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The 3D Vector Potential, Magnetic Field and Stored Energy in a Thin cos2θ Coil Array.*

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Abstract

The vector potential and the magnetic field have been derived for an arrays of quadrupole magnets with thin Cos(2θ) current sheet placed at r=R.[bc]. The field strength of each coil within the array, varies purely as a Fourier sinusoidal series of the longitudinal coordinate z in proportion to ω_mz, where $\omega_m = \frac{(2m-1)\pi}{L}$, L denotes the *half-period*, and m=1,2,3 etc. The analysis is based on the expansion of the vector potential in the region external to the windings of a linear 3D quad, and a revision of that expansion by the application of the "Addition Theorem" from that around the coil center to that around any arbitrary point in space .

The expression of the current density J in a quad (a form that satisfies the conservation condition $\nabla \cdot \vec{J_s} = \frac{\partial J_z}{\partial z} + \frac{1}{R}\frac{\partial J_\theta}{\partial \theta} = 0$ as required), and its contribution to the vector-potential $\vec{A}$ and magnetic field $\vec{B}$ in an infinite array of quadrupole magnets, are listed below for the region r≤R (excluding the self field of the coil with its center at 0,0)

$$\vec{J}(\theta, z)|_{r=R} = \sum_{m=1} J_{0z,m}\left[\left(\frac{\omega_m R}{2}\right)\sin 2\theta \sin \omega_m z \hat{e}_\theta + \cos 2\theta \cos \omega_m z \hat{e}_z\right]$$

$$J_{0z,m} = -\frac{1}{\mu_0}\frac{8RG_{2,m}}{(\omega_m R)^3 K_2'(\omega_m R)} \quad (\mathbf{A/m}) \quad \text{and} \quad G_{2,m} \text{ gradient at } z=0$$

$$\omega_m = \frac{(2m-1)\pi}{L} \quad \text{where L denotes the half period.}$$

$$\vec{A}_r = \sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m} R(4k-3)!}{2\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}}\left[C^+_{k,m} I'_{2(2k-1)}(\omega_m r) + 2(2k-1)C^-_{k,m}\frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r}\right]\cos 2(2k-1)\theta \sin \omega_m z$$

$$\vec{A}_\theta = -\sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m} R(4k-3)!}{2\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}}\left[C^-_{k,m} I'_{2(2k-1)}(\omega_m r) + 2(2k-1)C^+_{k,m}\frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r}\right]\sin 2(2k-1)\theta \sin \omega_m z$$

$$\vec{A}_z = \sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m} R(4k-3)!}{2\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} C^+_{k,m} I_{2(2k-1)}(\omega_m r)\cos 2(2k-1)\theta \cos \omega_m z$$

$$C^+_{k,m} = \sum_{i=1}\sum_{j=1}\left[K_{4(k-1)}(\omega_m S_{i,j})\cos 4(k-1)\theta_{0i,j} + K_{4k}(\omega_m S_{i,j})\cos 4k\theta_{0i,j}\right]\frac{16 I_2(\omega_m R)}{(4k-3)!}\left(\frac{\omega_m R}{2}\right)^{4k-2}$$

$$C^-_{k,m} = \sum_{i=1}\sum_{j=1}\left[K_{4(k-1)}(\omega_m S_{i,j})\cos 4(k-1)\theta_{0i,j} - K_{4k}(\omega_m S_{i,j})\cos 4k\theta_{0i,j}\right]\frac{16 I'_2(\omega_m R)}{(4k-3)!}\left(\frac{\omega_m R}{2}\right)^{4k-1}$$

Where I_n and K_n are the "modified" Bessel functions of the first and second kind of order n, and the prime denotes differentiation with respect to the argument. The summation i,j is over the infinite number of quads in the first octant of the array.

b "Multipoles in Cos(2θ) Coil Arrays – Type I", SC-MAG-583, LBID-2203, May 1997.

c "Multipoles in Cos(2θ) Coil Arrays – Type II", SC-MAG-596, June 1997.

The magnetic field components are,

$$B_r = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} (4k-3)! C^-_{k,m} I'_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z$$

$$B_\theta = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} 2(2k-1)(4k-3)! C^-_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \cos 2(2k-1)\theta \cos \omega_m z$$

$$B_z = -\sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} (4k-3)! C^-_{k,m} I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \sin \omega_m z$$

The format used here for A and B was specifically chosen to avoid a singularity that may rise when L is large (e.g. when the 3d problem reduces to 2d). Appendix B contains plots of both C^+ and C^-.

In calculating the values of the multipole coefficients we need to distinguish between coils placed on lines of symmetry and coils placed elsewhere.

θ=45°

On the 45 degree line of symmetry we use,

$$\omega_m S_{i,j} = (2m-1) 2\sqrt{2} i \frac{\pi S}{L}$$

$$C^+_{k,m} = -\frac{1}{2} \sum_{i=1} (-1)^k \left[-K_{4(k-1)}(\omega_m S_{i,j}) + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}\right] \frac{16 I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-2}$$

$$C^-_{k,m} = -\frac{1}{2} \sum_{i=1} (-1)^{k+1} \left[K_{4(k-1)}(\omega_m S_{i,j}) + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}\right] \frac{16 I'_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-1}$$

θ=0°

Summing on this line of symmetry requires ,

$$\omega_m S_{i,j} = 2(2m-1) i \frac{\pi S}{L}$$

and

$$C^+_{k,m} = -\frac{1}{2} \sum_{i=1} (-1)^i \left[K_{4(k-1)}(\omega_m S_{i,j}) + K_{4k}(\omega_m S_{i,j})\right] \frac{16 I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-2}$$

$$C^-_{k,m} = -\frac{1}{2} \sum_{i=1} (-1)^i \left[K_{4(k-1)}(\omega_m S_{i,j}) - K_{4k}(\omega_m S_{i,j})\right] \frac{16 I'_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-1}$$

Coils with centers in the first octant but NOT on lines of symmetry

This part also requires,

$$\omega_m S_{i,j} = 2(2m-1)\sqrt{(i+j)^2 + j^2}\frac{\pi S}{L}$$
$$\theta_{0i,j} = \tan^{-1}\frac{j}{(i+j)}$$

and

$$C^+_{k,m} = -\sum_{i=1}\sum_{j}(-1)^{i+2j}\left[K_{4(k-1)}(\omega_m S_{i,j})\cos 4(k-1)\theta_{0i,j} + K_{4k}(\omega_m S_{i,j})\cos 4k\theta_{0i,j}\right]\frac{16 I_2(\omega_m R)}{(4k-3)!}\left(\frac{\omega_m R}{2}\right)^{4k-2}$$
$$C^-_{k,m} = -\sum_{i=1}\sum_{j}(-1)^{i+2j}\left[K_{4(k-1)}(\omega_m S_{i,j})\cos 4(k-1)\theta_{0i,j} - K_{4k}(\omega_m S_{i,j})\cos 4k\theta_{0i,j}\right]\frac{16 I_2'(\omega_m R)}{(4k-3)!}\left(\frac{\omega_m R}{2}\right)^{4k-1}$$

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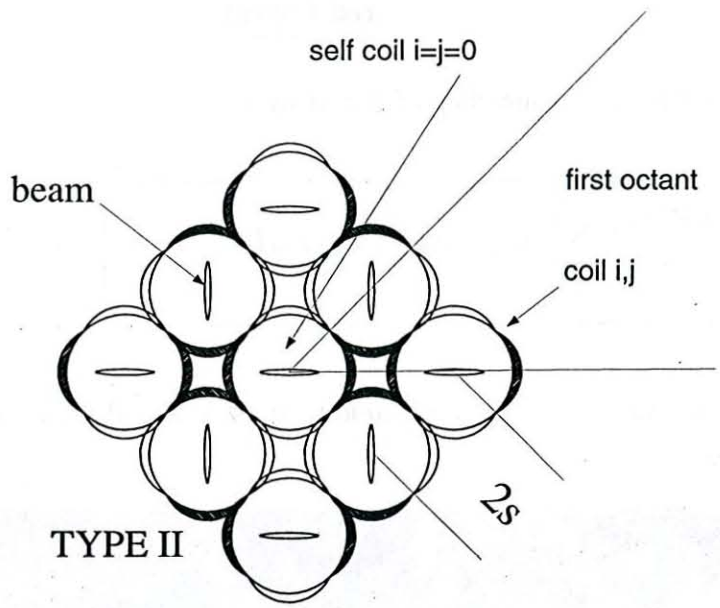

Figure 1 Cross section showing current density arrangement

From Figures 6–8 in Appendix B we can estimate the effect the geometry has on the values of $C^+_{k,m}$ and $C^-_{k,m}$. If we just use the fundamental term m=1 and vary the value of k, we notice that for k=1, which is the contribution to quadrupole term, the C's drop to 1/2 of their maximum value when S/R changes from 1.0 to 1.2. For the same change in S/R the dodecapole (k=2) and the 20's pole (k=3) will be reduced by a factor of 1/4 and 1/8 respectively. All harmonic terms do not vary significantly as long as $\frac{\pi R}{L} \leq 0.5$

Contributions from the self coil at i-0 j=0, r≤R

$$\vec{A}_r = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4}(\omega_m R)[K_3(\omega_m R) I_3(\omega_m r) - K_1(\omega_m R) I_1(\omega_m r)] \cos 2\theta \sin \omega_m z$$

$$\vec{A}_\theta = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4}(\omega_m R)[K_3(\omega_m R) I_3(\omega_m r) + K_1(\omega_m R) I_1(\omega_m r)] \sin 2\theta \sin \omega_m z$$

$$\vec{A}_z = \sum_{m=1} \mu_0 J_{0z,m} R K_2(\omega_m R) I_2(\omega_m r) \cos 2\theta \cos \omega_m z$$

$$B_r = \sum_{m=1} \frac{\mu_0 J_{0z,m}}{2}(\omega_m R)^2 K_2^{'}(\omega_m R) I_2^{'}(\omega_m r) \sin 2\theta \cos \omega_m z$$

$$B_\theta = \sum_{m=1} \mu_0 J_{0z,m} (\omega_m R)^2 K_2^{'}(\omega_m R) \frac{I_2(\omega_m r)}{(\omega_m r)} \cos 2\theta \cos \omega_m z$$

$$B_z = -\sum_{m=1} \frac{\mu_0 J_{0z,m}}{2}(\omega_m R)^2 K_2^{'}(\omega_m R) I_2(\omega_m r) \sin 2\theta \sin \omega_m z$$

Stored Energy

The energy stored in a single coil member of the array is,

$$E_{total} = -\frac{\pi R^2 L \mu_0}{8} \sum_{m=1} J_{0z,m}^2 (\omega_m R)^2 K_2^{'}(\omega_m R) I_2^{'}(\omega_m R) \left[1 + \frac{4C_{1,m}^{-}}{(\omega_m R)^3 K_2^{'}(\omega_m R)}\right]$$

where the first term in the square bracket corresponds to the self coil energy and the second term arises from all other coils in the array.

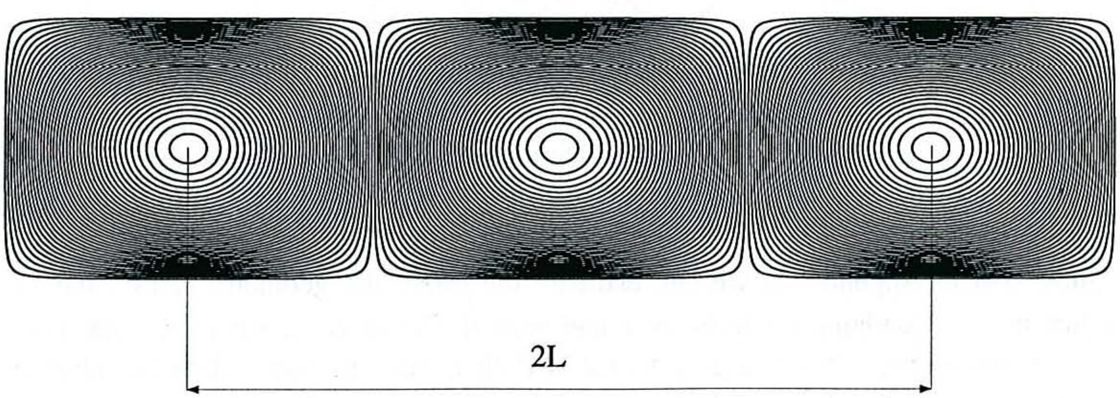

Figure 2 Top view of a single periodic coil array of a single function (m=1) quad.

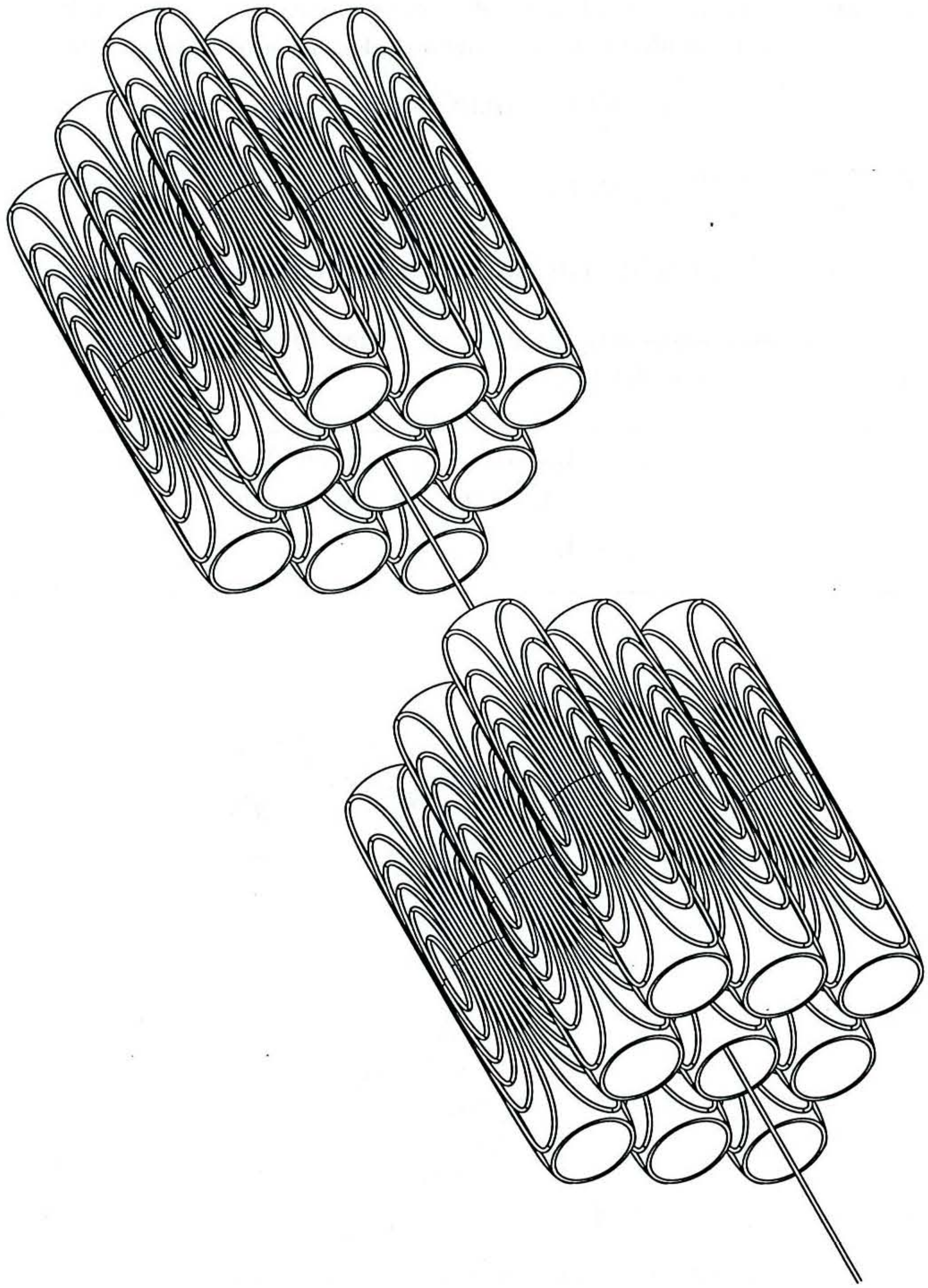

Figure 3 View of a 3x3 quadrupole array. The windings (of constant current) correspond to three terms m=1,2,3 which provide free space between coils.[d]

[d] This 3d CAD model was made on ProE by Ken Chow.

Analysis

Consider a quadrupole with its center located at $(S\cos\theta_0, S\sin\theta_0)$ as shown in Fig. 4. The expansion of the vector potential[e] in the region outside the current sheet (ρ>R) and around that center is,

$$\vec{A}_\rho = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R)[I_3(\omega_m R) K_3(\omega_m \rho) - I_1(\omega_m R) K_1(\omega_m \rho)] \cos 2\beta \sin \omega_m z$$
$$\vec{A}_\beta = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R)[I_3(\omega_m R) K_3(\omega_m \rho) + I_1(\omega_m R) K_1(\omega_m \rho)] \sin 2\beta \sin \omega_m z$$
$$\vec{A}_z = \sum_{m=1} \mu_0 J_{0z,m} R I_2(\omega_m R) K_2(\omega_m \rho) \cos 2\beta \cos \omega_m z$$

The relations between the components of the vector around (0,0) — (A_r, A_θ, A_z) and the above components (A_ρ, A_β, A_z) around $(S\cos\theta_0, S\sin\theta_0)$ are,

$$A_r = A_\rho \cos(\theta - \beta) + A_\beta \sin(\theta - \beta)$$
$$A_\theta = -A_\rho \sin(\theta - \beta) + A_\beta \cos(\theta - \beta)$$
$$A_z = A_z$$

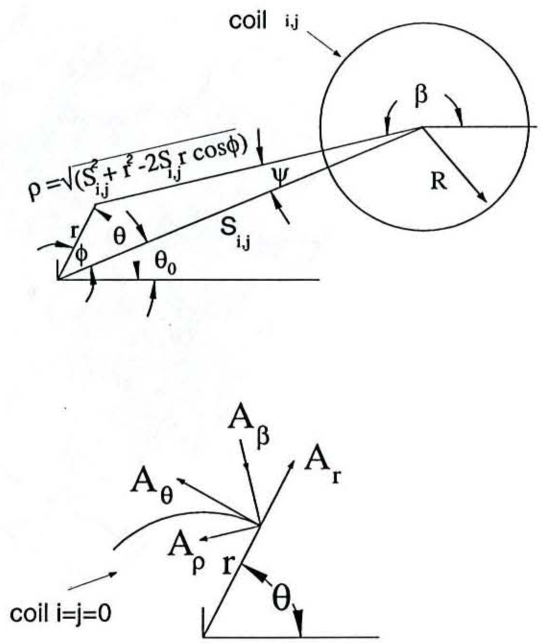

Figure 4 The geometry associated with the Addition Theorem — top, and components of the vector potential — bottom.

Proceeding, we shall make use of the following relationships,

The unit vector relations are:

$$\hat{e}_\rho = -\cos(\beta - \pi)\hat{e}_x - \sin(\beta - \pi)\hat{e}_y = \cos\beta\hat{e}_x + \sin\beta\hat{e}_y$$
$$\hat{e}_\beta = \sin(\beta - \pi)\hat{e}_x - \cos(\beta - \pi)\hat{e}_y = -\sin\beta\hat{e}_x + \cos\beta\hat{e}_y$$
$$\hat{e}_x = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta$$
$$\hat{e}_y = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$$

[e] "Combined Right and Left Hand Helical Function Magnets", S.Caspi, SC-MAG-534, January 1996.

Therefore,

$$\hat{e}_\rho = \cos(\theta-\beta)\hat{e}_r - \sin(\theta-\beta)\hat{e}_\theta$$
$$\hat{e}_\beta = \sin(\theta-\beta)\hat{e}_r + \cos(\theta-\beta)\hat{e}_\theta$$
or
$$\hat{e}_r = \cos(\theta-\beta)\hat{e}_\rho + \sin(\theta-\beta)\hat{e}_\beta$$
$$\hat{e}_\theta = -\sin(\theta-\beta)\hat{e}_\rho + \cos(\theta-\beta)\hat{e}_\beta$$

Addition Theorem

In deriving the expansion around (0,0) we shall make use of the Addition Theorem as described in Reference[f]

$$K_3(\omega_m\rho)\begin{Bmatrix}\cos 3\psi\\ \sin 3\psi\end{Bmatrix} = \sum_{k=-\infty}^{\infty} K_{3+k}(\omega_m S) I_k(\omega_m r)\begin{Bmatrix}\cos k\phi\\ \sin k\phi\end{Bmatrix}$$
$$K_2(\omega_m\rho)\begin{Bmatrix}\cos 2\psi\\ \sin 2\psi\end{Bmatrix} = \sum_{k=-\infty}^{\infty} K_{2+k}(\omega_m S) I_k(\omega_m r)\begin{Bmatrix}\cos k\phi\\ \sin k\phi\end{Bmatrix}$$
$$K_1(\omega_m\rho)\begin{Bmatrix}\cos \psi\\ \sin \psi\end{Bmatrix} = \sum_{k=-\infty}^{\infty} K_{1+k}(\omega_m S) I_k(\omega_m r)\begin{Bmatrix}\cos k\phi\\ \sin k\phi\end{Bmatrix}$$
$$\rho = \sqrt{S^2 + r^2 - 2Sr\cos\phi}$$

for simplicity we have dropped the i,j index for the parameter S.

We proceed in the derivation of the vector potential using the above relations and,

$$\phi = \theta - \theta_0$$
$$\beta = \pi + \theta_0 - \psi$$
$$\theta - \beta = \phi + \psi - \pi$$
$$\cos(3\beta-\theta) = -\cos(3\theta_0-\theta)\cos 3\psi - \sin(3\theta_0-\theta)\sin 3\psi$$
$$\sin(3\beta-\theta) = -\sin(3\theta_0-\theta)\cos 3\psi + \cos(3\theta_0-\theta)\sin 3\psi$$
$$\cos(\beta+\theta) = -\cos(\theta_0+\theta)\cos\psi - \sin(\theta_0+\theta)\sin\psi$$
$$\sin(\beta+\theta) = -\sin(\theta_0+\theta)\cos\psi + \cos(\theta_0+\theta)\sin\psi$$
$$\cos 2\beta = \cos 2\theta_0\cos 2\psi + \sin 2\theta_0 \sin 2\psi$$

$\vec{A}_r$

$$\vec{A}_r = A_\rho\cos(\theta-\beta) + A_\beta\sin(\theta-\beta)$$
$$\vec{A}_r = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4}(\omega_m R)[I_3(\omega_m R)K_3(\omega_m\rho)\cos(3\beta-\theta) - I_1(\omega_m R)K_1(\omega_m\rho)\cos(\beta+\theta)]\sin\omega_m z$$

f "Theory of Bessel Functions", G.N.Eatson, Cambridge University Press, page 361

$$\vec{A}_r = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4}(\omega_m R) \sum_{k=-\infty}^{\infty} \left\{ \begin{array}{l} \{-I_3(\omega_m R) K_{3+k}(\omega_m S) \cos[(k+3)\theta_0 - (k+1)\theta]\} + \\ + \{I_1(\omega_m R) K_{1+k}(\omega_m S) \cos[(k+1)\theta_0 - (k-1)\theta]\} \end{array} \right\} I_k(\omega_m r) \sin \omega_m z$$

The above is a contribution from a single quad with its center at $Se^{-i\theta_0}$. If we wish to assume an arrangement of 8 quads with full symmetry around 0,0 and centers located at $Se^{-i(\pm\theta_0)}$, $Se^{-i(\pm\frac{\pi}{2}\pm\theta_0)}$, $Se^{-i(-\pi\pm\theta_0)}$(quads on the symmetry line will require a weight factor of 1/2).

$$\sum_{\theta_0} \cos(k+3)\theta_0 = 2\cos(k+3)\theta_0 \left[1 + 2\cos\frac{(k+3)\pi}{2} + \cos(k+3)\pi\right]$$

$$\sum_{\theta_0} \sin(k+3)\theta_0 = 0$$

$$\sum_{\theta_0} \cos(k+1)\theta_0 = 2\cos(k+1)\theta_0 \left[1 + 2\cos\frac{(k+1)\pi}{2} + \cos(k+1)\pi\right]$$

$$\sum_{\theta_0} \sin(k+1)\theta_0 = 0$$

With the help of the relations,

$$I_{4k-3} = I'_{2(2k-1)} + \frac{2(2k-1)}{\omega_m r} I_{2(2k-1)}$$

$$I_{4k-1} = I'_{2(2k-1)} - \frac{2(2k-1)}{\omega_m r} I_{2(2k-1)}$$

and additional algebraic manipulation and the introduction of a normalization factor,

$$\vec{A}_r = \sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2(\frac{\omega_m R}{2})^{2(2k-1)}} \left[C^+_{k,m} I'_{2(2k-1)}(\omega_m r) + 2(2k-1) C^-_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r}\right] \cos 2(2k-1)\theta \sin \omega_m z$$

$$C^+_{k,m} = \sum_{i=1}\sum_{j=1} \left[K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}\right] \frac{16 I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-2}$$

$$C^-_{k,m} = \sum_{i=1}\sum_{j=1} \left[K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} - K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}\right] \frac{16 I'_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-1}$$

The summation in i,j is carried out over all coils in the first octant.

$\vec{A}_\theta$

$$\vec{A}_\theta = -A_\rho \sin(\theta - \beta) + A_\beta \cos(\theta - \beta)$$

$$\vec{A}_\theta = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4}(\omega_m R)[I_3(\omega_m R) K_3(\omega_m \rho) \sin(3\beta - \theta) + I_1(\omega_m R) K_1(\omega_m \rho) \sin(\beta + \theta)] \sin \omega_m z$$

$$\vec{A}_\theta = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) \sum_{k=-\infty}^{\infty} \left\{ \begin{array}{l} I_3(\omega_m R) K_{3+k}(\omega_m S) \sin\left[(k+1)\theta - (k+3)\theta_0\right] + \\ + I_1(\omega_m R) K_{1+k}(\omega_m S) \sin\left[(k-1)\theta - (k+1)\theta_0\right] \end{array} \right\} I_k(\omega_m r) \sin \omega_m z$$

The above expression is for the contribution from a single quad with its center at $S\mathrm{e}^{-\mathrm{i}\theta_0}$. As before if we now include contributions from 8 quads placed with full symmetry, we shall need to add the vector potential of coils with their center at $S\mathrm{e}^{-\mathrm{i}(\pm\theta_0)}$, $S\mathrm{e}^{-\mathrm{i}(\pm\frac{\pi}{2}\pm\theta_0)}$, $S\mathrm{e}^{-\mathrm{i}(-\pi\pm\theta_0)}$. Making use of the trigonometric relations as shown for A_r

$$\boxed{\vec{A}_\theta = -\sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left[C^-_{k,m} I'_{2(2k-1)}(\omega_m r) + 2(2k-1) C^+_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] \sin 2(2k-1)\theta \sin \omega_m z}$$

$\vec{A}_z$

$$\vec{A}_z = \sum_{m=1} \mu_0 J_{0z,m} R I_2(\omega_m R) K_2(\omega_m \rho) \cos 2\beta \cos \omega_m z$$

$$K_2(\omega_m \rho) \cos 2\beta = \sum_{k=-\infty}^{\infty} K_{2+k}(\omega_m S) I_k(\omega_m r) [\cos 2\theta_0 \cos k(\theta - \theta_0) + \sin 2\theta_0 \sin k(\theta - \theta_0)]$$

The above expression is for the contribution from a single quad with its center at $S\mathrm{e}^{-\mathrm{i}\theta_0}$. As before if we now include contributions from 8 quads and make use of the relations as shown for A_r,

$$\sum_{\theta_0} \cos 2\theta_0 \cos k(\theta - \theta_0) = 2 \cos 2\theta_0 \cos k\theta_0 \cos k\theta \left[1 + (-1)^k - 2\cos\frac{k\pi}{2} \right]$$

$$\sum_{\theta_0} \sin 2\theta_0 \sin k(\theta - \theta_0) = -2 \sin 2\theta_0 \sin k\theta_0 \cos k\theta \left[1 + (-1)^k - 2\cos\frac{k\pi}{2} \right]$$

$$K_2(\omega_m \rho) \cos 2\beta = 2 \sum_{k=-\infty}^{\infty} K_{2+k}(\omega_m S) I_k(\omega_m r) \cos(k+2)\theta_0 \left[1 + (-1)^k - 2\cos\frac{k\pi}{2} \right] \cos k\theta$$

$$\boxed{\vec{A}_z = \sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} C^+_{k,m} I_{2(2k-1)}(\omega_m r) \cos 2(2k-1)\theta \cos \omega_m z}$$

Check $\nabla \cdot A = 0$

A way of checking the results is to assure the divergence is zero anywhere in space.

$$\nabla \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\theta}{r\partial \theta} + \frac{\partial A_z}{\partial z} = 0$$

$$\frac{\partial (rA_r)}{\partial r} = \sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m} R(4k-3)!}{2\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left\{ \begin{array}{l} C^+_{k,m}\left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r)\right] + \\ + 2(2k-1)C^-_{k,m} I'_{2(2k-1)}(\omega_m r) \end{array} \right\} \cos 2(2k-1)\theta \sin \omega_m z$$

$$\frac{\partial A_\theta}{\partial \theta} = -\sum_{m=1}\sum_{k=1} (2k-1)\frac{\mu_0 J_{0z,m} R(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left[\begin{array}{l} C^-_{k,m} I'_{2(2k-1)}(\omega_m r) + \\ + 2(2k-1)C^+_{k,m} \dfrac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \end{array} \right] \cos 2(2k-1)\theta \sin \omega_m z$$

$$\frac{\partial A_z}{\partial z} = -\sum_{m=1}\sum_{k=1} \frac{\mu_0 J_{0z,m}(\omega_m R)(4k-3)!}{2\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} C^+_{k,m} I_{2(2k-1)}(\omega_m r) \cos 2(2k-1)\theta \sin \omega_m z$$

$$\nabla \vec{A} \propto \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left\{ \begin{array}{l} -\dfrac{C^+_{k,m}}{\omega_m r} \left[\begin{array}{l} I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - \\ - [2(2k-1)]^2 \dfrac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \end{array} \right] + \\ + C^+_{k,m} I_{2(2k-1)}(\omega_m r) \end{array} \right\} \cos 2(2k-1)\theta \sin \omega_m z$$

and since

$$\frac{1}{\omega_m r}\left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - [2(2k-1)]^2 \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] = I_{2(2k-1)}(\omega_m r)$$

the divergence

$$\nabla \vec{A} = -\sum_{m} \mu_0 J_{0z,m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ \begin{array}{l} -C^+_{k,m} I_{2(2k-1)}(\omega_m r) + \\ + C^+_{k,m} I_{2(2k-1)}(\omega_m r) \end{array} \right\} \cos 2(2k-1)\theta \sin \omega_m z = 0!$$

as it should be.

The Magnetic Field Components.

The r component of $\vec{B}$

From the vector potential we derive the radial magnetic field component in the region r≤R,

$$B_r = \left(\nabla \times \vec{A}\right)_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}$$

$$\frac{\partial A_z}{\partial \theta} = -\sum_{m=1} \frac{\mu_0 J_{0z,m} R}{2} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} 2(2k-1) C^+_{k,m} I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z$$

$$\frac{\partial A_\theta}{\partial z} = -\sum_{m=1} \frac{\mu_0 J_{0z,m}}{2}(\omega_m R) \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left[\begin{array}{l} C^-_{k,m} I'_{2(2k-1)}(\omega_m r) + \\ + 2(2k-1)C^+_{k,m} \dfrac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \end{array} \right] \sin 2(2k-1)\theta \cos \omega_m z$$

and B_r reduces to,

$$B_r = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} (4k-3)! C^-_{k,m} I'_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z$$

We need to add the self field contribution from the coil at i=j=0. For r≤R we add,

$$B_r = \sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} I'_2(\omega_m r) \sin 2\theta \cos \omega_m z$$

and for r≥R we add,

$$B_r = \sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I'_2(\omega_m R)}{K'_2(\omega_m R)} K'_2(\omega_m r) \sin 2\theta \cos \omega_m z$$

The θ component of $\vec{B}$

$$B_\theta = \left(\nabla \times \vec{A}\right)_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$\frac{\partial A_r}{\partial z} = \sum_{m=1} \mu_0 J_{0z,m} \left(\frac{\omega_m R}{2}\right) \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \begin{bmatrix} C^+_{k,m} I'_{2(2k-1)}(\omega_m r) + \\ + 2(2k-1) C^-_{k,m} \dfrac{I_{2(2k-1)}}{\omega_m r} \end{bmatrix} \cos 2(2k-1)\theta \cos \omega_m z$$

$$\frac{\partial A_z}{\partial r} = \sum_{m=1} \mu_0 J_{0z,m} \left(\frac{\omega_m R}{2}\right) \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} C^+_{k,m} I'_{2(2k-1)}(\omega_m r) \cos 2(2k-1)\theta \cos \omega_m z$$

$$B_\theta = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \frac{2(2k-1)(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} C^-_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \cos 2(2k-1)\theta \cos \omega_m z$$

In the region r≤R we add the self field,

$$B_\theta = \sum_{m=1} \frac{4\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I_2(\omega_m r)}{\omega_m r} \cos 2\theta \cos \omega_m z$$

and similarly for r≥R we add,

$$B_\theta = \sum_{m=1} \frac{4\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I_2'(\omega_m R)}{K_2'(\omega_m R)} \frac{K_2(\omega_m r)}{\omega_m r} \cos 2\theta \cos \omega_m z$$

The z component of $\vec{B}$

r≤R

$$B_z = \left(\nabla \times \vec{A}\right)_z = \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

$$\frac{\partial (rA_\theta)}{\partial r} = -\sum_{m=1} \frac{\mu_0 J_{0z,m}}{\omega_m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ \begin{array}{l} C_{k,m}^{-}\left[I_{2(2k-1)}'(\omega_m r) + \omega_m r I_{2(2k-1)}''(\omega_m r)\right] + \\ + 2(2k-1) C_{k,m}^{+} I_{2(2k-1)}'(\omega_m r) \end{array} \right\} \sin 2(2k-1)\theta \sin \omega_m z$$

$$\frac{\partial A_r}{\partial \theta} = -\sum_{m=1} \frac{\mu_0 J_{0z,m}}{\omega_m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ \begin{array}{l} 2(2k-1) C_{k,m}^{+} I_{2(2k-1)}'(\omega_m r) + \\ + [2(2k-1)]^2 C_{k,m}^{-} \frac{I_{2(2k-1)}'(\omega_m r)}{\omega_m r} \end{array} \right\} \sin 2(2k-1)\theta \sin \omega_m z$$

but

$$\frac{1}{\omega_m r}\left[I_{2(2k-1)}'(\omega_m r) + \omega_m r I_{2(2k-1)}''(\omega_m r) - [2(2k-1)]^2 \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r}\right] = I_{2(2k-1)}(\omega_m r)$$

Therefore :

$$B_z = -\sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} (4k-3)! C_{k,m}^{-} I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \sin \omega_m z$$

in the region r≤R we supplement the field with the self field,

$$B_z = -\sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} I_2(\omega_m r) \sin 2\theta \sin \omega_m z$$

and in the outer region r≥R we add

$$B_z = -\sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I_2'(\omega_m R)}{K_2'(\omega_m R)} K_2(\omega_m r) \sin 2\theta \sin \omega_m z$$

Check $\nabla \cdot B = 0$

As was done for vector potential we make sure the divergence of B is zero,

$$\nabla \vec{B} = \frac{1}{r}\frac{\partial (rB_r)}{\partial r} + \frac{\partial B_\theta}{r\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial (rB_r)}{\partial r} = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} C^-_{k,m} \left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r)\right] \sin 2(2k-1)\theta \cos \omega_m z$$

$$\frac{\partial B_\theta}{\partial \theta} = -\sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} [2(2k-1)]^2 C^-_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \sin 2(2k-1)\theta \cos \omega_m z$$

$$\frac{\partial B_z}{\partial z} = -\sum_{m=1} \mu_0 J_{0z,m}\omega_m \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} C^-_{k,m} I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z$$

$$\nabla \vec{B} \propto \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ \begin{array}{l} -\dfrac{C^-_{k,m}}{\omega_m r} \left[\begin{array}{l} I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - \\ -[2(2k-1)]^2 \dfrac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \end{array} \right] + \\ + C^-_{k,m} I_{2(2k-1)}(\omega_m r) \end{array} \right\} \sin 2(2k-1)\theta \cos \omega_m z$$

and since

$$\frac{1}{\omega_m r}\left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - [2(2k-1)]^2 \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r}\right] = I_{2(2k-1)}(\omega_m r)$$

$$\boxed{\nabla \vec{B} = -\sum_{m} \mu_0 J_{0z,m}\omega_m \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left[\begin{array}{l} -C^-_{k,m} I_{2(2k-1)}(\omega_m r) + \\ + C^-_{k,m} I_{2(2k-1)}(\omega_m r) \end{array} \right] \sin 2(2k-1)\theta \cos \omega_m z \stackrel{\cdot}{=} 0!}$$

The stored energy

In calculating the stored energy we start from,

$$E = \frac{1}{2}\int\int\int \vec{J} \cdot \vec{A} dv$$

and integrate the product on the surface current only over a full period[g] :

$$E = \frac{1}{2}\int_0^{2\pi}\int_{-L}^{L} \vec{J} \cdot \vec{A} d\sigma = \frac{1}{2}\int_0^{2\pi}\int_{-L}^{L} \vec{J} \cdot \vec{A} R d\theta dz$$

(the current density is per unit length and the unit of energy is $J = T \cdot A \cdot m^2$).

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[g] "Forces and Stored Energy in Thin Cosine(nθ) Accelerator magnets." S.Caspi, SC-MAG-546, LBL-38500, March 1996.

We shall calculate the contributions arising from all external coil arrays followed by the self coil contribution. The product of the current density and the vector potential is :

$$\vec{J} \cdot \vec{A}|_{r=R} = J_\theta A_\theta + J_z A_z$$

In performing the integration we shall make use of the following orthogonality relations:

$$\int_0^{2\pi} \sin 2\theta \sin 2(2k-1)\theta d\theta = \begin{Bmatrix} 0 & k \neq 1 \\ \pi & k = 1 \end{Bmatrix}.$$

$$\int_0^{2\pi} \cos 2\theta \cos 2(2k-1)\theta d\theta = \begin{Bmatrix} 0 & k \neq 1 \\ \pi & k = 1 \end{Bmatrix}$$

$$\int_{-L}^{L} \sin \omega_m z \sin \omega_j z dz = \begin{Bmatrix} 0 & m \neq j \\ L & m = j \end{Bmatrix}$$

$$\int_{-L}^{L} \cos \omega_m z \cos \omega_j z dz = \begin{Bmatrix} 0 & m \neq j \\ L & m = j \end{Bmatrix}$$

As a result the stored energy is reduced to,

$$E_{ext.} = \frac{1}{2}\int_0^{2\pi}\int_{-L}^{L} J \cdot A R d\theta dz = \begin{cases} -\frac{\pi R^2 L \mu_0}{2} \sum\limits_{m=1} \frac{J_{0z,m}^2}{\omega_m R}\left[C_{1,m}^- I_2'(\omega_m R) + 2C_{1,m}^+ \frac{I_2(\omega_m R)}{\omega_m R}\right] & + \\ +\pi R^2 L \mu_0 \sum\limits_{m=1} \frac{J_{0z,m}^2}{(\omega_m R)^2} C_{1,m}^+ I_2(\omega_m R) \end{cases}$$

or,

$$E_{ext.} = -\frac{\pi R^2 L \mu_0}{2} \sum_{m=1} \frac{J_{0z,m}^2}{\omega_m R} C_{1,m}^- I_2'(\omega_m R)$$

where,

$$C_{1,m}^- = \sum_{i=1}\sum_{j=1}[K_0(\omega_m S_{i,j}) - K_4(\omega_m S_{i,j}) \cos 4\theta_{0i,j}] 16 I_2'(\omega_m R)\left(\frac{\omega_m R}{2}\right)^3$$

the stored energy of the self coil is :

$$E_{self} = -\frac{\pi R^2 L \mu_0}{8} \sum_m J_{0z,m}^2 (\omega_m R)^2 K_2'(\omega_m R) I_2'(\omega_m R)$$

and the total energy is,

$$E_{total} = -\frac{\pi R^2 L \mu_0}{8} \sum_{m=1} J_{0z,m}^2 (\omega_m R)^2 K_2'(\omega_m R) I_2'(\omega_m R)\left[1 + \frac{4C_{1,m}^-}{(\omega_m R)^3 K_2'(\omega_m R)}\right]$$

Simulation of Current density and flow lines

To generate flow lines we make use of a technic first demonstrated by J.Laslett and W. Fawley of this laboratory. The character of the flow lines for a single function magnet n, will follow from the differential equation,

$$\frac{Rd\theta}{dz} = \frac{J_\theta}{J_z}$$

Assuming the current density for magnet type n=2 as,

$$\vec{J} = \sum_{m=1} J_{0z,m}\left[\left(\frac{\omega_m R}{2}\right)\sin 2\theta \sin \omega_m z \hat{e}_\theta + \cos 2\theta \cos \omega_m z \hat{e}_z\right]$$

it follows that,

$$\frac{Rd\theta}{dz} = \frac{\sum_{m=1} J_{0z,m}(\omega_m R)\sin 2\theta \sin \omega_m z}{2\sum_{m=1} J_{0z,m}\cos 2\theta \cos \omega_m z}$$

and

$$\frac{\cos 2\theta}{\sin 2\theta}d\theta - \frac{\sum_{m=1} J_{0z,m}(\omega_m R)\sin \omega_m z}{2R\sum_{m=1} J_{0z,m}\cos \omega_m z}dz = 0$$

or,

$$\ln(\sin 2\theta) + \ln\left(2\sum_{m=1} J_{0z,m}\cos \omega_m z\right) = \text{const.}$$

So that,

$$\sin 2\theta = \frac{const.}{2\sum_{m=1} J_{0z,m}\cos \omega_m z}$$

and the flow lines are therefore,

$$\boxed{\sin 2\theta = \frac{\sum_{m=1} J_{0z,m}}{\sum_{m=1} J_{0z,m}\cos \omega_m z}\sin 2\theta_0}$$

where θ_0 denotes the value of θ at z=0.

In a special case, we may choose special values for $J_{02,m}$ such that,

$$J_{02,m} = \frac{J_{02}}{2^{2(M-1)}}\binom{2M-1}{M-m} = J_{02}\frac{1}{2^{2(M-1)}}\frac{(2M-1)!}{(M+m-1)!(M-m)!}$$

where M is the number of m terms used in a particular case and J_{02} is a constant.

We note that in this particular case

$$\frac{1}{2^{2(M-1)}}\sum_{m=1}^{M}\frac{(2M-1)!}{(M+m-1)!(M-m)!}\cos \omega_m z = \cos^{2M-1}\omega_1 z$$

$$\frac{1}{2^{2(M-1)}}\sum_{m=1}^{M}\frac{(2M-1)!}{(M+m-1)!(M-m)!} = 1$$

and therefore the current density is,

$$\sum_{m=1} J_{02,m} \cos \omega_m z = J_{02} \sum_{m=1}^{M} \frac{1}{2^{2(M-1)}} \frac{(2M-1)!}{(M+m-1)!(M-m)!} \cos \omega_m z = J_{02} \cos^{2M-1} \omega_1 z$$

With that, the flow lines reduce to the simple expression,

$$\sin 2\theta = \frac{1}{\cos^{2M-1} \omega_1 z} \sin 2\theta_0$$
$$\omega_1 = \frac{\pi}{L}$$

and the components of current density are,

$$\vec{J}(\theta, z)|_{r=R} = J_{02} \left\{ \begin{array}{l} 0\hat{e}_r \\ \frac{\pi R}{2L}(2M-1)\cos^{2(M-1)} \frac{\pi z}{L} \sin \frac{\pi z}{L} \sin 2\theta \hat{e}_\theta \\ \cos^{2M-1} \frac{\pi z}{L} \cos 2\theta \hat{e}_z \end{array} \right\}$$

m=1

$$J_\theta = \frac{\pi R}{2L} \sin \frac{\pi z}{L} \sin 2\theta$$
$$J_z = \cos \frac{\pi z}{L} \cos 2\theta$$
$$\text{Flow lines}: \quad \sin 2\theta = \frac{\sin 2\theta_0}{\cos \frac{\pi z}{L}}$$

m=1 and m=2

$$J_\theta = 3\frac{\pi R}{2L}\left(\cos \frac{\pi z}{L}\right)^2 \sin \frac{\pi z}{L} \sin 2\theta$$
$$J_z = \left(\cos \frac{\pi z}{L}\right)^3 \cos 2\theta$$
$$\text{Flow lines}: \quad \sin 2\theta = \frac{\sin 2\theta_0}{\left(\cos \frac{\pi z}{L}\right)^3}$$

m=1, m=2 and m=3

$$J_\theta = 5\frac{\pi R}{2L}\left(\cos\frac{\pi z}{L}\right)^4 \sin\frac{\pi z}{L}\sin 2\theta$$

$$J_z = \left(\cos\frac{\pi z}{L}\right)^5 \cos 2\theta$$

$$\text{Flow lines :} \qquad \sin 2\theta = \frac{\sin 2\theta_0}{\left(\cos\frac{\pi z}{L}\right)^5}$$

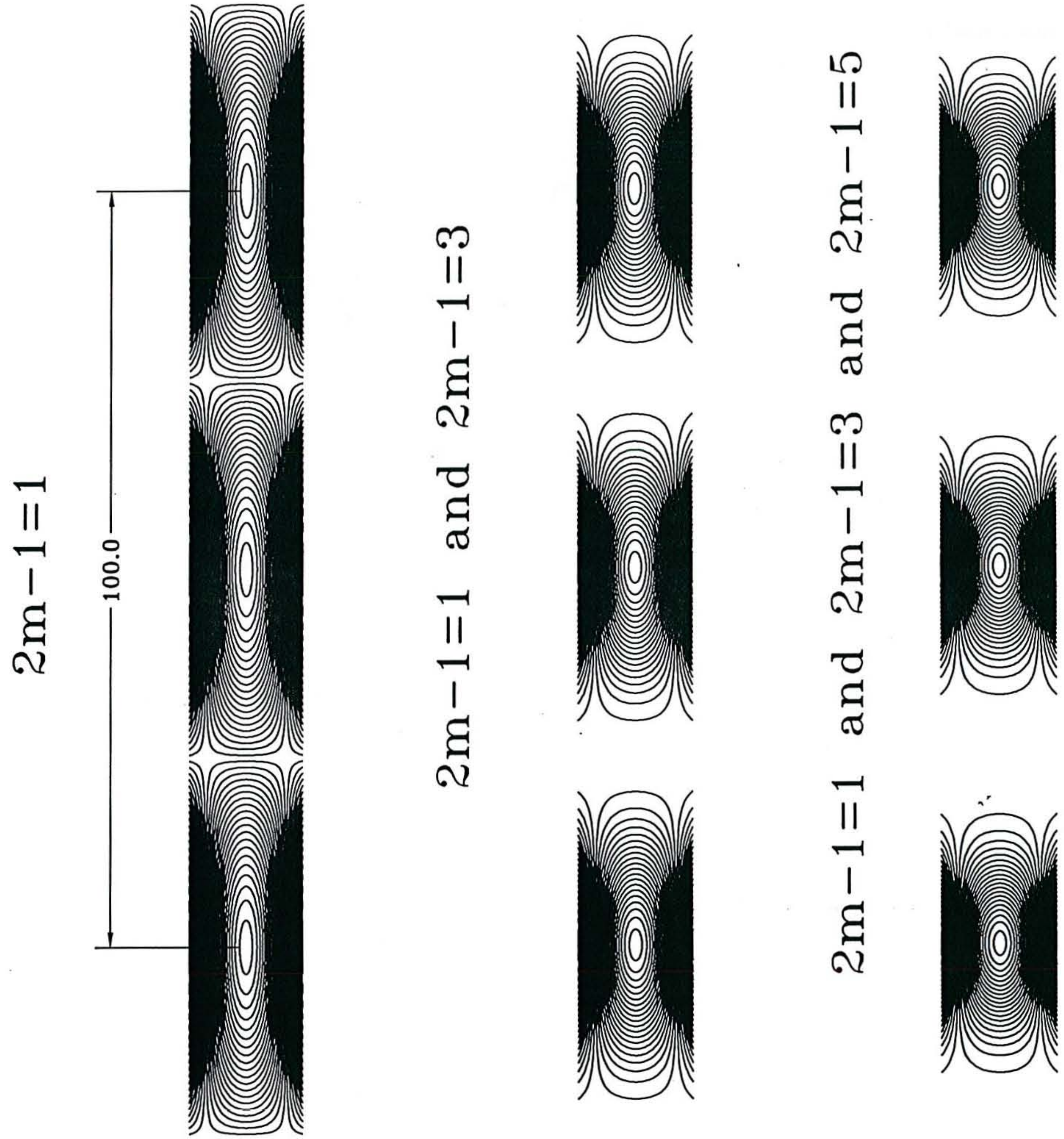

Figure 5 View of a full period array of a quad with m=1 only, a summation over m=1,2 , and m=1,2,3. These special cases revel the reduction in crowding between magnets at the expense of an increased non-linear field.

The limiting 2 dimensional case

As a farther simplification and check, we reduce the results obtained for the 3D array by extending the periodicity to infinity, $lim_{L\to\infty}\omega_m = 0$, and compare those with more familiar 2D cases of multipole magnets as cited in the abstract.

$$s \to 0$$
$$L \to \infty$$

$$\begin{cases} \lim_{s\to 0} I_2(s) \to \frac{s^2}{8} & \lim_{s\to 0} I_2'(s) \to \frac{s}{4}. \\ \lim_{s\to 0} K_2(s) \to \frac{2}{s^2} & \lim_{s\to 0} K_2'(s) \to -\frac{4}{s^3} \\ & \lim_{s\to 0} \frac{I_2'(s)}{K_2'(s)} \to -\frac{s^4}{16} \\ \lim_{s\to 0} I_{2(2k-1)}(s) \to \frac{1}{[2(2k-1)]!}\left(\frac{s}{2}\right)^{2(2k-1)} & \lim_{s\to 0} I'_{2(2k-1)}(s) \to \frac{1}{2(4k-3)!}\left(\frac{s}{2}\right)^{4k-3} \\ \lim_{s\to 0} K_{4(k-1)}(s) \to \frac{(4k-5)!}{2}\left(\frac{s}{2}\right)^{-4(k-1)} & \lim_{s\to 0} K_{4k}(s) \to \frac{(4k-1)!}{2}\left(\frac{s}{2}\right)^{-4k} \end{cases}$$

$$\vec{J}_z = J_0 \cos 2\theta \ ; \quad J_0 = \frac{2RG_{2,m}}{\mu_0}$$

The 3D vector-potential reduces to the 2D expression (including self coil):

$$\vec{A}_r = 0$$
$$\vec{A}_\theta = 0$$
$$\vec{A}_z = \sum_m \frac{\mu_0 J_{0z,m}}{4}\left(\frac{r}{R}\right)^2 \cos 2\theta + 2\sum_{m=1}\mu_0 J_{0z,m} R \sum_{k=1}(4k-1)\cos 4k\theta_{0i,j}\left(\frac{R}{S_{i,j}}\right)^{4k}\left(\frac{r}{R}\right)^{4k-2}\cos 2(2k-1)\theta$$

where the first term corresponds to the self field contribution from coil i=j=0. The field-not including the self field, is :

$$\vec{B}_r = -4\sum_{m=1}\mu_0 J_{0z,m}\sum_{k=1}(2k-1)(4k-1)\cos 4k\theta_{0i,j}\left(\frac{R}{S_{i,j}}\right)^{4k}\left(\frac{r}{R}\right)^{4k-3}\sin 2(2k-1)\theta$$
$$\vec{B}_\theta = -4\sum_{m=1}\mu_0 J_{0z,m}\sum_{k=1}(2k-1)(4k-1)\cos 4k\theta_{0i,j}\left(\frac{R}{S_{i,j}}\right)^{4k}\left(\frac{r}{R}\right)^{4k-3}\cos 2(2k-1)\theta$$
$$\vec{B}_z = 0$$

in agreement with the references cited in the abstract. The self field is :

$$\vec{B}_r = -\sum_{m=1}\frac{\mu_0 J_{0z,m}}{2}\left(\frac{r}{R}\right)\sin 2\theta$$
$$\vec{B}_\theta = -\sum_{m=1}\frac{\mu_0 J_{0z,m}}{2}\left(\frac{r}{R}\right)\cos 2\theta$$
$$\vec{B}_z = 0$$

The array 2D multipole coefficients are equal to their contribution to the stored energy.

$$E^{2D}_{total} = \frac{\pi R^2 L\mu_0}{8}\sum_m J_0^2\left(1 - C_m^{-2D}\right) = \frac{\pi R^2 L\mu_0}{8}\sum_m J_0^2\sum_i\sum_j\left[1 + 24\left(\frac{R}{S_{i,j}}\right)^4\cos 4\theta_{0i,j}\right]$$

Appendix A Region outside the coil r≥R

The contribution of the self field (centered at 0,0) in the region outside the current sheet is listed below,

Contributions from the coil at i=0, j=0 in the region r≥R

$$\vec{A}_r = \sum_{m=1} \frac{\mu_0 J_{0z,m}}{4}(\omega_m R)[I_3(\omega_m R)K_3(\omega_m r) - I_1(\omega_m R)K_1(\omega_m r)] \cos 2\theta \sin \omega_m z$$

$$\vec{A}_\theta = \sum_{m=1} \frac{\mu_0 J_{0z,m}}{4}(\omega_m R)[I_3(\omega_m R)K_3(\omega_m r) + I_1(\omega_m R)K_1(\omega_m r)] \sin 2\theta \sin \omega_m z$$

$$\vec{A}_z = \sum_{m=1} \mu_0 J_{0z,m} R I_2(\omega_m R) K_2(\omega_m r) \cos 2\theta \cos \omega_m z$$

$$B_r = \sum_{m=1} \frac{J_{0z_m}\mu_0}{2}(\omega_m R)^2 I_2'(\omega_m R) K_2'(\omega_m r) \sin 2\theta \cos \omega_m z$$

$$B_\theta = \sum_{m=1} \frac{J_{0z_m}\mu_0}{2}(\omega_m R)^2 I_2'(\omega_m R) \frac{2K_2(\omega_m r)}{\omega_m r} \cos 2\theta \cos \omega_m z$$

$$B_z = -\sum_{m=1} \frac{J_{0z_m}\mu_0}{2}(\omega_m R)^2 I_2'(\omega_m R) K_2(\omega_m r) \sin 2\theta \sin \omega_m z$$

$$G_{2,m} = -\frac{J_{0z_m}\mu_0}{8R}(\omega_m R)^3 K_2'(\omega_m R)$$

Appendix B 3 Dimensional Coefficients

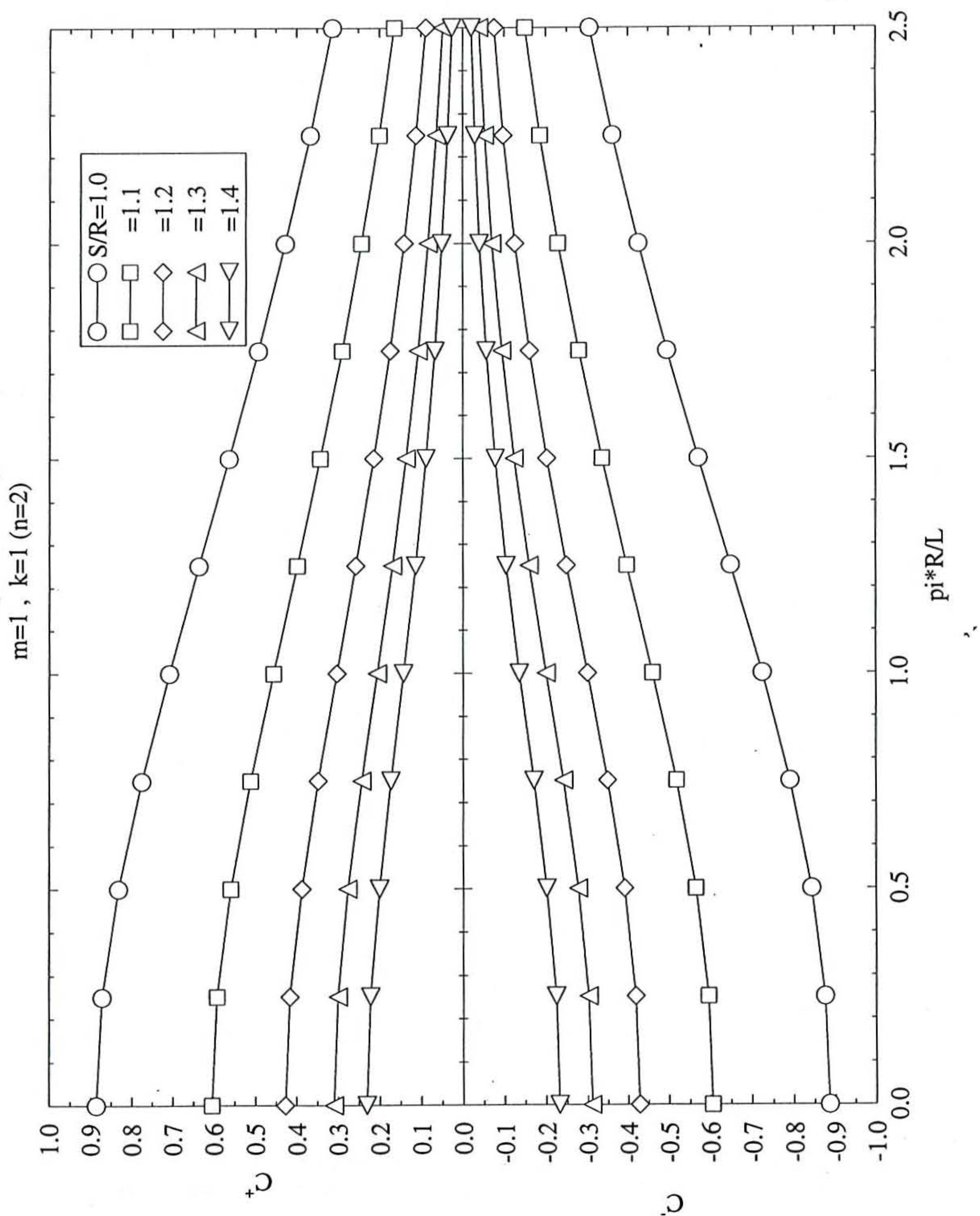

Figure 6 The values associated with the multipole coefficients for m=1 and the quad term k=1. Values at pi*R/L = 0 correspond to the 2 dimensional case.

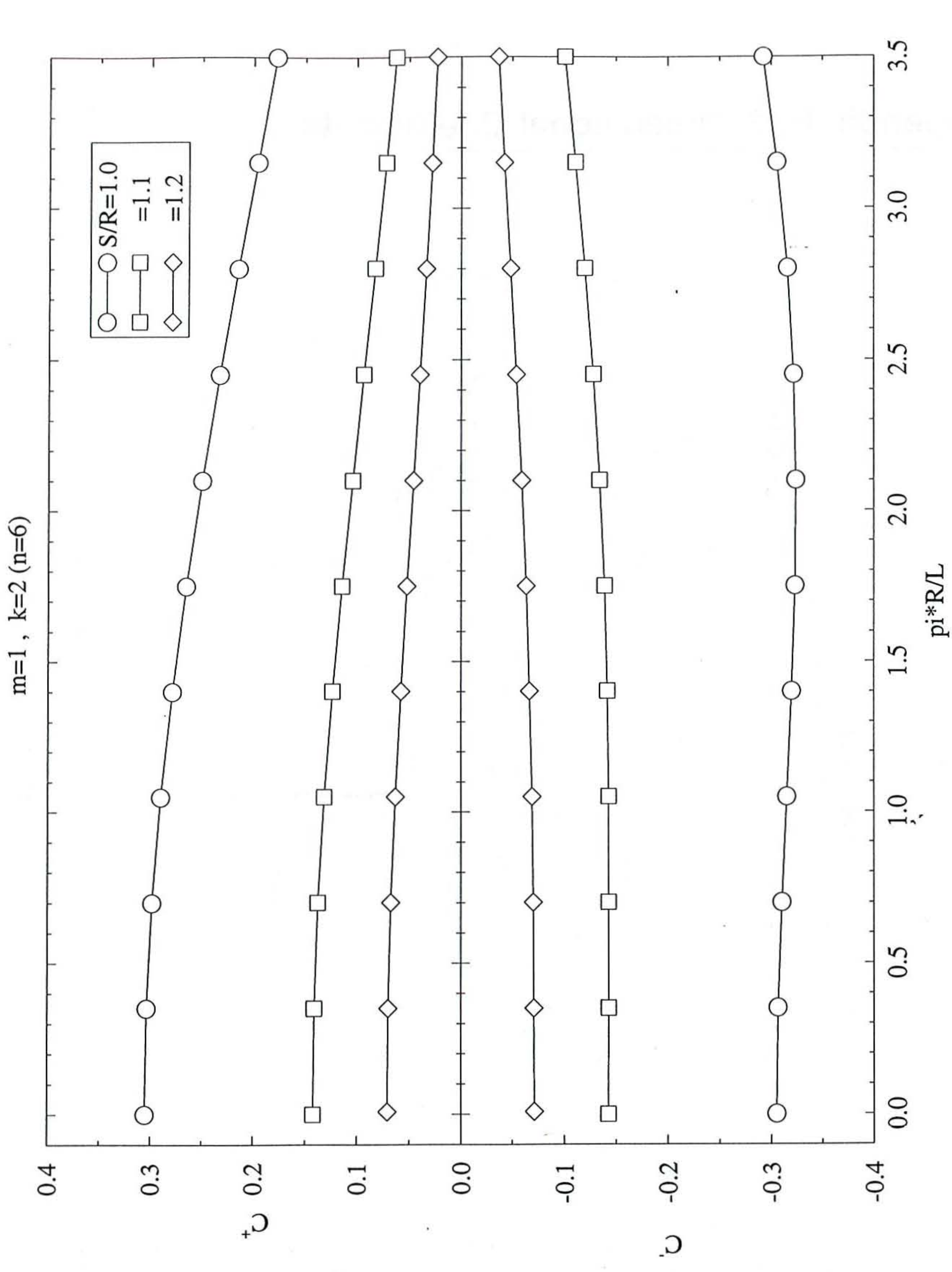

Figure 7 The values associated with the multipole coefficients for m=1 and the dodecapole term k=2. Values at pi*R/L = 0 correspond to the 2 dimensional case.

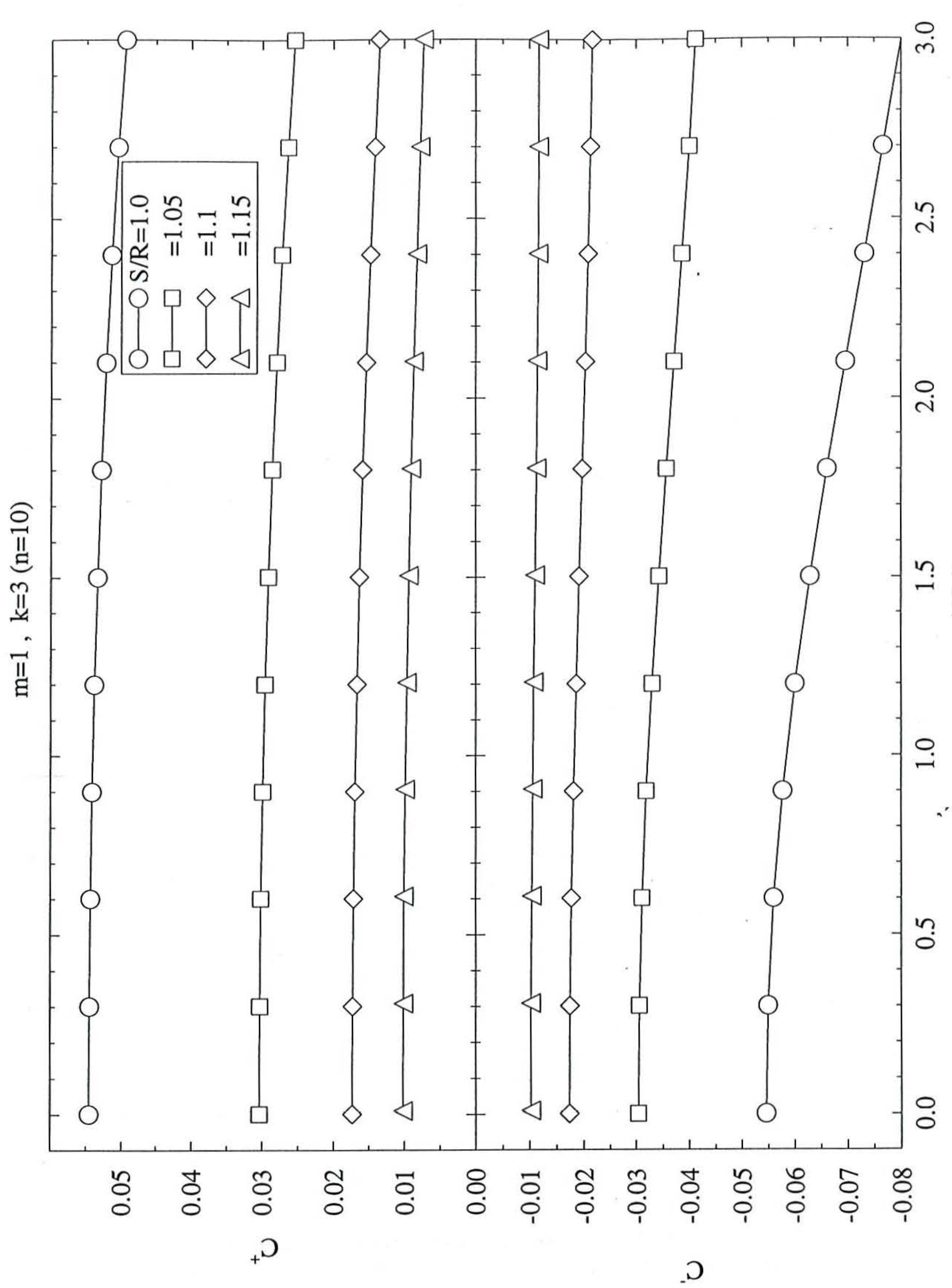

Figure 8 The values associated with the multipole coefficients for m=1 and the 20's pole k=3. Values at pi*R/L = 0 correspond to the 2 dimensional case.

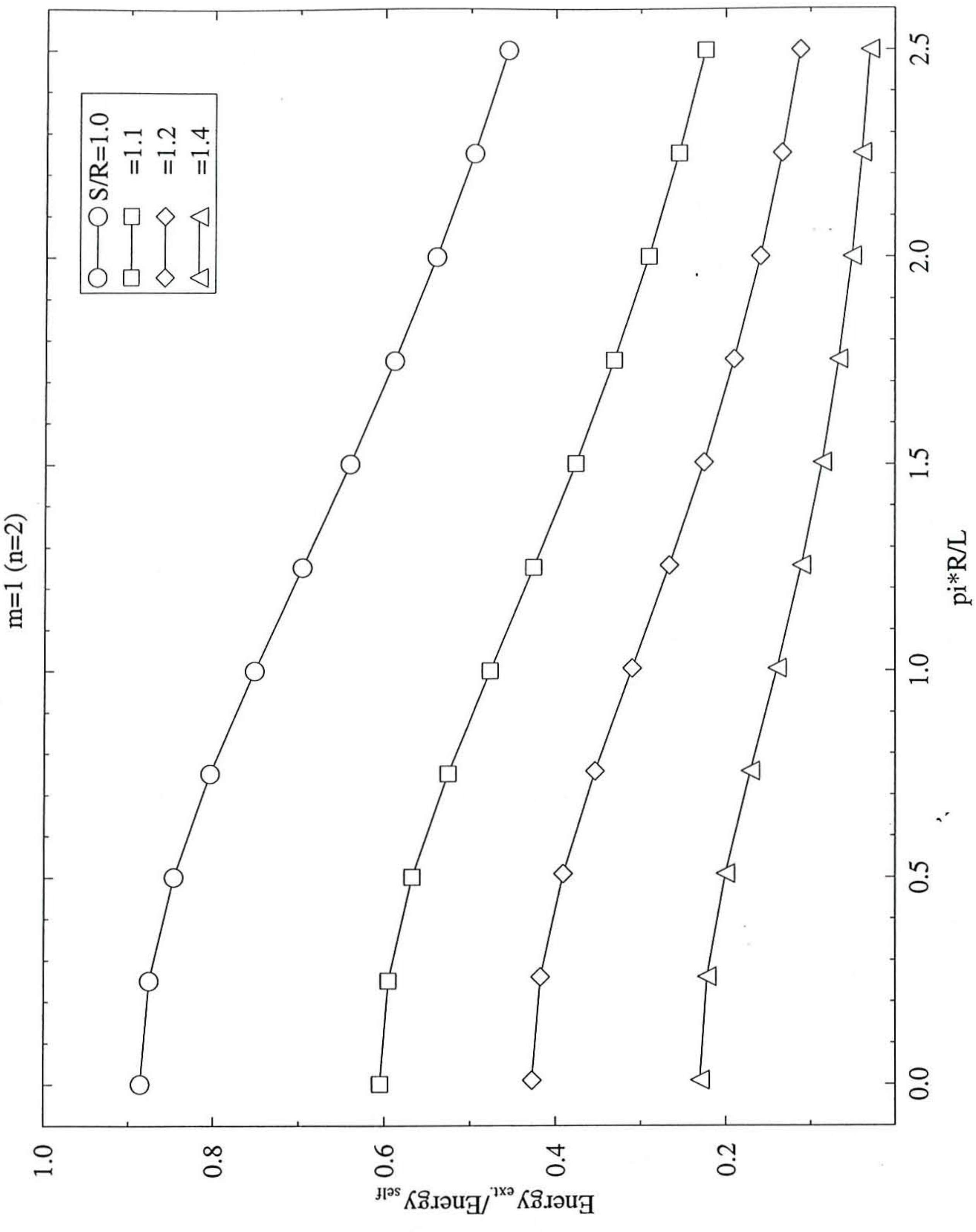

Figure 9 Ratio of stored energy contributed by external coils for m=1 and the quad term k=1. Values at pi*R/L = 0 correspond to the 2 dimensional case.