Conf-9108185--1

LBL-31325

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics Division

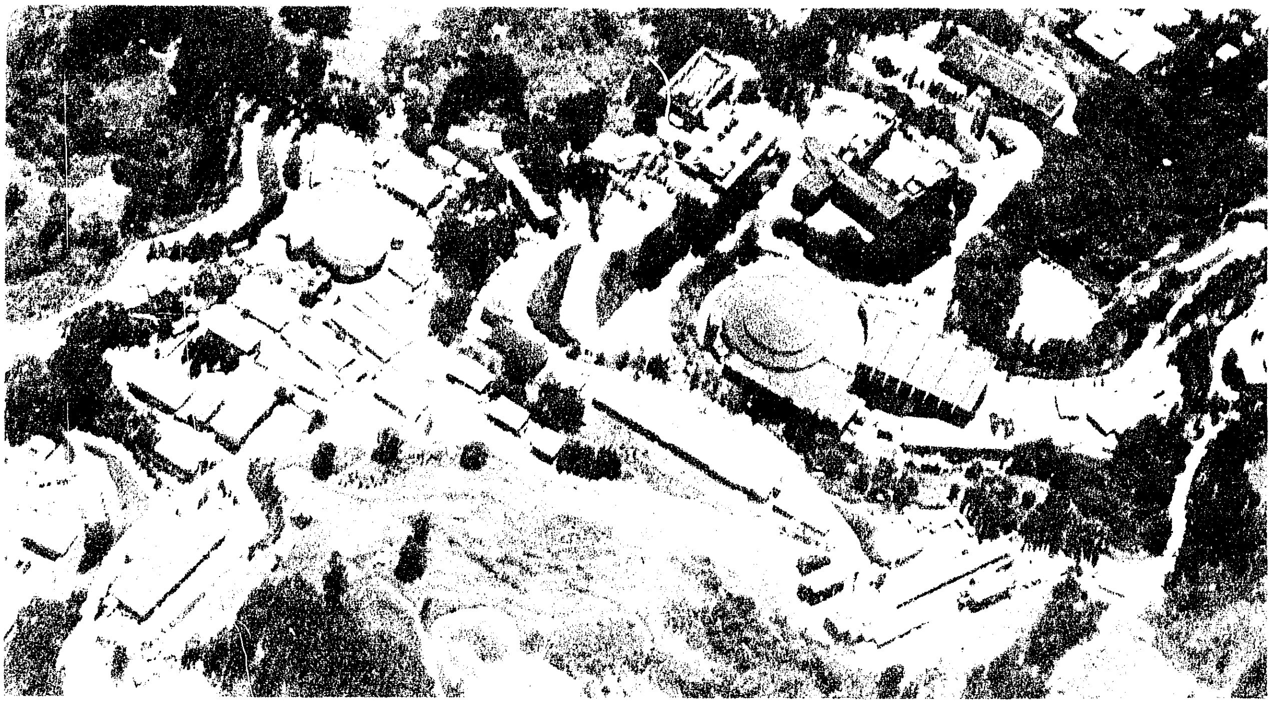
DEC 17 1991

Presented at the Workshop on Superstrings and Related Topics, Trieste, Italy, August 1991, and to be published in the Proceedings

Superspace WZW Models and Black Holes

M. Roček, C. Ahn, K. Schoutens, and A. Sevrin

October 1991



Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

LBL--31325

Oct. 1991

IASSNS-HEP-91/
LBL-31325,

DE92 004169

SUPERSPACE WZW MODELS AND BLACK HOLES[*†‡]

M. Roček[§]

School of Natural Sciences
Institute for Advanced Study
Olden Lane, Princeton NJ 08540

C. Ahn, K. Schoutens

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840

and

A. Sevrin

Department of Physics, University of California
and
Theoretical Physics Group, Lawrence Berkeley Laboratory
Berkeley, California 94720

We show how to write an off-shell action for the $SU(2) \times U(1)$ supersymmetric WZW model in terms of $N = 2$ chiral and twisted chiral multiplets. We discuss the $N = 4$ supersymmetry of this model and exhibit the $N = 4$ superconformal current algebra. Finally, we show that the off-shell formulation makes it possible to perform a duality transformation, which leads to a supersymmetric sigma model on a manifold with a black hole type singularity.

*Contribution to the *Workshop on Superstrings and Related Topics*, Trieste, Aug. 1991. Talk given by K. Schoutens.

†Work supported in part by grant NSF-91-08054

‡This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY90-21139.

§On leave from ITP, SUNY at Stony Brook

Off-shell formulations of WZW-models are known in $N = 0, 1$ superspace [1, 2]. For example, the $N = 1$ action is

$$S = \int d^2 z \nabla_+ \nabla_- \left[(g_{ij} + b_{ij}) \nabla_+ \phi^i \nabla_- \phi^j \right] , \tag{1}$$

where ϕ^i is a unconstrained scalar superfield that coordinatizes the group manifold, g_{ij} is the metric, and b_{ij} is the potential for the parallelizing torsion. It is known that any even dimensional group allows for an $N = 2$ super Kac-Moody symmetry, and a subset of these models have an $N = 4$ symmetry [3]. On dimensional grounds, it is clear that $N = 2, 4$ superspace actions are simply functions of the superfields without any derivatives; hence it is not evident how one can write g and b terms separately. For example, if one writes an action that depends on the most familiar $N = 2$ scalar multiplet, a chiral superfield, then one finds that g is necessarily Kähler and $b = 0$ [4]. For WZW models, g is *never* Kähler and $b \neq 0$, so chiral superfields are not enough. This is a new feature of extended supersymmetry: the dynamics is not determined entirely by the form of the action, but also by the kinematical nature of the superfields. A particular example of a variant (twisted) scalar multiplet was introduced by Gates, Hull, and Roček [5] (see also [6]). In a recent paper [7], we showed that the $SU(2) \times U(1)$ super WZW model can be formulated in $N = 2, 4$ superspace using a usual chiral and a twisted chiral superfield. We also showed that all other WZW models require more exotic representations.

In this paper we will first briefly review the results of [7] for the off-shell formulation of the $SU(2) \times U(1)$ super WZW model in $N = 2$ superspace. We will then focus on the on-shell current algebra, and, working in chiral $N = 2$ superspace, explicitly show how the $N = 2$ superconformal algebra can be extended to $N = 4$. Finally, we will go back to the classical level and perform a duality transformation which leads to a dual sigma model. The latter has the interpretation of a black hole solution to two-dimensional string theory.

In $N = 2$ superspace, we work with complex left and right handed spinor derivatives $D_\pm$ satisfying the algebra

$$\{D_\pm, \overline{D}_\pm\} = \partial_{\pm\pm} , \tag{2}$$

all other anticommutators vanish. Here $\partial_{++} = \partial_z$, etc. Chiral superfields obey:

$$\overline{D}_\pm \Phi = 0\,, \qquad D_\pm \overline{\Phi} = 0\,. \tag{3}$$

In contrast, twisted chiral superfields obey [5]

$$\overline{D}_+ \Lambda = 0\,,\; D_- \Lambda = 0\,,\; D_+ \overline{\Lambda} = 0\,,\; \overline{D}_- \overline{\Lambda} = 0\,. \tag{4}$$

Both superfields can be reduced to $N = 1$ superfields as follows: We define real $N = 1$ spinor derivatives $\nabla_\pm = D_\pm + \overline{D}_\pm$ and "extra" supersymmetry generators $\tilde{Q}_\pm = i(\overline{D}_\pm - D_\pm)$. The resulting $N = 1$ superfields ϕ, λ are unconstrained scalars with the following transformations under the extra supersymmetry:

$$\begin{aligned} \tilde{Q}_\pm \phi = -i\nabla_\pm \phi\,, \qquad\qquad & \tilde{Q}_\pm \overline{\phi} = +i\nabla_\pm \overline{\phi}\,, \\ \tilde{Q}_+ \lambda = -i\nabla_+ \lambda\,,\; \tilde{Q}_- \lambda = +i\nabla_- \lambda\,, \qquad & \tilde{Q}_+ \overline{\lambda} = +i\nabla_+ \overline{\lambda}\,,\; \tilde{Q}_- \overline{\lambda} = -i\nabla_- \overline{\lambda}\,. \end{aligned} \tag{5}$$

However, it is known that extra supersymmetries can be written in $N = 1$ superspace as [5]

$$\tilde{Q}_\pm \phi^i = J_{(\pm)}{}^i{}_j \nabla_\pm \phi^j\,. \tag{6}$$

Comparing (5) with (6), we can read off $J_{(\pm)}$, and find that they are both constant, distinct, commuting complex structures. This is a general feature of complex structures on models constructed with only chiral and twisted chiral multiplets: the resulting left and right complex structures must commute [5]. In [7] it was shown that such commuting structures exist on $SU(2) \times U(1)$, but not on other group manifolds.

A supersymmetric non-linear σ-model has N left and right handed supersymmetries when there exist two sets of $N-1$ covariantly constant complex structures [8, 3]. All the complex structures within each set anticommute, and the metric has to be hermitian with respect to all of them. When the connection has torsion, integrability requires the vanishing of the Nijenhuis tensors and the left handed (right handed) complex structures have to be covariantly constant with respect to the connection consisting of the metric connection plus (minus) the torsion ($\Gamma_\pm = \{\} \pm T$).

In the case of supersymmetric WZW models, these conditions were completely solved in [3]. A complex structure is in one to one correspondence with a Cartan decomposition of the Lie algebra. On the rootspace, the complex structure is diagonalized and has eigenvalue i or $-i$, when the root is positive or negative, resp.; the Cartan subalgebra is mapped to itself. The existence of a second complex structure, anticommuting with the first one, implies a third complex structure (the product of the first two), *i.e.,* $N = 3$ implies $N = 4$ supersymmetry. It turns out that $N = 4$ is only possible on a restricted set of group manifolds. These group manifolds are such that they can be written as a product of coset spaces which have the following structure. Given a group G with Lie algebra g and a Cartan decomposition, we consider the highest root θ. Then $E_{\pm\theta}$ and $\theta \cdot H$ form a $su(2)$ subalgebra, which we call $su(2)_\theta$. The remainder of the Cartan subalgebra together with all roots perpendicular to θ form another subalgebra $H_\perp$. The coset space $W = G/H_\perp \times SU(2)_\theta$ is a Wolf space [9]. An $N = 4$ group manifold can be decomposed as products of coset spaces of the form $W \times SU(2)_\theta \times U(1)$. The second complex structure acts within each of these coset spaces. The action on W is clear as it decomposes in doublets under $SU(2)_\theta$. The action on $SU(2) \times U(1)$ is such that $E_{\pm\theta}$ get mapped to the Cartan subalgebra and vice versa. More details are given in [3, 10, 11].

We now analyze the case of $SU(2) \times U(1)$ in detail. Following the discussion above, we have essentially unique candidates for $J_{(\pm)}$:

$$\begin{aligned} J_{(\pm)}E_+ &= iE_+\,, & J_{(\pm)}E_- &= -iE_-\,, \\ J_{(\pm)}(H_0 + iH_3) &= \pm i(H_0 + iH_3)\,, & J_{(\pm)}(H_0 - iH_3) &= \mp i(H_0 - iH_3)\,, \end{aligned} \tag{7}$$

where H_0 generates $U(1)$ transformation and $E_\pm$, H_3 are the generators of $SU(2)$. The form is fixed by the condition that $J_{(+)}$ and $J_{(-)}$ commute. Eq. (6) implies analogous relations for the Lie algebra valued currents:

$$(g^{-1}\tilde{Q}_+ g)^a = J_{(+)}{}^a{}_b (g^{-1}\nabla_+ g)^b\,, \quad (\tilde{Q}_- g\, g^{-1})^a = J_{(-)}{}^a{}_b (\nabla_- g\, g^{-1})^b\,. \tag{8}$$

Using the explicit form of $J_{(\pm)}$ (7), and the relation to the $N = 2$ derivatives $D = \frac{1}{2}(\nabla + i\tilde{Q})$, $\overline{D} = \frac{1}{2}(\nabla - i\tilde{Q})$, we can lift the relations (8) to $N = 2$ superspace. This leads to the following parametrization of g in terms of a chiral superfield Φ and a twisted chiral superfield Λ:

$$g = \frac{e^{i\theta}}{\sqrt{\Phi\overline{\Phi} + \Lambda\overline{\Lambda}}} \begin{pmatrix} \Lambda & \overline{\Phi} \\ -\Phi & \overline{\Lambda} \end{pmatrix}, \tag{9}$$

where $\theta = -\frac{1}{2}\ln(\Phi\overline{\Phi} + \Lambda\overline{\Lambda})$. This gives an off-shell $N = 2$ formulation of the group $SU(2) \times U(1)$. In these coordinates, the metric on the group manifold is:

$$ds^2 = \frac{d\Phi d\overline{\Phi} + d\Lambda d\overline{\Lambda}}{\Phi\overline{\Phi} + \Lambda\overline{\Lambda}}. \tag{10}$$

In [5], it was shown that the metric can be expressed in terms of a potential function (analogous to a Kähler potential in the case without torsion): $ds^2 = K_{\Phi\overline{\Phi}} d\Phi d\overline{\Phi} - K_{\Lambda\overline{\Lambda}} d\Lambda d\overline{\Lambda}$. Here, we find

$$K(\Phi, \overline{\Phi}, \Lambda, \overline{\Lambda}) = -\int^{\frac{\Lambda\overline{\Lambda}}{\Phi\overline{\Phi}}} \frac{dx}{x} \ln(1+x) + \ln\Phi \ln\overline{\Phi}. \tag{11}$$

This is the $N = 2$ superspace lagrangian. We can read off the torsion potential from $K_{\Phi\overline{\Lambda}}$, etc. (see [5]).

As noted above, $SU(2) \times U(1)$ actually admits $N = 4$ supersymmetry. In $N = 2$ superspace, the necessary condition for $N = 4$ supersymmetry is $K_{\Phi\bar{\Phi}} + K_{\Lambda\overline{\Lambda}} = 0$, [5], which is clearly satisfied in this case. In [5, 12], the general $N = 4$ superspace description is given. For the case at hand, this was further worked out in [7].

The existence of a fully off-shell formulation of the model has an important consequence: it is straightforward to deform the model while maintaining full $N = 4$ supersymmetry, and hence conformal invariance [7]. Such CFT's have recently been proposed as a stringy instanton solutions [13].

We will now take a brief look at the quantum theory and discuss the $N = 4$ superconformal symmetry at the level of on-shell current algebra.

Let us first say a few general things about the on-shell current algebra in $N = 2$ superspace for the supersymmetric WZW model, with level k, on a group G of even dimension. This theory was first worked out by Hull and Spence in [14].

We pick a complex basis for the Lie algebra, labelled by a, $\bar{a}$, $a = 1, 2, \ldots$, $\frac{1}{2}\dim G$, which is such that the complex structure related to the second supersymmetry has eigenvalue $+i$ on the generators T_a and $-i$ on the generators $T_{\bar{a}}$. The $N = 2$ affine Kac-Moody currents Q^a and $Q^{\bar{a}}$ can then be characterized by the following constraints (we will only discuss the currents that are chiral in the sense that they are annihilated by D_- and $\overline{D}_-$; for brevity we will write D for D_+ and $\overline{D}$

for $\overline{D}_+$)

$$DQ^a = -\frac{1}{2(k+\tilde{h})} f^a{}_{bc} Q^b Q^c, \quad \overline{D} Q^{\bar{a}} = -\frac{1}{2(k+\tilde{h})} f^{\bar{a}}{}_{\bar{b}\bar{c}} Q^{\bar{b}} Q^{\bar{c}}. \tag{12}$$

In here $\tilde{h}$ is the dual Coxeter number of G and the the f's are the structure constants in the complex basis. The fundamental OPE's of these $N=2$ superfields are

$$\begin{aligned}
Q^a(Z_1)Q^b(Z_2) &= \frac{\bar{\theta}_{12}}{z_{12}} f^{ab}{}_c Q^c(Z_2) + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}} \frac{1}{k+\tilde{h}} f^a{}_{ec} f^{be}{}_d Q^c Q^d(Z_2) \\
Q^{\bar{a}}(Z_1)Q^{\bar{b}}(Z_2) &= \frac{\theta_{12}}{z_{12}} f^{\bar{a}\bar{b}}{}_{\bar{c}} Q^{\bar{c}}(Z_2) - \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}} \frac{1}{k+\tilde{h}} f^{\bar{a}}{}_{\bar{e}\bar{c}} f^{\bar{b}\bar{e}}{}_{\bar{d}} Q^{\bar{c}} Q^{\bar{d}}(Z_2) \\
Q^a(Z_1)Q^{\bar{b}}(Z_2) &= (k+\tilde{h}) \left[\frac{g^{a\bar{b}}}{z_{12}} + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} \left(\frac{1}{2} g^{a\bar{b}} + \frac{1}{2(k+\tilde{h})} f^a{}_{cd} f^{\bar{b}cd} \right) \right] \\
&\quad + \frac{\theta_{12}}{z_{12}} f^{a\bar{b}}{}_c Q^c(Z_2) + \frac{\bar{\theta}_{12}}{z_{12}} f^{a\bar{b}}{}_{\bar{c}} Q^{\bar{c}}(Z_2) \\
&\quad + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}} \left[f^{a\bar{b}}{}_c \bar{D} Q^c(Z_2) + \frac{1}{k+\tilde{h}} f^{a\bar{c}}{}_d f^{\bar{b}}{}_{\bar{c}\bar{e}} Q^d Q^{\bar{e}}(Z_2) \right],
\end{aligned} \tag{13}$$

where

$$\theta_{12} = \theta_1 - \theta_2, \; \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2, \text{ and } z_{12} = z_1 - z_2 - \frac{1}{2}(\theta_1\bar{\theta}_2 + \bar{\theta}_1\theta_2). \tag{14}$$

Let us now focus on the $N=2$ superconformal algebra. The appropriate generalization to $N=2$ superspace of the well-known Sugawara construction gives the following formula for the $N=2$ super stress tensor in terms of the super Kac-Moody currents Q^a and $Q^{\bar{a}}$ ([14])

$$\mathcal{T} = \frac{i}{k+\tilde{h}} g_{a\bar{b}} (Q^a Q^{\bar{b}}) - \frac{1}{k+\tilde{h}} (f_{\bar{a}} D Q^{\bar{a}} + f_a \overline{D} Q^a), \tag{15}$$

where $g_{a\bar{b}} = \delta_{a\bar{b}}$, $f_a = g_{b\bar{c}} f_a{}^{b\bar{c}}$ and $f_{\bar{a}} = g_{b\bar{c}} f_{\bar{a}}{}^{b\bar{c}}$. It satisfies the OPE

$$\mathcal{T}(Z_1)\mathcal{T}(Z_2) = -\frac{1}{z_{12}^2}\frac{c_k}{3} - i\left[\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} + \frac{\theta_{12}}{z_{12}} D - \frac{\bar{\theta}_{12}}{z_{12}} \overline{D} + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}} \partial_2\right] \mathcal{T}(Z_2). \tag{16}$$

The total central charge is the sum of contributions $c_k' = \frac{3 \dim G}{2}(1 - \frac{2\tilde{h}}{3(k+\tilde{h})})$ for each simple factor of G. Of course, the $N=2$ superfield $\mathcal{T}$ has as its component fields the bosonic stress tensor T, two supercurrents G and $\overline{G}$ and the $U(1)$ current J, which together form the familiar $N=2$ current algebra.

In the example of $G = SU(2)\times U(1)$, the affine Kac-Moody currents Q^a and $Q^{\bar{a}}$ can be expressed in the coordinate fields Λ, $\overline{\Lambda}$, Φ, and $\overline{\Phi}$ as follows

$$
\begin{aligned}
Q^1 &= \frac{(k+2)}{r^2}(\Phi D\Lambda - \Lambda D\Phi) \\
Q^2 &= -\frac{i(k+2)}{r^2}(\overline{\Lambda} D\Lambda + \overline{\Phi} D\Phi) \\
Q^{\bar{1}} &= -\frac{(k+2)}{r^2}(\overline{\Phi D\Lambda} - \overline{\Lambda D\Phi}) \\
Q^{\bar{2}} &= -\frac{i(k+2)}{r^2}(\Lambda \overline{D\Lambda} + \Phi\overline{D\Phi})\,,
\end{aligned} \tag{17}
$$

where

$$
r^2 = \Lambda\overline{\Lambda} + \Phi\overline{\Phi}\,. \tag{18}
$$

Via the above they lead to an $N=2$ superconformal algebra of central charge $c_k = \frac{9}{2}(1-\frac{4}{3(k+2)}) + \frac{3}{2} = 6\,\frac{k+1}{k+2}$, which is $c=4$ for $k=1$ and approaches $c=6$ if $k\to\infty$.

The above makes manifest the $N=2$ superconformal symmetry of our model. However, we already mentioned that the model actually posseses a $N=4$ superconformal symmetry. The appropriate algebra is the so-called 'large' $N=4$ superconformal algebra [15]-[18]. This algebra has 16 generators, which are: the spin-2 stress tensor T, 4 spin-3/2 supercurrents G^i, 7 spin-1 currents generating the affine extension of $SU(2)\times SU(2)\times U(1)$ and 4 spin-1/2 currents Γ^i. The unitary representations of this algebra can be characterized by two integers k_+ and k_-, with a corresponding central charge equal to $c(k_+,k_-) = 6\,k_+k_-/(k_++k_-)$. The parameter $\alpha = \frac{1}{2}\frac{k_+-k_-}{k_++k_-}$ is a measure for the asymmetry between two affine $SU(2)$ subalgebras, which have level k_+ and k_-, respectively. The projective subalgebra is isomorphic to $D(2,1;\alpha-\frac{1}{2})$.

It was shown in [18, 19] that the level k $SU(2)\times U(1)$ WZW model gives a realization of this $N=4$ superconformal algebra with $k_+ = (k+1)$, $k_- = 1$. (For $k=0$ the bosonic $SU(2)$ WZW model decouples and this realization reduces to the $c=3$ realization with one free boson and four free fermions which was first discussed in [16]). We will now derive explicit formulas for the generators of the full $N=4$ algebra in terms of the fundamental superfields Λ and Φ of the model.

When written in (chiral) $N=2$ superspace, the full $N=4$ algebra is generated by (i) the super stress tensor $\mathcal{T}$, which has conformal spin 1, (ii) two spin-1/2 superfields $\mathcal{A}$ and $\mathcal{B}$ and (iii) a spin-0 superfield $\mathcal{I}$. Each of these provides four component fields, so that we find the correct total number of 16 currents.

To determine the extra currents $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{I}$, we will use the results of [11], where the relation between the affine currents and the $N=4$ superconformal algebra was worked out in detail in $N=1$ superspace. The explicit relation between the $N=1$ super Kac-Moody currents Q^a, $Q^{\bar{a}}$ in chiral $N=1$ superspace (z,θ^1) [2] and the $N=2$ super Kac-Moody currents is as follows (there are actually equal numbers of both since the $N=2$ affine currents are constrained, see (12)), [14]

$$\begin{aligned} \mathcal{Q}^a &= Q^a - i\theta^2(\nabla Q^a + \frac{1}{k+\tilde{h}} f^a{}_{bc} Q^b Q^c) \\ \mathcal{Q}^{\bar{a}} &= Q^{\bar{a}} + i\theta^2(\nabla Q^{\bar{a}} + \frac{1}{k+\tilde{h}} f^{\bar{a}}{}_{\bar{b}\bar{c}} Q^{\bar{b}} Q^{\bar{c}}). \end{aligned} \tag{19}$$

In here, θ^2 is the second fermionic coordinate. These relations together with the results in [11] make it possible to determine the extra currents. We find

$$\begin{aligned} D\mathcal{I} &= \frac{1}{\sqrt{1-4\alpha^2}}\frac{i}{k+2}\mathcal{Q}^2, \qquad & \overline{D}\mathcal{I} &= \frac{1}{\sqrt{1-4\alpha^2}}\frac{i}{k+2}\mathcal{Q}^{\bar{2}} \\ \mathcal{A} &= -\frac{1}{2}(\mathcal{Q}^1 - \mathcal{Q}^{\bar{1}}), \qquad & \mathcal{B} &= \frac{i}{2}(\mathcal{Q}^1 + \mathcal{Q}^{\bar{1}}), \end{aligned} \tag{20}$$

where $\alpha = \frac{k}{2(k+2)}$. Together with the expression for the stress energy tensor,

$$\mathcal{T} = \frac{i}{k+2}(\mathcal{Q}^1\mathcal{Q}^{\bar{1}} + \mathcal{Q}^2\mathcal{Q}^{\bar{2}}) - \frac{1}{k+2}(D\mathcal{Q}^{\bar{2}} - \overline{D}\mathcal{Q}^2), \tag{21}$$

these relations express all the generators of the $N=4$ superconformal algebra in terms of the $N=2$ affine Kac-Moody currents and thereby, through (17), in terms of the fundamental fields. Comparing with (17), we may conclude that $\mathcal{I}$ is given by

$$\mathcal{I} = \frac{1}{\sqrt{1-4\alpha^2}}\ln r^2 . \tag{22}$$

The OPE's of the currents $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{I}$ are given by (16) and

$$\mathcal{T}(Z_1)\mathcal{I}(Z_2) \;=\; -i\left[\frac{\theta_{12}}{z_{12}}D - \frac{\bar{\theta}_{12}}{z_{12}}\overline{D} + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\partial_2\right]\mathcal{I}(Z_2)$$

$$
\begin{aligned}
\mathcal{T}(Z_1)\mathcal{A}(Z_2) &= -i\left[\frac{\theta_{12}\bar\theta_{12}}{z_{12}^2}\frac{1}{2}+\frac{\theta_{12}}{z_{12}}D-\frac{\bar\theta_{12}}{z_{12}}\overline{D}+\frac{\theta_{12}\bar\theta_{12}}{z_{12}}\partial_2\right]\mathcal{A}(Z_2)-\frac{2\alpha}{z_{12}}\mathcal{B}(Z_2)\\
\mathcal{T}(Z_1)\mathcal{B}(Z_2) &= -i\left[\frac{\theta_{12}\bar\theta_{12}}{z_{12}^2}\frac{1}{2}+\frac{\theta_{12}}{z_{12}}D-\frac{\bar\theta_{12}}{z_{12}}\overline{D}+\frac{\theta_{12}\bar\theta_{12}}{z_{12}}\partial_2\right]\mathcal{B}(Z_2)+\frac{2\alpha}{z_{12}}\mathcal{A}(Z_2)\\
\mathcal{A}(Z_1)\mathcal{A}(Z_2) &= -\frac{c_1}{3}\frac{1}{z_{12}}-\frac{\theta_{12}\bar\theta_{12}}{z_{12}}\frac{i}{2}\left[\mathcal{T}(Z_2)+\frac{2\alpha i}{\sqrt{1-4\alpha^2}}[D,\overline{D}]\mathcal{I}(Z_2)\right]\\
\mathcal{B}(Z_1)\mathcal{B}(Z_2) &= -\frac{c_1}{3}\frac{1}{z_{12}}-\frac{\theta_{12}\bar\theta_{12}}{z_{12}}\frac{i}{2}\left[\mathcal{T}(Z_2)+\frac{2\alpha i}{\sqrt{1-4\alpha^2}}[D,\overline{D}]\mathcal{I}(Z_2)\right]\\
\mathcal{A}(Z_1)\mathcal{B}(Z_2) &= -\frac{c_2}{3}\frac{\theta_{12}\bar\theta_{12}}{z_{12}^2}\frac{i}{2}\\
&\quad -i\left[\frac{\theta_{12}}{z_{12}}D-\frac{\bar\theta_{12}}{z_{12}}\overline{D}+\frac{1}{2}\frac{\theta_{12}\bar\theta_{12}}{z_{12}}\partial_2\right]\frac{1}{\sqrt{1-4\alpha^2}}\mathcal{I}(Z_2)\\
\mathcal{I}(Z_1)\mathcal{A}(Z_2) &= \frac{\theta_{12}\bar\theta_{12}}{z_{12}}\frac{i}{2}\sqrt{1-4\alpha^2}\,\mathcal{B}(Z_2)\\
\mathcal{I}(Z_1)\mathcal{B}(Z_2) &= -\frac{\theta_{12}\bar\theta_{12}}{z_{12}}\frac{i}{2}\sqrt{1-4\alpha^2}\,\mathcal{A}(Z_2)\\
\mathcal{I}(Z_1)\mathcal{I}(Z_2) &= -\frac{c_k}{3}\ln z_{12}\,,
\end{aligned}
\tag{23}
$$

where $c_1=\frac{1}{1-4\alpha^2}c_k=\frac{3}{2}(k+2)$ and $c_2=\frac{2\alpha}{1-4\alpha^2}c_k=\frac{3}{2}k$.

We now return to the off-shell $N=2$ superspace action given in (11). This action has the form that admits a duality transformation [20, 5, 21]. From the general theory, we know that after the transformation, all the superfields will be chiral, and the manifold will therefore be Kähler (with vanishing torsion). We will now explicitly compute the metric of this manifold.

The first step of the duality transformation is to rewrite the action (11) in a first order form. The first order lagrangian depends on the chiral superfields Φ, $\overline{\Phi}$ and η and $\bar\eta$ and on the real quantity X. We define

$$
\begin{aligned}
\widetilde{K}_X(\Phi,\overline{\Phi},\eta,\bar\eta) &= -\int^{e^X}\frac{dx}{x}\ln(1+x)+\ln\Phi\ln\overline{\Phi}\\
&\quad +a\,[X+\ln(\Phi\overline{\Phi})](\eta+\bar\eta)\,,
\end{aligned}
\tag{24}
$$

where a is a constant $\neq 0$. When varying the first order action w.r.t. η and $\bar\eta$, we should keep in mind that these are *constrained* superfields. It can be shown that the most general expression for X that is compatible with the η, $\bar{\eta}$ field equations is

$$X = \ln(\Lambda\overline{\Lambda}) - \ln(\Phi\overline{\Phi})\,, \tag{25}$$

where Λ, $\overline{\Lambda}$ is a twisted chiral superfield. Substituting this back into the first-order action (24) one finds back the original action (11).

Let us now treat the first order action differently, and use the field equations of the field X instead of those of η, $\bar{\eta}$. They lead to

$$1 + e^X = e^{a\,(\eta+\bar{\eta})}\,. \tag{26}$$

We now define the following variables

$$\chi = e^{a\,\eta}, \quad \tilde{\Phi} = \ln\Phi + a\eta\,. \tag{27}$$

Notice that both χ and $\tilde{\Phi}$ are *chiral* $N = 2$ superfields. Substituting the above into the first order action (24), we arrive at the following second order action

$$\begin{aligned} \widetilde{K}(\tilde{\Phi},\bar{\tilde{\Phi}},\chi,\overline{\chi}) &= -\int^{\chi\overline{\chi}-1}\frac{dx}{x}\ln(1+x) + \ln(\chi\overline{\chi}-1)\ln(\chi\overline{\chi}) \\ &\quad -\frac{1}{2}(\ln(\chi\overline{\chi}))^2 + \tilde{\Phi}\bar{\tilde{\Phi}}\,. \end{aligned} \tag{28}$$

This lagrangian describes a theory which is dual to the original theory. Although both theories are equivalent at the level of the classical equations of motion, their geometric interpretation is very much different: the original WZW model describes a group manifold (with torsion), whereas the dual model describes a Kähler geometry (without torsion) with the Kähler potential given by (28).

The geometry associated with the dual model clearly splits as a product of a torus (with coordinates $\tilde{\Phi}$ and $\bar{\tilde{\Phi}}$) and a disk bearing the singular metric

$$ds^2 = \widetilde{K}_{\chi\overline{\chi}}\,d\chi\,d\overline{\chi} = \frac{d\chi d\overline{\chi}}{\chi\overline{\chi}(\chi\overline{\chi}-1)}\,. \tag{29}$$

In terms of the coordinates $u = 1/\chi$ this metric takes the simple form

$$ds^2 = \frac{du\,d\overline{u}}{1-u\overline{u}}\,. \tag{30}$$

If one follows the process of passing from the original to the dual formulation at the level of the functional integral, one finds that, apart from the change of metric, the transition leads to a non-vanishing dilaton field in the dual formulation (see [21] for a careful discussion). In our case the dilaton field is given by

$$\phi = \ln(1 - u\bar{u}) . \tag{31}$$

It can be observed that the above combination of metric and dilaton fields is such that the sigma model is conformally invariant. Due to this, this geometry can serve as a consistent background for a string theory with a two-dimensional target space-time. This observation has been worked out by Witten [22], who proposed the interpretation of this geometry as a back hole solution to $D = 2$ string theory. We would like to remark that our derivation of this geometry (through a duality transformation in $N = 2$ superspace) is similar to, although independent from, Witten's derivation, which is based on a gauging a $U(1)$ subgroup in the $SU(2)$ WZW model. (The relation has recently been clarified in [23].)

It would be interesting to work out the duality transformation at the quantum level. The central charge c_k can be written as $c_k = 3 + \frac{3k}{k+2}$, where the $c = 3$ part corresponds to the free fields $\tilde{\Phi}$, $\tilde{\tilde{\Phi}}$ and the remaining part describes the interacting sigma-model with metric (30). The fate of the $N = 4$ superconformal symmetry in the dual model is not yet clear. On first inspection, one finds that the duality breaks the $N = 4$ supersymmetry, but one still expects that some remnant of it could survive and might have some interesting applications in the (super)string theory interpretation of the model. We leave these issues for further study.

References

[1] E. Witten, Comm. Math. Phys. **92** (1984) 455.

[2] P. di Vecchia, V.G. Knizhnik, J.L. Petersen and P. Rossi, Nucl. Phys. **B253** (1985) 701.

[3] Ph. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, Phys. Lett. **206B** (1988) 71; Nucl. Phys. **B308** (1988) 662.

[4] B. Zumino, Phys. Lett. **87B** (1979) 203.

[5] S.J. Gates, C.M. Hull and M. Roček, Nucl. Phys. **B248** (1984) 157.

[6] E.A. Ivanov and S.O. Krivonos, J.Phys. A: Math. and Gen. **17** (1984) L671, E.A. Ivanov, S.O. Krivonos, and V.M. Leviant, Nucl. Phys. **304** (1988) 601, O. Gorovy and E. Ivanov, "Superfield Actions for N=4 WZNW-Liouville Systems", Dubna preprint (1991).

[7] M. Roček, K. Schoutens and A. Sevrin, Phys. Lett. **265B** (1991) 303.

[8] L. Alvarez-Gaumé and D. Freedman, Comm. Math. Phys. **80** (1981) 443.

[9] J.A. Wolf, J. Math. Mech. **14** (1965) 1033;
D.V. Alekseevskii, Math. USSR Izv. **9** (1975) 297.

[10] A. Van Proeyen, Class. Quant. Grav. **6** (1989) 1501.

[11] A. Sevrin and G. Theodoridis, Nucl. Phys. **B332** (1990) 380.

[12] T. Buscher, U. Lindström and M. Roček, Phys. Lett. **202B** (1988) 94.

[13] C. Callan, J. Harvey and A. Strominger, "Worldsheet Approach to Heterotic Instantons and Solitons", Princeton preprint PUPT-1233.

[14] C.M. Hull and B. Spence, Phys. Lett. **241B** (1990) 357.

[15] M. Ademollo *et al.* Phys. Lett. **62B** (1976) 75.

[16] K. Schoutens, Phys. Lett. **194B** (1987) 75.

[17] K. Schoutens, Nucl. Phys. **B295** (1988) 634.

[18] A. Sevrin, W. Troost, and A. Van Proeyen, Phys. Lett. **208B** (1988) 447.

[19] A. Sevrin, W. Troost, A. Van Proeyen and Ph. Spindel, Nucl. Phys. **311B** (1988/1989) 465.

[20] U. Lindström and M. Roček, Nucl. Phys. **B222** (1983) 285.

[21] T. Buscher, Phys. Lett. **159B** (1985) 127, **194B** (1987) 59, **201B** (1988) 466.

[22] E. Witten, Phys. Rev. **D44** (1991) 314.

[23] M. Roček and E. Verlinde, "Duality, Quotients, and Currents", IAS preprint IASSNS-HEP-91/68.

END

DATE FILMED

01 /27/92