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Title: Advanced Numerical Methods in Mesh Generation and Mesh Adaptation

Author(s): Lipnikov, Konstantin - LANL
Agouzal, A. University of Lyon 1, France
Danilov, A. Institute of Numerical Mathematics, Moscow Russia
Vassilevski, Y. - Institute of Numerical Mathematics, Moscow Russia

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Advanced Numerical Methods in Mesh Generation and Mesh Adaptation

A. Agouzal [1]
A. Danilov [2]
K. Lipnikov [3]
Yu. Vassilevski [2]

[1] University of Lyon 1, France
[2] Institute of Numerical Mathematics, Moscow, Russia
[3] Los Alamos National Laboratory, New Mexico, USA

Keywords: advancing front technique, Delaunay triangulation, mesh adaptation, tensor metric, quasi-optimal mesh.

Numerical solution of partial differential equations requires appropriate meshes, efficient solvers and robust and reliable error estimates. Generation of high-quality meshes for complex engineering models is a non-trivial task. This task is made more difficult when the mesh has to be adapted to a problem solution. This article is focused on a synergistic approach to the mesh generation and mesh adaptation, where best properties of various mesh generation methods are combined to build efficiently simplicial meshes.

First, the advancing front technique (AFT) is combined with the incremental Delaunay triangulation (DT) to build an initial mesh [1]. Second, the metric-based mesh adaptation (MBA) method [2] is employed to improve quality of the generated mesh and/or to adapt it to a problem solution. We demonstrate with numerical experiments that combination of all three methods is required for robust meshing of complex engineering models.

The key to successful mesh generation is the high-quality of the triangles in the initial front. We use a black-box technique to improve surface meshes exported from an unattainable CAD system. The initial surface mesh is refined into a shape-regular triangulation which approximates the boundary with the same accuracy as the CAD mesh. The DT method adds robustness to the AFT. The resulting mesh is topologically correct but may contain a few slivers. The MBA uses seven local operations to modify the mesh topology. It improves significantly the mesh quality.

The MBA method is also used to adapt the mesh to a problem solution to minimize computational resources required for solving the problem. The MBA has a solid theoretical background [2]. In the first two experiments, we consider the convection-diffusion and elasticity problems. We demonstrate the optimal reduction rate of the discretization error on a sequence of adaptive strongly anisotropic meshes. The key element of the MBA method is construction of a tensor

metric from hierarchical edge-based error estimates. We conclude that the quasi-optimal mesh must be quasi-uniform in this metric.

All numerical experiments are based on the publicly available Ani3D package [3], the collection of advanced numerical instruments.

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Abstract

We describe a synergistic approach, where best properties of various mesh generation methods are combined to build efficiently simplicial meshes. First, the advancing front technique is combined with the Delaunay triangulation to build an initial mesh. Second, the metric-based mesh generation is employed to improve quality of this mesh and/or to adapt it to a problem solution. We describe relevant features of these methods and illustrate them with application examples involving robust mesh generation of complex engineering models and mesh adaptation for minimization of a discretization error.

Keywords: advancing front technique, Delaunay triangulation, mesh adaptation, tensor metric, quasi-optimal mesh.

1 Introduction

Numerical solution of partial differential equations (PDE) requires appropriate meshes, efficient solvers and robust and reliable error estimates. Generation of high-quality meshes for complex engineering models is a non-trivial task. This task is made more difficult when the mesh has to be adapted to a problem solution. This article is focused on a synergistic approach to the mesh generation and mesh adaptation.

We present a few mesh generation methods that are part of the Ani3D package [1], the collection of advanced numerical instruments. The package implements the advancing front technique (AFT) and the Delaunay triangulation to generate an initial mesh [2, 3] and the metric-based mesh re-generation [4, 5] to improve the mesh quality and to adapt it to a problem solution. The supporting utilities, implemented as independent libraries, include hierarchical uniform and local mesh refinement preserving mesh quality [6, 7], metric recovery algorithms [4, 8] and Krylov subspace solvers with incomplete LU preconditioners [9, 10].

Describing the mesh generation methods, we put focus on how well they satisfy the following four requirements: (a) generation of meshes with high-quality elements; (b) efficiency of underlying algorithms; (c) robustness of the underlying algorithms; (d) adaptation to a problem solution.

The AFT adds tetrahedra to the mesh by advancing the initial front inside the computational domain. The user can control the local mesh size. The mesh size can be automatically adjusted by the AFT based on the initial front mesh size. The key to successful mesh generation is the high-quality of the triangles in the initial front. The Ani3D package uses a black-box technique to improve surface meshes exported from an unattainable CAD system. The initial surface mesh is refined into a shape-regular triangulation which approximates the boundary with the same accuracy as the CAD mesh [2, 11].

The AFT has low cost but may produce a small number of low-quality tetrahedra (slivers) and may leave small isolated parts (lacunas) of the computational domain unmeshed. The unmeshed lacunas are partitioned into tetrahedra by the incremental Delaunay triangulation (DT) algorithm [2, 12, 13, 14, 15]. The DT method adds robustness to the mesh generation. The resulting mesh is topologically correct but may contain a few slivers. This mesh is post-processed with the metric-based mesh regeneration method. The same method is used to adapt the mesh to a problem solution.

The mesh adaptation technology [5, 16, 17, 18] allows the user to minimize computational resources in solving systems of PDEs. The described methods have solid theoretical background developed in [8, 19]. In these papers we developed analysis of quasi-optimal meshes minimizing L^p-norm of the gradient of the P_1-interpolation error, where $p > 0$. We demonstrated theoretically and numerically that the discretization error is proportional to $N^{-1/d}$, where N is the number of simplexes and d is the space dimension. The adapted meshes generated in Section 3.1 are quasi-optimal for the L^2-norm of gradient of the finite element (FE) discretization error. The key element of this technology is construction of a tensor metric from edge-based error estimates. A posteriori hierarchical error estimators [20] due to Deuflhard, Leinen and Yserentant are used to obtain the edge-based errors. A different mesh adaptation based on similar a posteriori error estimates is considered in [21].

We demonstrate the power of mesh adaptation with a two-dimensional convection-diffusion problem and a three-dimensional elasticity problem. We illustrate with numerical experiments that a posteriori error estimates provide reasonable bounds for the discretization error on strongly anisotropic meshes aligned with the problem solution. We also demonstrate importance of the MBA method in improving mesh quality in meshing of a complex engineering model.

The paper outline is as follows. In Section 2, we describe key mesh generation methods implemented in the Ani3D package. In Section 3, we illustrate the performance of these methods with three numerical experiments. The final remarks are in Section 4.

2 Mesh generation for solving PDEs

Efficient solution of partial differential equations requires specially designed meshes. In this article, we address four requirements which are stated frequently in generation of computational meshes:

Quality. A mesh generator must produce meshes with high-quality elements.

Efficiency. The employed algorithms must be efficient and scale linearly or sublinearly with the number of mesh elements.

Robustness. The employed algorithms must be reliable and robust.

Adaptivity. A mesh generator must produce meshes adapted to a problem solution.

None of the existing mesh generation methods has been proved to satisfy all four requirements. Our strategy is based on synergy of various methods described below. This allows us to design a technology which satisfies these requirements. The technology is implemented in the publicly available package Ani3D [1].

2.1 Advancing front technique

We employ the advancing front technique (AFT) for initial mesh generation. We define the initial front as a set of oriented triangular faces forming a closed conformal surface mesh. The idea of the AFT is to construct new tetrahedra by advancing this front inside the computational domain. The front actually divides the domain into two parts: already meshed one and the remaining part. At each step, a new tetrahedron is constructed and the front is advanced. The AFT algorithm terminates when the front becomes empty. Details of this algorithm are presented in [2]. Here, we discuss main features of the AFT and how it can be modified to comply with some of the requirements formulated above.

Quality. By design, majority of tetrahedra produced by the AFT have high-quality in the Euclidean metric provided that the initial front consists of high-quality triangles. Let h_Δ denote the diameter of element Δ and $|e|$ denote the length of edge e. The AFT guarantees that for each edge e of each element Δ, we have

$$\rho_1 h_\Delta \leq |e| \leq \rho_2 h_\Delta,$$

where positive constants ρ_1 and ρ_2 are independent of the mesh. Thus, the resulting mesh does not have small edges but may contain nearly flat tetrahedra (slivers). Additional checks prevent the AFT from generating elements with acute dihedral angles.

A good surface mesh is the key to successful mesh generation. If the initial front contains low-quality triangles, we start with improving their quality. In [11], we introduced a new technique for surface mesh modification which is implemented in the Ani3D package. The basic idea is to split the surface into several nearly flat polygons, and re-mesh them. In order to construct a nearly flat polygon, we fix a flatness criterion, and use it to add triangles to the polygon. The criterion says that triangle T lies

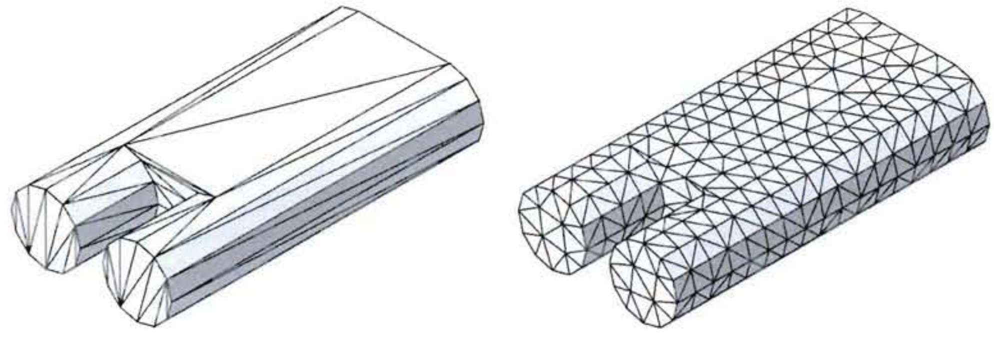

Figure 1: Left picture shows the surface mesh exported from a CAD system. Right picture shows the re-meshed surface mesh.

nearly in plane P, if (a) an angle between the normals to T and P is small enough, and (b) the distance from T to P is also small. The user can control the degree of the admissible deviation. After the surface is split into several nearly flat polygons, they are re-meshed with high-quality meshes. In order to provide a conformal surface mesh we first re-mesh the interfaces between these polygons and then re-mesh their interior. The polygon is projected onto a plane, where the robust planar AFT is applied to produce a high-quality triangulation. The new mesh is projected back onto the original surface. Since the original polygon was nearly flat, a surface distortion caused by the projection will be insignificant. An example of surface mesh improvement is presented in Figure 1.

Efficiency. The major advantage of the AFT is its low cost. For quasi-uniform tetrahedral meshes, the cost of the AFT is roughly $N \log(H/h)$, where N is the total number of constructed tetrahedra, H is the characteristic size of the computational domain, and h is the mesh size. The logarithmic complexity is achieved by using an octree-based search tree for searching faces in the three-dimensional space. In each vertex of the tree, we keep a list of front triangles assigned to the vertex. This data structure allows us to perform efficiently basic operations of the AFT.

Consider, for example, one of the main operations in the AFT, an intersection check between a tetrahedron Δ and the current front. Let B_Δ be its circumscribed sphere of radius R_Δ and B_T be the sphere centered at the barycenter of the front triangle T with the radius R_T equal to the maximum distance from the barycenter to the vertices of T. If T intersects with Δ, then B_T intersects with B_Δ and the distance d_T between centers of the two spheres is less than $R_T + R_\Delta$. The octree structure allows us to find quickly all triangles T satisfying $d_T \leq R_T + R_\Delta$. The cost of this search is proportional to $(\log M + K)$, where M is the total number of triangles in the octree, and K is the number of triangles satisfying the search conditions.

Robustness. The major drawback of the AFT is lack of robustness. It may fail to construct a mesh for the entire computational domain. The unmeshed volume is

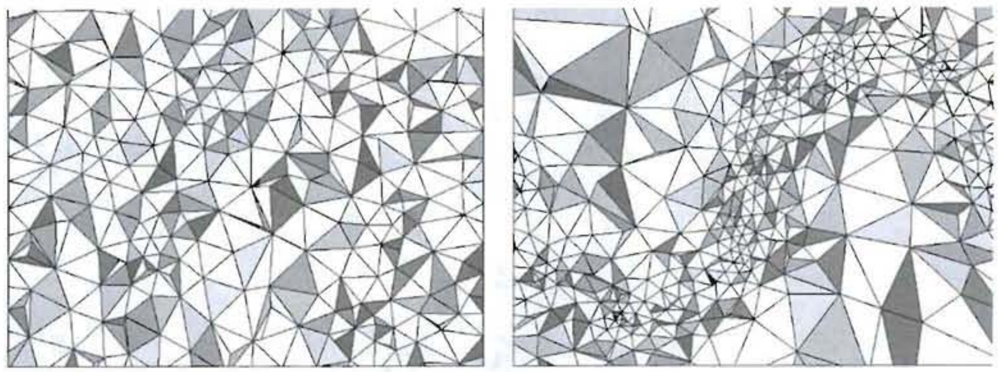

Figure 2: A cut through meshes generated with constant (left) and non-constant (right) mesh size functions.

usually less than 1% of the domain volume. Since only the intersection tests depend heavily on the arithmetic precision, we use a fail-safe intersection test to increase robustness. It may wrongly report non-intersecting triangles as intersecting due to round-off issues. However, it will never report actually intersecting triangles as non-intersecting ones. Thus, we slightly narrow the possibility for front advance, but we will always have a valid conformal mesh at the end.

Adaptivity. The AFT algorithm implemented in the Ani3D package allows the user to provide a size function $h(x)$ to control the local size of mesh elements (see Figure 2 for two examples of $h(x)$). The size function can be defined on the basis of a posteriori error estimates. It must be bounded from below to avoid an infinite refinement loop.

Summarizing, the AFT complies with the scalability and the adaptivity requirements. However, it may produce a small number of low-quality tetrahedra and may leave some lacunas of the computational domain unmeshed. Next step is to employ more sophisticated mesh generation methods in order to split the unmeshed lacunas into tetrahedra.

2.2 Delaunay triangulation

If the AFT fails to mesh the entire domain, we launch the Delaunay triangulation (DT) method to mesh the remaining lacunas and to preserve their boundary faces. The general idea of this method has been proposed in [13]. Here, we discuss main features of the DT method and its necessary modifications.

Quality. By design, the DT method tends to produce high-quality tetrahedra provided that the mesh vertices are appropriately placed. In exact arithmetic, the DT methods works for arbitrary fronts [13] but may create slivers. The appearance of slivers is usually induced by irregular distribution of mesh nodes at the boundaries of lacunas.

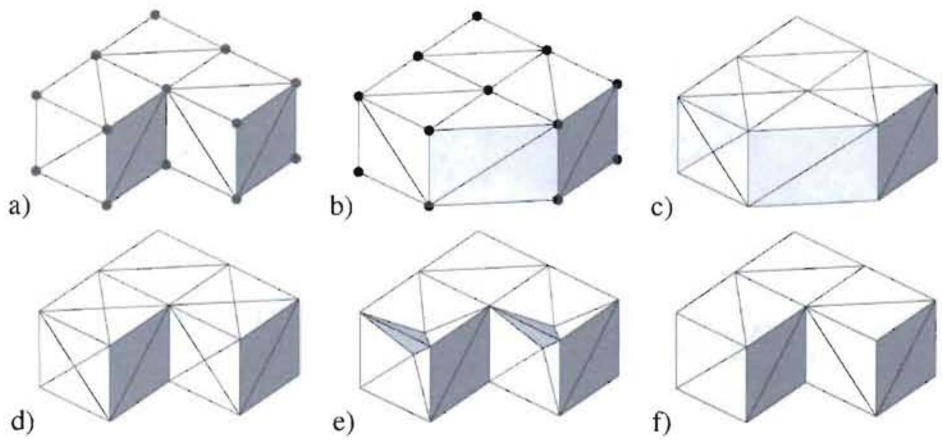

Figure 3: Main steps of the DT method: a) initial front; b) DT mesh of a convex hull; c) refined mesh after intersection with the front; d) restored geometry of the mesh; e) intersection points are shifted inside lacunas; f) final mesh.

Efficiency. Using Figure 3, we describe basic steps in the modified DT method implemented in the Ani3D package (see [2] for more details). First, we take points from the front left by the AFT and apply the conventional incremental algorithm [12] (Figure 3b). This gives two different meshes with the same sets of vertices: the DT mesh of a convex hull and the triangular surface mesh on lacunas boundary. Second, we intersect the DT mesh with the surface mesh (Figure 3c), and remove tetrahedra lying outside the lacunas (Figure 3d). Note that the surface of the modified DT mesh may no longer match the original surface mesh of the lacunas, due to the intersection points. Third, the intersection points are removed from the surface by shifting them inside the lacunas (Figure 3e). The boundary conformity is restored by filling the surface dents with tetrahedra (Figure 3f).

All these operations scale linearly with the number of nodes at the boundary of lacuna. Moreover, in all our experiments, the DT method produced less than 5% of tetrahedra, which allows us to neglect its contribution to the total mesh generation time.

Robustness. In the DT method, round-off errors may lead to wrong edge-to-face intersection, and also may result in degenerated elements after point movements. Intersection problems are mainly due to badly shaped front left after the AFT method. However, tedious implementation resolves these problems robustly, at least from the practical standpoint.

Adaptivity. Since the nodes at the boundaries of lacunas are given, the DT method does not provide enough flexibility of the tetrahedral partition in lacunas to be adapted to a problem solution. Fortunately, the DT method is used only in very small parts of the domain; therefore, we can neglect local violation of adaptation criteria.

Summarizing, the DT method adds robustness to the AFT. The resulting mesh still may have a small number of slivers which are the low-quality tetrahedra. Next step is to employ the robust mesh re-generator to remove these slivers.

2.3 Metric-based mesh re-generation and adaptation

We improve the mesh quality using a sequence of local changes of mesh topology. The local topological operations provide a robust way to adapt the mesh to the problem solution and to gain additional accuracy for the same number of mesh elements. The general idea of the metric-based adaptation (MBA) is described in [4, 22]. Here, we discuss main features of the MBA and derive a tensor metric for minimizing the energy norm of the finite element discretization error.

Quality. Let $Q(\Delta)$ be a quality of tetrahedron Δ. We define this quality as a function depending on the tetrahedron shape and given metric such that

$$0 \leq Q(\Delta) \leq 1.$$

The higher value of $Q(\Delta)$ corresponds to a more shape-regular element and $Q(\Delta) = 1$ corresponds to an equilateral tetrahedron. The mesh quality $Q(\Omega_h)$ is defined as the quality of the worst element in the mesh. By design, the MBA increases the mesh quality after each local change of the mesh topology. The mesh always remains conformal, and the method terminates when $Q(\Omega_h)$ reaches a user-specified threshold. The MBA produces a mesh which is quasi-uniform in a user-defined metric. We use the Euclidean metric to remove slivers. For a stronger mesh adaptation, we use full (possibly anisotropic) tensor metrics.

Let $\mathfrak{M}$ be the space tensor metric. To define a particular tetrahedron quality, we recall definitions of the volume of Δ and the total length of its edges in the metric $\mathfrak{M}$ that are denoted by $|\Delta|_{\mathfrak{M}}$ and $|\partial\Delta|_{\mathfrak{M}}$, respectively:

$$|\Delta|_{\mathfrak{M}} = \int_{\Delta} \sqrt{\det(\mathfrak{M}(\mathbf{x}))}\, \mathrm{d}x, \qquad |\partial\Delta|_{\mathfrak{M}} = \sum_{k=1}^{6} (\mathfrak{M}\, \mathbf{e}_k, \mathbf{e}_k)^{1/2},$$

where $\mathbf{e}_k$ is a vector representation of the tetrahedron edge e_k. The orientation of $\mathbf{e}_k$ is not important for the quality definition. In the Ani3D package, we set

$$Q(\Delta) = 6^4 \sqrt{2} \frac{|\Delta|_{\mathfrak{M}}}{|\partial\Delta|_{\mathfrak{M}}^3}\, f\left(\frac{|\partial\Delta|_{\mathfrak{M}}}{6h_*}\right),$$

where $f(a)$ is a smooth positive function with the only maximum at $a = 1$ and $f(a)$ tends to zero when $a \to 0$ or when $a \to \infty$. The parameter h_* is the size of the targeted uniform mesh in metric $\mathfrak{M}$. Note that other mesh qualities can be also used, see for example [23, 24].

Efficiency. Complete mesh re-generation may require 5-15 sweeps of the MBA operations through the mesh, which makes it rather costly. Practical experience shows that meshes with up to one million nodes may be processed successfully on a PC. In

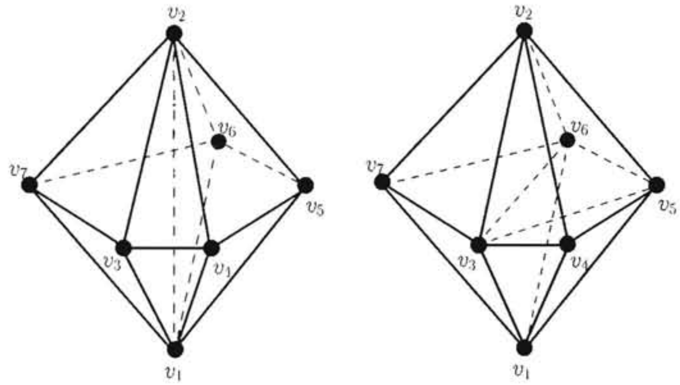

Figure 4: Generalized edge-based swapping: replacement of edge $v_1 - v_2$ by triangulated polygon $v_3 - v_4 - v_5 - v_6 - v_7$.

many cases, it is sufficient to focus the work only on low-quality elements and a few layers of their neighbors. The Ani3D package allows to freeze selected elements.

Robustness. Local changes of the mesh topology is the key to the method robustness. The package Ani3D has seven local topological operations including edge-face swapping, generalized edge-faces swapping, node deletion, node insertion, edge collapse, and node movement. Most of the operations are well-known in the meshing community. The less popular is the generalized edge-face swapping operation that provides additional topological flexibility. Basic idea of this operation is shown in Figure 4: removal of the edge $v_1 - v_2$ results in a possibly non-convex polyhedron which is split into tetrahedra by triangulating non-flat polygon $v_3 - v_4 - v_5 - v_6 - v_7$. This operation improves mesh quality in cases where no other operation can do it [25].

Adaptivity. Ability to adapt the mesh to a problem solution is the key feature of the MBA. The adapted mesh is quasi-uniform in a tensor metric $\mathfrak{M}$ recovered from a discrete solution. In order to generate the $\mathfrak{M}$-quasi-uniform mesh, we use Algorithm 1. For a steady-state problem, generation of a mesh adapted to a solution requires to solve the PDE a few times, at least twice, to generate a proper metric. For a time-dependent problem, solution from a previous time step can be used to generate a metric.

Algorithm 1 Adaptive mesh generation for a steady-state PDE

1: Generate an initial mesh Ω_h, solve the PDE, and compute the metric $\mathfrak{M}$.
2: **loop**
3: Generate a new $\mathfrak{M}$-quasi-uniform mesh Ω_h with the prescribed number of elements.
4: Solve the PDE and recompute the metric $\mathfrak{M}$.
5: If Ω_h is $\mathfrak{M}$-quasi-uniform, then exit the loop.
6: **end loop**

The corner stone of the adaptation technique is the design of a tensor metric which controls the mesh properties. The conventional approach is based on the recovery of the discrete Hessian of the problem solution [4, 5, 22]. In [8, 19, 26, 27], we developed the new approach applicable to approximate solution of a PDE, for example, to the minimization of the finite element discretization error.

Let u be a continuous function and u_h be the piecewise linear finite-element solution on mesh Ω_h. First, we find an approximation of the discretization error $\varepsilon = u - u_h$ on every mesh edge. Then, we derive a metric $\mathfrak{M}$ that allows us to minimize the energy norm of the discretization error. These steps are described in the next two subsections.

2.3.1 Hierarchical error estimates

We search for approximation of the discretization error using the hierarchical enrichment of the primary continuous piecewise linear finite element basis [20]:

$$\varepsilon \approx \varepsilon_h = \sum_{edges\ e_k} \gamma_k\, b_k, \tag{1}$$

where b_k is the bubble function associated with the k-th edge of the mesh.

The straightforward and the most expensive method for calculating the coefficients γ_k is based on the solution of a larger finite element problem. If the primary finite element method results in the algebraic problem $A_{LL}U_L^* = F_L$, the enriched method requires to solve

$$\begin{bmatrix} A_{LL} & A_{LQ} \\ A_{QL} & A_{QQ} \end{bmatrix} \begin{bmatrix} U_L \\ U_Q \end{bmatrix} = \begin{bmatrix} F_L \\ F_Q \end{bmatrix},$$

where subscripts L and Q stand for linear and quadratic terms. If an approximate solution U_L^* of the primary problem is known, the following algebraic problem for the error has to be solved:

$$\begin{bmatrix} A_{LL} & A_{LQ} \\ A_{QL} & A_{QQ} \end{bmatrix} \begin{bmatrix} D_L \\ D_Q \end{bmatrix} = \begin{bmatrix} F_L - A_{LL}U_L^* \\ F_Q - A_{QL}U_L^* \end{bmatrix}. \tag{2}$$

The entries of vector D_Q are the sought coefficients γ_k in Equation (1). Since the exact solution of Equation (2) is too expensive, we make additional approximation. Analysis of spectral properties of finite element matrices performed in [20] for elliptic problems and quasi-uniform meshes suggests to estimate coefficients γ_k by solving the reduced algebraic problem with the sparse and well-conditioned matrix A_{QQ}:

$$A_{QQ}\,\tilde{D}_Q = F_Q - A_{QL}U_L^*. \tag{3}$$

The entries of vector $\tilde{D}_Q$ are the approximations of coefficients γ_k in Equation (1). The numerical analysis performed in [27] shows that the reduced problem provides accurate estimates of the coefficients γ_k on strongly anisotropic meshes aligned with the solution.

2.3.2 Derivation of metric and adaptive loop

Consider a tetrahedron Δ with six edges e_k and six associated coefficients γ_k. The L^2-norm of the gradient of ε_h is given by

$$\|\nabla \varepsilon_h\|^2_{L^2(\Delta)} = \|\sum_{k=1}^{6} \gamma_k \nabla b_k\|^2_{L^2(\Delta)} = |\Delta|(\mathbb{B}\boldsymbol{\gamma}, \boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma}$ is the vector with 6 components γ_k and $\mathbb{B}$ is the 6×6 symmetric positive definite matrix with entries $\mathbb{B}_{k,l} = |\Delta|^{-1} \int_\Delta \nabla b_k \cdot \nabla b_l \, dx$. This error is only a number; therefore, it does not provide any directional information. To recover this information, we split this error into 6 edge-based error estimates $\alpha_k \geq 0$ such that

$$\|\nabla \varepsilon_h\|^2_{L^2(\Delta)} = |\Delta| \sum_{k=1}^{6} \alpha_k, \qquad \alpha_k = |\gamma_k| \, (\mathbb{B}\boldsymbol{\gamma}, \boldsymbol{\gamma}) \Big(\sum_{k=1}^{6} |\gamma_k| \Big)^{-1}. \tag{4}$$

The six numbers α_k are sufficient to define six entries of a constant metric tensor $\mathfrak{M}_\Delta$. Let us consider the quadratic function

$$v_2 = -\frac{1}{2} \sum_{k=1}^{6} \alpha_k b_k$$

and denote its Hessian by $\mathfrak{H}_2$. If $det(\mathfrak{H}_2) \neq 0$, we set $\mathfrak{M}_\Delta = |\mathfrak{H}_2|$, where $|\mathfrak{H}_2|$ is the spectral module of $\mathfrak{H}_2$. Otherwise, we increase slightly the largest α_k so that the modified function v_2 has a non-singular Hessian. In practice, increase by 1% was sufficient in all numerical experiments.

The derived metric $\mathfrak{M}_\Delta$ connects the gradient error from Equation (4) with geometry of element Δ. The following estimate is proved in [8, 19]:

$$(0.3)^{1/3} \, |\Delta|_{\mathfrak{M}_\Delta} |\Delta|^{2/3}_{\mathfrak{M}_\Delta} \leq \|\nabla \varepsilon_h\|_{L^2(\Delta)} \leq |\Delta|_{\mathfrak{M}_\Delta} |\partial \Delta|^2_{\mathfrak{M}_\Delta}. \tag{5}$$

The error equidistribution principle and bounds in Equation (5) suggest to balance $\mathfrak{M}_\Delta$-volumes and $\mathfrak{M}_\Delta$-perimeters. We define the global space metric $\mathfrak{M}$ as the piecewise constant metric with values $\mathfrak{M}_\Delta$. A mesh Ω_h with $N(\Omega_h)$ elements satisfying

$$|\Delta|_{\mathfrak{M}_\Delta} \sim |\partial \Delta|^3_{\mathfrak{M}_\Delta} \sim \frac{1}{N(\Omega_h)} |\Omega_h|_{\mathfrak{M}}$$

is $\mathfrak{M}$-quasi-uniform and is shown numerically to provide the asymptotically optimal rate of error reduction:

$$\|\nabla \varepsilon_h\|_{L^2(\Omega)} \sim N(\Omega_h)^{-1/3}.$$

Analysis of two-dimensional problems gives different asymptotics, $N(\Omega_h)^{-1/2}$, see for example [8, 19].

In practice, Algorithm 1 converges faster when the metric is continuous. To define a continuous metric at nodes of Ω_h, we use the method of shifts. For every node $\mathbf{v}_i$ in Ω_h, we define the superelement σ_i as the union of all tetrahedra sharing $\mathbf{v}_i$. Then, $\mathfrak{M}(\mathbf{v}_i)$ is defined as one of the metrics in σ_i with the largest determinant. Thus, the method of shifts always chooses the worst metric in the superelement.

3 Application examples

The methods described above can be applied for solving two-dimensional boundary value problems [28]. The essential simplification is that the DT method is no longer needed: no lacunas are left by the two-dimensional AFT method.

In the following subsections, we consider three different applications of the mesh generators described above. In the first application, the MBA is used to generate a quasi-optimal mesh minimizing the energy-norm of the discretization error. In the second application, we MBA is used to generate a quasi-optimal mesh minimizing the maximum norm of the discretization error. In the third application, synergy of all three mesh generation methods allows us to build a high-quality mesh for a complex engineering model.

3.1 2D convection-diffusion problem

Let Ω be the unit square. We consider the following boundary value problem:

$$\begin{aligned} -\mathrm{div}(\nu \nabla u - \mathbf{v}u) &= f \qquad \text{in } \Omega, \\ u &= u_0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\nu = 10^{-2}$ and $\mathbf{v} = (2,3)^T$. The right-hand side and the Dirichlet boundary data are calculated using the exact solution proposed in [29]:

$$u(x,y) = \left(x - \exp\left(\frac{2(x-1)}{\nu}\right)\right)\left(y^2 - \exp\left(\frac{3(y-1)}{\nu}\right)\right).$$

The viscosity coefficient ν characterizes thickness of the boundary layer in the top-right corner of Ω.

Figure 5 compares the piecewise linear (P_1) FE solutions on quasi-uniform and adaptive meshes with roughly 4000 elements. Solution on the quasi-uniform mesh exhibits numerical instabilities that can be eliminated using the SUPG technique. Still, the boundary layer will not be well resolved. The adaptive strategy allows us to solve the problem without using the SUPG technique by relocating mesh points to the boundary layers and stretching the near-boundary triangles. The maximal ratio of radii of superscribed to inscribed circles in the stretched triangles reaches 293. It takes 6 iterations of Algorithm 1 to reduce the initial discretization error to a 5% neighborhood of the final error. Solution of Equation (3) requires only 2-3 iterations.

Table 1 shows the error reduction on the sequence of quasi-optimal meshes. The expected half-order convergence rate is observed. Moreover, the hierarchical a posteriori error estimator $\|\nabla\varepsilon_h\|_{L^2(\Omega)}$ is in the good agreement with the true error $\|\nabla\varepsilon\|_{L^2(\Omega)}$.

3.2 3D elasticity problem

Let Ω be the union of two parallelepipeds rotated about z-axis (see Figure 6). The barycenter of Ω has coordinates $(0,0,0)$. We use the AFT to build a coarse mesh with

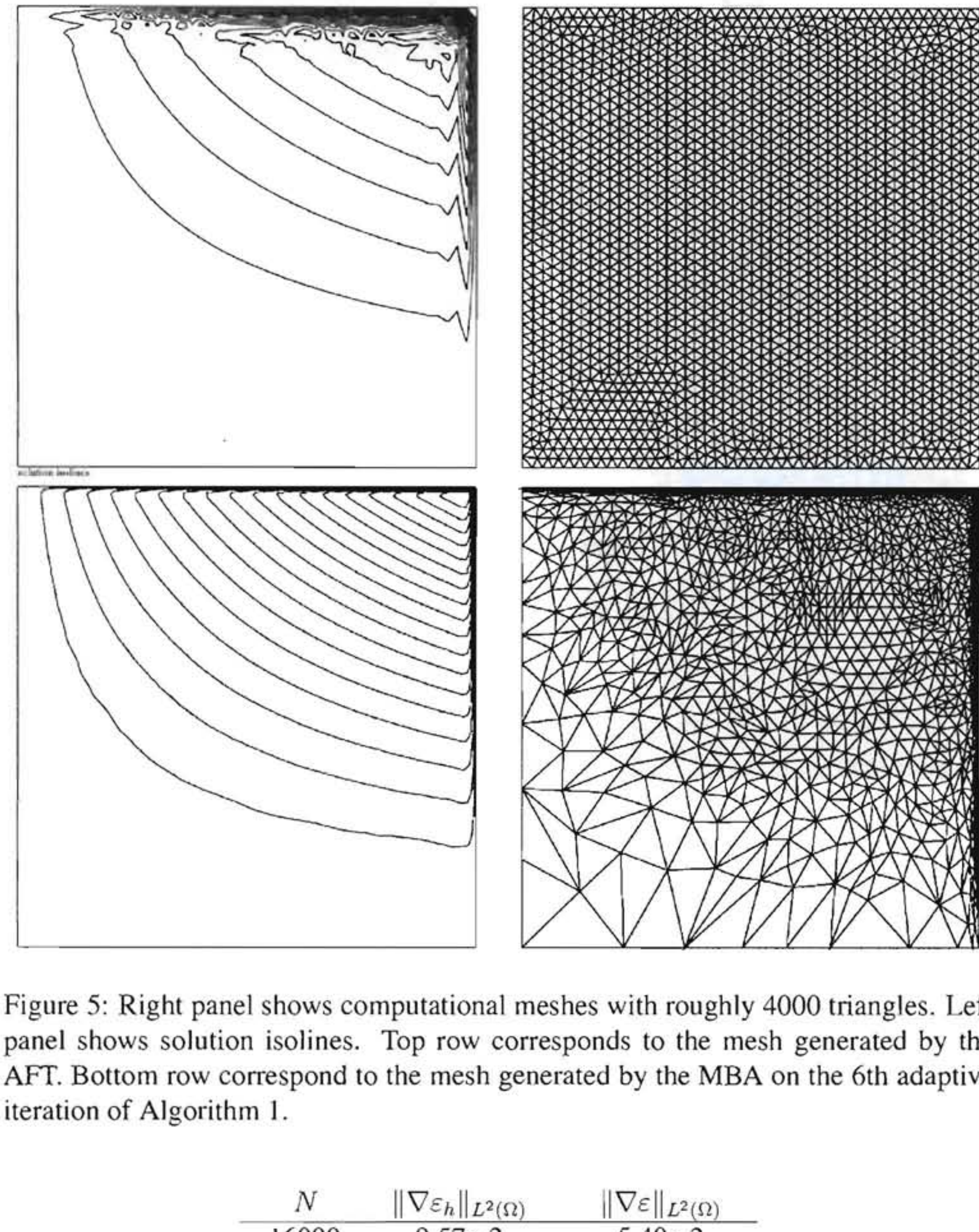

Figure 5: Right panel shows computational meshes with roughly 4000 triangles. Left panel shows solution isolines. Top row corresponds to the mesh generated by the AFT. Bottom row correspond to the mesh generated by the MBA on the 6th adaptive iteration of Algorithm 1.

N	$\|\nabla\varepsilon_h\|_{L^2(\Omega)}$	$\|\nabla\varepsilon\|_{L^2(\Omega)}$
16000	9.57e-2	5.40e-2
64000	5.12e-2	3.01e-2
256000	2.69e-2	1.65e-2
rate	0.46	0.43

Table 1: The error estimates and the true discretization errors.

1177 vertices and 4317 tetrahedra. Then, we employ another supporting utility of the Ani3D package to refine uniformly this mesh. The final mesh has 7307 vertices, 5096

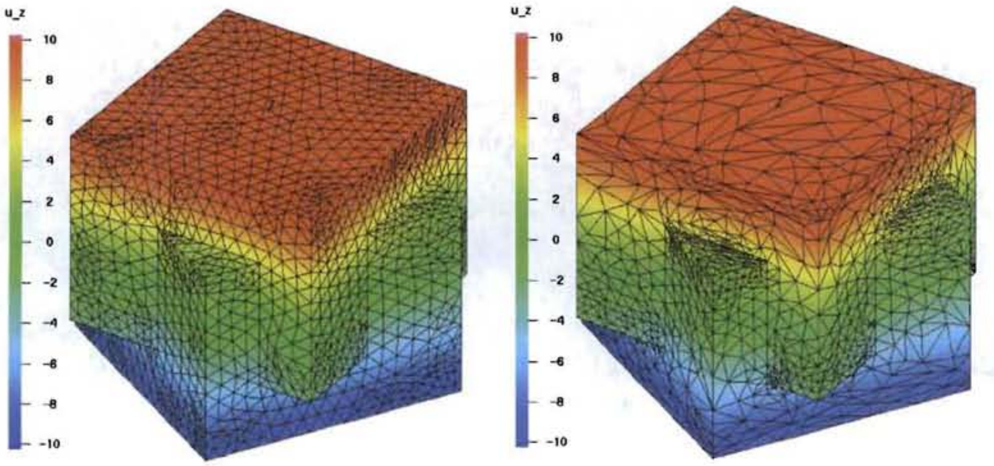

Figure 6: Displacement on the initial mesh (left) and the adaptive mesh after four loops of Algorithm 1.

surface triangles and 34536 tetrahedra.

We consider the following boundary value problem for displacement $\mathbf{u}$:

$$\begin{aligned} \mathrm{div}\mu\nabla\mathbf{u} + (\mu+\lambda)\nabla\mathrm{div}\mathbf{u} &= 0 \quad \text{in } i \quad \Omega, \\ \mathbf{u} &= \mathbf{u}_0 \quad \text{on} \quad \Gamma, \end{aligned}$$

where μ and λ are the Lame coefficients, and Γ is the union of two opposite square faces parallel to plane $x - y$. The Neumann boundary condition is imposed on the remaining boundary. We set $\mu = 3$, $\lambda = 1$, and $\mathbf{u}_0 = (0, 0, z)^T$. The Dirichlet boundary conditions pull the faces in opposite directions. This is a trivial model of a stress analysis.

Our goal is to adapt the mesh in such a way to control the maximum norm of the discretization error in the displacement. We assume that the discretization error can be controlled by the interpolation error, so that the conventional Hessian-based metric can be employed. After four adaptive loops of Algorithm 1, we obtain the mesh shown on the right in Figure 6. The mesh has 7352 points, 4518 triangular faces and 34709 tetrahedra. The MBA relocates mesh vertices in regions where potential tearing up of the model may occur.

3.3 Gear model

The Ani3D package can be interfaced with the OpenCASCADE CAD kernel, which natively supports BREP, STEP and IGES file formats. We consider the STEP model of the gear. It has 422 vertices, 636 curvilinear edges, and 217 curvilinear faces.

We successfully completed generation of meshes with two different resolutions. For coarser resolution, the surface was meshed into a quasi-uniform mesh with 3530

method	$Q(\Omega_h)$	N_T	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
AFT	$1.011 \cdot 10^{-3}$	140442	139684	750	8	—	—
AFT+DT	$1.596 \cdot 10^{-5}$	140723	139817	854	37	5	10
AFT+DT+MBA	$2.000 \cdot 10^{-1}$	156538	156538	—	—	—	—

Table 2: Distribution of tetrahedra by their quality for the mesh with finer resolution.

vertices and 7084 triangles (see Figure 7). For finer resolution, the surface was meshed into a quasi-uniform mesh with 19405 vertices and 38834 triangles. We did not observe any fundamental difference between the two meshes; therefore, the second one is not shown.

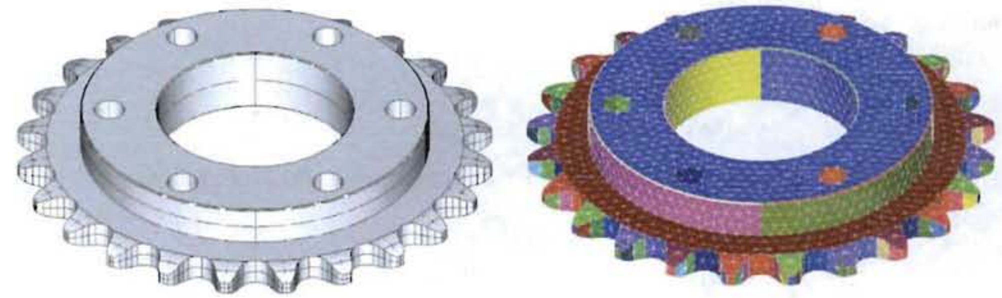

Figure 7: Gear model and the initial surface mesh with 7084 triangles.

The combination of the AFT and DT methods is used for volume meshing. At the first stage, we apply the AFT with quasi-uniform mesh size. For both resolutions, the AFT meshes only 99.96% of the volume. For the finer resolution, it constructs 140442 tetrahedra and terminates leaving 352 triangles in the front. The worst element quality is equal to $1.011 \cdot 10^{-3}$. At the second stage, we apply the DT method in order to mesh the remaining part. The minimal tetrahedra quality after both the AFT and DT methods is $1.596 \cdot 10^{-5}$. The mesh quality distribution is shown in Table 2.

At the post-processing stage, we use the MBA, and the mesh quality is significantly improved. The minimal cell quality is now $2.0 \cdot 10^{-1}$. The final mesh has 38368 vertices, 38834 boundary faces, and 156335 tetrahedra.

Total time of mesh construction is 9 minutes and 33 seconds. Time distribution is as follows: surface meshing from CAD model is 8 minutes and 40 seconds, volume meshing with the AFT and DT is 41 seconds, and post-processing improvement is 12 seconds.

Similar behavior of the mesh generators was observed in constructing the mesh with the coarser resolution. The final quality of this mesh was $1.439 \cdot 10^{-1}$. In Figure 8 we present a cross section of this mesh.

Figure 8: A cut through the volume mesh of a gear. The mesh has 6870 vertices, 7084 boundary faces, and 22456 tetrahedra.

4 Conclusion

We presented a synergistic approach to generation of simplicial meshes and adaptive finite element solution of boundary value problems. The initial mesh generation employs robust combination of the advancing front technique and the Delaunay triangulation. The metric based mesh re-generation method provides essential improvement of the mesh quality in complex engineering models. It also provides mesh adaptation to a problem solution and minimization of various norms of the discretization error.

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