing the equation (51) with the gravitational field equation

$$
\sigma^{BB'\nu}\,F_{BC}{}^{\mu}{}_{\nu} - \frac{1}{2}\,\sigma^{B'}_{C}{}^{\mu}\left(\sigma^{MM'\alpha}\,\sigma^{N}_{M'}{}^{\beta}\,F_{MN\alpha\beta}\right) = 0, \tag{52}
$$

obtained by Jacobson and Smolin[9] from the Lagrangian variational principle we conclude that this equation is the statement in spinor language of the fact that in vacuum the E. Cartan moment of rotation is equal to zero, as it is supposed to be.

VI. Conclusion.

We have demonstrated that the construction of the E. Cartan moment of rotation can be extended to the Ashtekar's theory of gravity. Despite the fact that the self–dual curvature is complex even in real relativity the E. Cartan moment of rotation in this case is real. The imaginary part of the E. Cartan moment of rotation vanishes due to the cyclic symmetries of the Riemann tensor. We have also shown that the two–spinor components of the E. Cartan moment of rotation can be expressed in terms of the curvature of the unprimed spinor connection. Such an expression coincides with the left hand side of the Lagrangian equation obtained by Jacobson and Smolin from variational principles. This leads us to an elegant and transparent geometric interpretation of their result.

To put it differently, the E. Cartan moment of rotation and, together with it the whole structure of classical general relativity appears to be not sensitive to the choice of variables used to describe the gravity field. It can be described equally well in standard metric variables, in tetrad variables as well as in Ashtekar's variables. The present report does not provide any indications whether an alternate choice of variables (as opposed to standard metric variables) can provide any advantages in gravity quantization.

The answer to this question can be obtained only via a careful analysis of the initial value problem in different variables and a study of separation of truly dynamic gravitational degrees of freedom in different approaches.

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