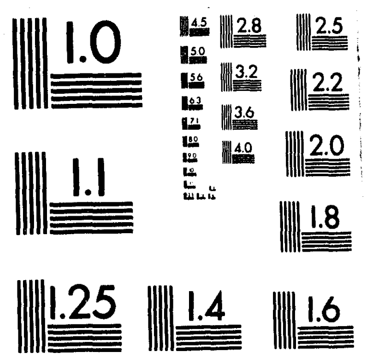
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TITLE: **HYSTERESIS AND NONLINEAR ELASTICITY IN ROCKS**

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HYSTERESIS AND NONLINEAR ELASTICITY IN ROCKS

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INTRODUCTION

The macroscopic elastic properties of highly heterogeneous materials, such as rocks, are unusual and much more complex than those of the materials from which they are assembled. Equations of state for a typical rock, e.g., velocity vs pressure, show nonlinearity that is orders of magnitude greater than that of conventional materials [1,2]. Further, these equations of state are often hysteretic and possess memory features called discrete memory or end point memory [3–5]. The fundamental reason for the hysteretic nonlinear elastic behavior of rock is that rock contains an enormous variety of mesoscopic structural features (cracks, joints, contacts, etc., of typical size $\approx 1\mu$m) with elastic properties that are specific to their structure. It is these mesoscopic elastic units that dominate the response of the rock to both the external pressure used to find a quasi-static equation of state and to the internal pressure that accompanies an elastic wave.

The purpose of this paper is to describe a theory of the propagation of elastic waves in hysteretic nonlinear elastic materials, e.g., rock. In the next section, we introduce the Priesach-Mayergoyz (P-M) model [6,7] of hysteretic systems and adapt it to describe the hysteretic mesoscopic elastic units (HMEU) determining the elastic properties of a rock. We combine the P-M model with effective medium theory (EMT) [8] to find the elastic response of a rock that has experienced a specified pressure history. Next, we consider elastic wave propagation in a hysteretic nonlinear elastic system governed by a history dependent equation of state. We consider one-dimensional propagation of compressional waves. The equation of motion for the longitudinal displacement field contains the same hysteretic nonlinear interactions that characterize the equation of state. We solve the equation of motion using the Green function technique developed by McCall [9]. This solution lets us identify the qualitative features in harmonic generation that are signatures of nonlinearity and hysteresis.

EQUATION OF STATE

We take a rock's macroscopic elasticity to result from a system of many hysteretic mesoscopic elastic units (HMEU). Let the rock be modeled by a simple

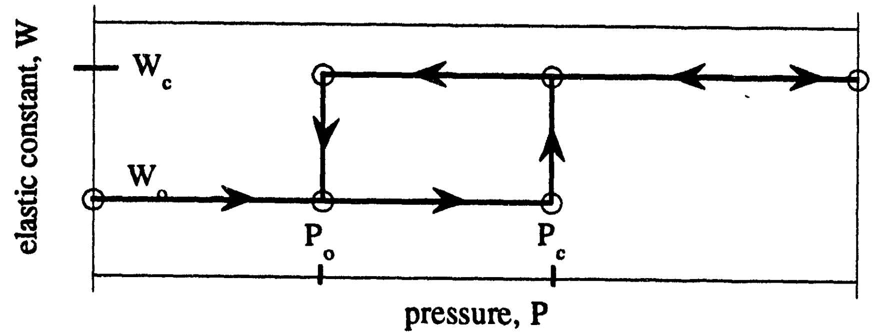

Figure 1. Hysteretic Mesoscopic Elastic Unit (HMEU). A HMEU is characterized by a pair of pressures (P_c, P_o) and a pair of elastic constants (W_c, W_o). At low pressure the elastic constant of the HMEU is W_o. Upon raising the pressure to P_c, the elastic constant of the HMEU becomes W_c. The elastic constant remains W_c until the pressure is reduced to below P_o.

cubic lattice of HMEU with lattice spacing nominally 10 μm. To each of the HMEU we assign two pairs of numbers, a pair of pressures (P_c, P_o), where $P_c \geq P_o$, and a pair of elastic (spring) constants (W_c, W_o), where $W_c \geq W_o$ (see Figure 1). Assume for illustrative purposes that the structural features we are describing with the HMEU are compliant cracks. Then, if the pressure applied to a unit is raised from zero, the unit responds with spring constant W_o (o meaning open) up to pressure P_c (c meaning closed). At P_c the spring constant of the unit changes to W_c, a value the unit retains for all higher pressures. If the pressure is then dropped from a value above P_c, the spring constant of the unit remains W_c until the pressure on the unit is $P_o \leq P_c$, at which time the spring constant becomes W_o. Each of the HMEU has hysteretic spring constant versus pressure behavior. For simplicity, we do not let the equilibrium spacing enforced by the springs change as the pressure changes. We may take (W_c, W_o) to have a statistical or deterministic connection to (P_c, P_o).

In Figure 2(a), we show the pairs (P_c, P_o) in P-M space from the example below. The density of HMEU in the space of (P_c, P_o) pairs is $\rho(P_c, P_o)$. A pressure protocol brings the rock from $P = 0$ to $P \neq 0$ with n pressure reversals. This history leads to a separation of P-M space into two parts bounded by the curve $E(P_c, h)$ [see Figure 2(a)], where h stands for the pressure history leading to the rock's current pressure state P. The history h of the rock, and therefore $E(P_c, h)$, depends on the points of pressure reversal $P_1 \ldots P_n$, the maximum pressure which the rock has experienced P_{max}, and the current pressure P. In Figure 2(a), the HMEU below and to the left of $E(P_c, h)$ are in their closed configuration; the HMEU above and to the right of $E(P_c, h)$ are in their open configuration. We calculate the macroscopic elastic constant using effective medium theory [8]. Thus, the effective spring constant $\overline{W}$ is found from

$$\int_0^{P_{max}} dP_c \int_0^{E(P_c)} dP_o\, \rho(P_c, P_o)\, F(W_c, \overline{W}) + \int_P^{\infty} dP_c \int_{E(P_c)}^{P_c} dP_o\, \rho(P_c, P_o)\, F(W_o, \overline{W}) = 0, \quad (1)$$

where

$$F(W_\nu, \overline{W}) = \frac{W_\nu(P_c, P_o) - \overline{W}[E]}{W_\nu(P_c, P_o) + 2\overline{W}[E]}, \tag{2}$$

and ν stands for c or o. The equation of state is completely determined when values are assigned to $W_\nu(P_c, P_o)$ for each of the HMEU. Given $\overline{W}$ from Equation (1), we take $c^2 = \overline{W}/\rho_r$, where ρ_r, the density of the rock, is a constant. Discrete memory in the number of closed or open HMEU is a consequence of the structure of P-M space and the nature of the possible curves $E(P_c, h)$ [10]. Discrete memory for the elastic constant and the sound velocity c follows from Equation (1). If $\rho(P_c, P_o)$ is strictly diagonal, the velocity of sound may be a complicated function of P but there is no hysteresis. The area of a hysteresis loop is related to the fraction of the density ρ that is off the diagonal.

Example

A set of 500 points (P_c, P_o) were generated according to the rules

$$P_c = 100\, r_c^{1/3}, \qquad P_o = 100\, P_c\, r_o^{1/3}, \tag{3}$$

where r_c and r_o are random numbers uniformly distributed between 0 and 1. Representative points (P_c, P_o) generated from Equation (3) are plotted in Figure 2(a). The rock is carried through the pressure protocol shown in Figure 2(b) in which the pressure was raised and lowered three times. Values of the pressure are in arbitrary units. We took (W_c, W_o) to be uncorrelated with (P_c, P_o) and to be separately distributed according to $p(W_\nu) = 4$, where $0.25 < W_o < 0.5$ and $0.75 < W_c < 1$.

At point A on the pressure protocol in Figure 2(b) the pressure history has included four points of pressure reversal, denoted $1 \ldots 4$. The corresponding separation curve $E[P_c, h(A)]$ is shown on Figure 2(a). In Figure 2(c) we show the elastic constant $\overline{W}$, calculated from Equation (1), as a function of the pressure history. The pressure protocol is such that each hysteresis loop is swept out in a counterclockwise direction. We take each loop to be made up of in-phase and out-of-phase components. The in-phase component is the average of the two values of $\overline{W}$ associated with each pressure. The out-of-phase component is the difference at each point in the loop between $\overline{W}$ and the appropriate in-phase value. In Figure 2(d) we show the in-phase and out-of-phase components for the largest loop in Figure 2(c). The other two loops look similar. The in-phase component is in phase with the pressure and represents the reversible part of the nonlinearity. The out-of-phase component is out of phase with the pressure, has the sense of the rate of change of the pressure and represents the hysteretic part of the nonlinearity.

ELASTIC WAVE PROPAGATION

In this section we apply a Green function formalism developed in Reference 9 to describe elastic wave propagation in rock. We wish to focus on the consequences of hysteresis and will therefore limit ourselves to the propagation of compressional waves in a single dimension.

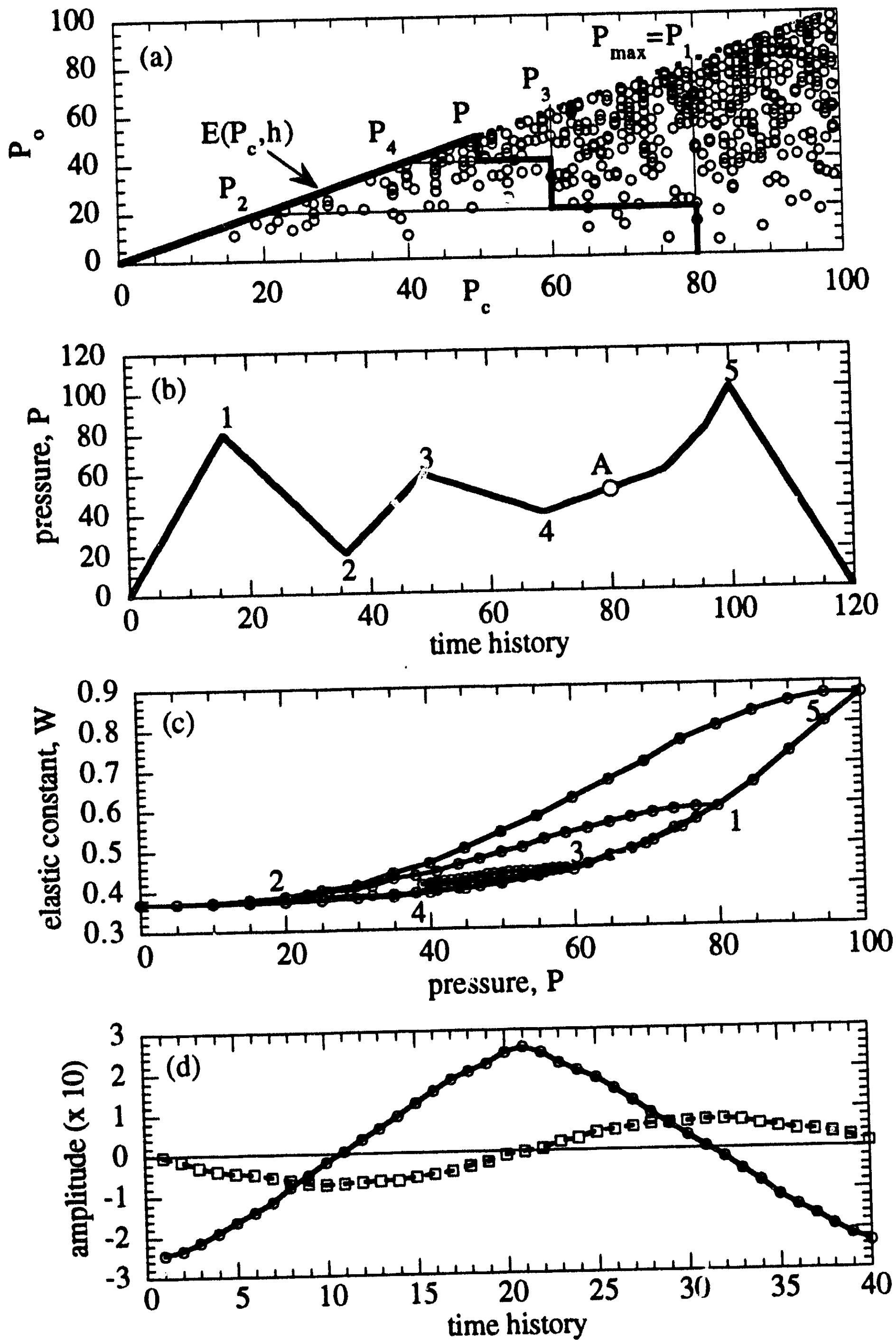

Figure 2. Example elastic equation of state. (a) The points (P_c, P_o) in Priesach-Mayergoyz (P-M) space; 500 points generated from Equation (3) are shown. Because $P_o \leq P_c$, the density of points $\rho(P_c, P_o)$ is non-zero on or below the diagonal. The heavy curve corresponds to $E(P_c, h)$ for the pressure protocol at point A in (b). (b) The pressure protocol followed in constructing the equation of state. (c) The elastic constant as a function of pressure for the pressure protocol shown in (b). (d) The in-phase (circles) and out-of-phase (squares) components of the largest loop shown in (c).

We begin with the equation of motion for the displacement field in a rock that has been brought to ambient pressure P by a prescribed pressure protocol

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial}{\partial x}\left\{[1+\kappa(x,t)]\frac{\partial u}{\partial x}\right\} + S(x), \tag{4}$$

where u is the x-component of the displacement field, $S(x)$ is the external source that drives the system, and c_0^2 is the velocity of sound at pressure P. The quantity $\kappa(x,t)$ is the modification of the velocity of sound induced by the pressure fluctuation $\delta P(x,t)$. Thus, $\kappa(x,t)$ is a functional of the pressure or the displacement field.

Using the Green function method, we develop a systematic treatment of Equation (4) without initially specifying $\kappa(x,t)$. The procedure is as follows (see Reference 9): (1) Specify the external disturbance. (2) Find the Green function $g(x,x',\omega)$ for the $\kappa(x,t)=0$ problem and the specific geometry to be studied. (3) Develop $u(x,t)$ and $\kappa(x,t)$ in powers of the strength of the source S. For the leading correction to the displacement field u_0 induced by $S(x,t)$ one finds

$$u_1(x,\omega) = \int dx' \int \frac{d\omega'}{2\pi} g(x,x',\omega)\frac{\partial}{\partial x'}\left[\kappa_0(x',\omega')\frac{\partial u_0(x',\phi)}{\partial x'}\right], \tag{5}$$

where $\phi = \omega - \omega'$, $\kappa_0(x,\omega) = \kappa[u_0(x,\omega)]$ and u_0 is

$$u_0(x,\omega) = \int dx' g(x,x',\omega) S(x',\omega). \tag{6}$$

For hysteretic materials such as rocks, we showed that pressure cycles cause a change in the elastic constant that is not an analytic function of $u(x,t)$ or $\delta P(x,t)$. The P-M space and EMT model lets us assess the effect on the elastic constant of fluctuations in $E(P_c,h)$ brought about by δP. The nonanalyticity of $\kappa(x,t)$, and therefore the velocity of sound, comes from the nonanalyticity of $E[P_c, h(P+\delta P)]$ at points of pressure reversal.

The elastic constant as a function of q the fraction of closed HMEU can be written

$$\overline{W}(q_0+\delta q) = \overline{W}(q_0) + \left.\frac{\partial \overline{W}}{\partial q}\right|_{q=q_0} \delta q(x,t). \tag{7}$$

Then, the velocity of sound and $\kappa(x,t)$ are given by

$$c^2 = c_0^2\,[1+\beta\,\delta q(x,t)], \qquad \kappa(x,t) = \beta\,\delta q(x,t), \tag{8}$$

where $c_0^2 = \overline{W}/\rho_r$ and $\beta = (\partial\overline{W}/\partial q)/\overline{W}$ evaluated at q_0. Fluctuation in the fraction of closed elastic units is the source of fluctuation in the velocity of sound. Let $\delta q_0(x,\omega)$ denote the fluctuation in q brought about by u_0. From the discussion of the in-phase and out-of-phase components of a hysteresis loop, we know that to leading order in the amplitude of δP, $\delta q_0(x,\omega)$ is a sum of components in phase and out of phase with δP_0. Thus, if $u_0 = U\sin(k_0 x - \omega_0 t)$ and $\delta P = -K(\partial u/\partial x)$, where K is the compressibility of the rock at pressure P, we can write $\delta q_0(x,\omega)$ as a Fourier series in $\tau = k_0 x - \omega_0 t$:

$$\delta q_0(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\tau + \sum_{n=1}^{\infty} b_n \sin n\tau, \tag{9}$$

where the a_n are the in-phase amplitudes and the b_n are the out-of-phase amplitudes. The amplitudes a_n and b_n are proportional to δP_0. We have chosen c_0^2 so that $a_0 = 0$.

Using the Green function for an infinite, homogeneous material, and $u_0 = U \sin\tau$, we find the perturbation displacement $u_1(x,t)$ is

$$u_1(x,t) = -\frac{\beta k_0 U x}{2} \cos\tau \sum_{n=1}^{\infty} c_n \cos(n\tau - \phi_n), \tag{10}$$

where $c_n = \sqrt{a_n^2 + b_n^2}$ and $\tan\phi_n = b_n/a_n$.

There are several observations of general validity about $u_1(x,t)$. (1) The amplitude at distance x from the source is proportional to x, independent of the choice of $\kappa_0(x,t)$. This proportionality represents the fact that nonlinear elastic waves interact in the system over the entire range between source and observer. (2) The amplitude is proportional to $(k_0 U)^2$. The first factor of $k_0 U$ comes from the incident elastic wave. The second factor comes about because $c_n \propto \delta P_0 \propto k_0 U$. (3) Hysteresis exhibits itself in the phase of the scattered wave.

We may reduce Equation (10) to a number of known results.

(a) Standard cubic anharmonicity:

$$\kappa_0(x,t) = \beta k_0 U \cos\tau. \tag{11}$$

This choice of κ vs δP is the solid line in Figure 3(a). In Equation (9), $a_n = k_0 U \delta_{n,1}$ and $b_n = 0$. Then u_1 in Equation (10) is the well known result

$$u_1(x,t) = -\frac{\beta k_0^2 U^2 x}{4} \left[\cos 2\tau + 1\right]. \tag{12}$$

(b) For an entirely hysteretic nonlinear elastic response, consider

$$\kappa_0(x,t) = \beta k_0 U \sin\tau, \tag{13}$$

the dashed circle in Figure 3(a). In this case, $a_n = 0$ and $b_n = k_0 U \delta_{n,1}$. For u_1 we find

$$u_1(x,t) = -\frac{\beta k_0^2 U^2 x}{4} \sin 2\tau. \tag{14}$$

Note the phase difference between these two results. The elastic wave response to a hysteretic nonlinearity is 90° out of phase with the response to a nonhysteretic nonlinearity.

(c) Choose a simple hysteretic function for κ which is composed of parabolic pieces:

$$\kappa_0(x,t) = \beta k_0 U \begin{cases} 1 - \frac{2}{\pi^2}\tau^2, & 0 < \tau < \pi \\ 1 - \frac{4}{\pi}\tau + \frac{2}{\pi^2}\tau^2, & \pi < \tau < 2\pi \end{cases}, \tag{15}$$

the short-dashed curve in Figure 3(a). In Equation (9), $a_n = 8k_0U/(\pi n)^2$ and $b_n = 16k_0U/(\pi n)^3$ for n odd; $a_n = b_n = 0$ for n even. The first order nonlinear displacement u_1 as a function of τ resulting from this choice of κ is shown in

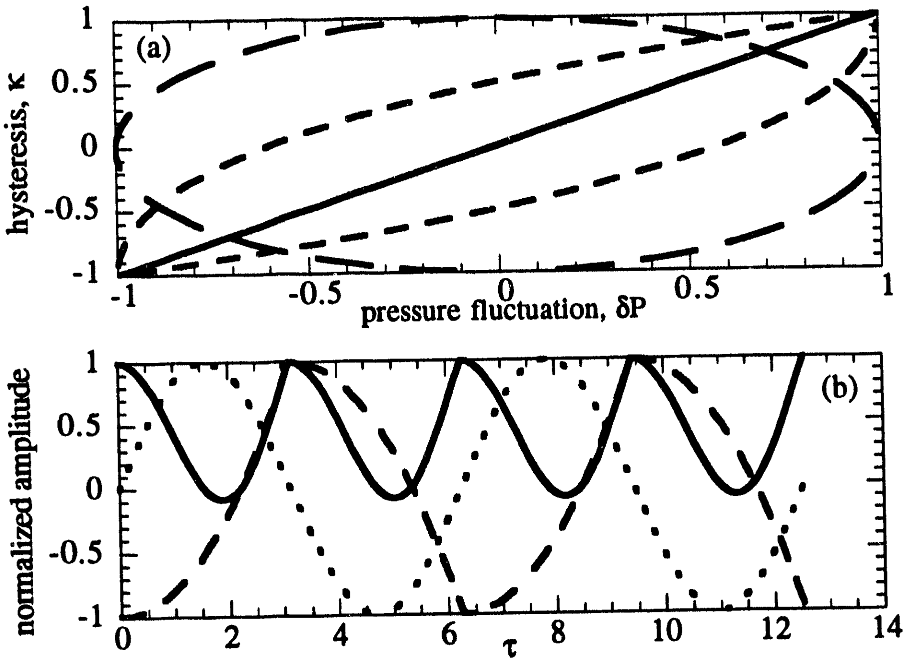

Figure 3. Models of $\kappa(x,t)$ for wave propagation. (a) κ as a function of the pressure fluctuation δP. The solid line is from Equation (11), the long-dashed line is from Equation (13), and the short-dashed line is from Equation (15). (b) $u_0(\tau)$ (short-dashed line), $\kappa(\tau)$ from Equation (15) (long-dashed line), and the resulting $u_1(\tau)$ (solid line).

Figure 3(b), along with κ and u_0. The displacement u_1 has a substantial dc offset, has a discontinuous derivative at $\tau = n\pi$, and oscillates at twice the frequency of u_0.

ENERGY LOSS; Q

The out-of-phase component of $\kappa(x,t)$ is described by the b_n terms in Equation (9) and contributes to the attenuation. We define Q by

$$\frac{1}{Q} = \frac{\Delta E}{\overline{E}}, \tag{16}$$

where ΔE is the energy loss per cycle and $\overline{E}$ is the average energy in the wave during a cycle. For ΔE we take

$$\Delta E = \oint \sigma d\epsilon, \tag{17}$$

where $\oint$ stands for integration over one cycle in time, σ is the stress, and ϵ is the strain. The stress σ is found from the first term on the right hand side of Equation (4),

$$\sigma = c_0^2 \left[1 + \kappa(x,t)\right] \frac{\partial u}{\partial x}. \tag{18}$$

We develop ΔE as a series in the strength of the nonlinearity β, in direct analogy with the method of solution to Equation (4) in Reference 9. We find to first order in β that $\Delta E = \Delta E_0 + \Delta E_1$, where ΔE_0 is the contribution to the energy loss due to the linear elastic response of the system and

$$\Delta E_1 = c_0^2 \oint \kappa(x,t) \frac{\partial u_0}{\partial x} \frac{\partial \dot{u}_0}{\partial x} dt = \frac{c_0^2 \omega_0 (k_0 U)^2}{2} \oint \kappa(x,t) \sin 2\tau dt. \qquad (19)$$

The integral around a cycle in time picks out the term in $\kappa(x,t)$ that is proportional to $\sin 2\tau$. It is the amplitude b_2 of the out-of-phase component of the nonlinear elasticity that is responsible for the attenuation. We have

$$\frac{1}{Q} - \frac{1}{Q_0} \propto \beta b_2. \qquad (20)$$

where $Q^{-1} = \Delta E / \overline{E}$, $Q_0^{-1} = \Delta E_0 / \overline{E}$ and $\overline{E} = \rho c_0^2 (k_0 U)^2$.

Recall that $b_2 \propto \delta P_0 \propto k_0 U$. The hysteretic part of the nonlinear elasticity is responsible for the amplitude dependent attenuation, i.e., the nonlinear attenuation. The coefficient b_2 is a measure of the size of this nonlinear attenuation. In the work of Day and Minster [11], nonlinear attenuation Q is found to be the cause of hysteresis. Here, in contrast, we find hysteresis to be the cause nonlinear attenuation.

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