Mechanics of Layered Anisotropic Poroelastic Media with Applications to Effective Stress for Fluid Permeability

James G. Berryman[1,*]

[1] *University of California, Lawrence Berkeley National Laboratory, One Cyclotron Road MS 90R1116, Berkeley, CA 94720, USA*

Abstract

The mechanics of vertically layered porous media has some similarities to and some differences from the more typical layered analysis for purely elastic media. Assuming welded solid contact at the solid-solid interfaces implies the usual continuity conditions, which are continuity of the vertical (layering direction) stress components and the horizontal strain components . These conditions are valid for both elastic and poroelastic media. Differences arise through the conditions for the pore pressure and the increment of fluid content in the context of fluid-saturated porous media. The two distinct conditions most often considered between any pair of contiguous layers are: (1) an undrained fluid condition at the interface, meaning that the increment of fluid content is zero (*i.e.*, $\delta\zeta = 0$), or (2) fluid pressure continuity at the interface, implying that the change in fluid pressure is zero across the interface (*i.e.*, $\delta p_f = 0$). Depending on the types of measurements being made on the system and the pertinent boundary conditions for these measurements, either (or neither) of these two conditions might be directly pertinent. But these conditions are sufficient nevertheless to be used as thought experiments to determine the expected values of all the poroelastic coefficients. For quasi-static mechanical changes over long time periods, we expect drained conditions to hold, so the pressure must then be continuous. For high frequency wave propagation, the pore-fluid typically acts as if it were undrained (or very nearly so), with vanishing of the fluid increment at the boundaries being appropriate. Poroelastic analysis of both these end-member cases is discussed, and the general equations for a variety of applications to heterogeneous porous media are developed. In particular, effective stress for the fluid permeability of such poroelastic systems is considered; fluid permeabilities characteristic of granular media or tubular pore shapes are treated in some detail, as are permeabilities of some of the simpler types of fractured materials.

* JGBerryman@LBL.GOV

1. Introduction

Studies of elastic systems containing holes, fractures, or pores possibly filled with fluids typically employ a single step of upscaling, considering the otherwise homogeneous solid and fluid constituents' properties, volume fractions, and spatial arrangements, and subsequently deducing an estimate of the likely macroscopic behavior of such systems [1-8]. In recent work [9], the author has shown in some detail how the poroelastic coefficients are related to the microstructural constants of the solid constituents when the overall behavior varies from isotropic to orthotropic. The focus of the present effort is on layered poroelastic materials, which clearly involves a second upscaling step needed to homogenize the macroscale system. The resulting theory again is anisotropic, in part due to the anisotropy of individual layers, and in part due to the layering procedure itself. The scale of interaction of the probes used on such media is important, and can be either very large scale (as might happen with low frequency seismic waves), or very fast (as might happen with ultrasonic waves). In this study, each layer is assumed to satisfy the assumptions of the class of problems considered in reference [9], and therefore the system is heterogeneous at the *poroelastic mesoscale*, as we move from layer to layer. Then, the layer-averaging step takes us to a *macroscale* level of analysis that simultaneously includes (via this effective-medium-style averaging) many of these poroelastic layers whose local properties may differ substantially from one to another.

One main issue addressed here concerns how the interface boundary conditions between anisotropic porous layers should be treated. For very low frequency (say quasi-static) analysis, this issue is clear since then the boundary conditions must be drained conditions and therefore the fluid pressure is continuous across the boundary. However, for high frequency wave propagation, it is expected to be more appropriate to treat the system as locally undrained (pore-fluid is trapped), since the pressure of the pore-fluid does not have time to equilibrate with the outside world via the drainage mechanism, and also since this can take much longer than is appropriate to these quasi-static analyses. The most accurate way to treat these situations in general is to consider the variables to be frequency dependent and complex (therefore viscoelastic and including dissipation mechanisms). This approach has been taken for example by Pride *et al.* [10-12] for some simple mixtures of isotropic poroelastic materials. But these problems become harder for the anisotropic case because – although there were simple exact results for the two-isotropic-component case – such simple results are not generally available for the anisotropic problems. And, more importantly, the interest in layered media is not just for two-component examples, but ultimately for multi-component layered media. So it is necessary for many applications of current practical interest to consider these cases separately, as is being done here.

This analysis is restricted to anisotropic systems composed of locally anisotropic poroelastic layers. The nature of the grains themselves composing the solid frame material of the porous medium will not be a focus of the present paper. This issue does matter too, but (as will be shown) it is most important for determining the relationship between the grain constants and the off-diagonal coefficients that are called the β's in this formulation. These issues have been fully addressed in an earlier contribution of the author [9], and will therefore not be treated again in such detail here. Our focus instead is on heterogeneous poroelastic media when the heterogeneity is well-represented via layered porous-medium modeling.

2. Basics of Anisotropic Poroelasticity

2.1 Orthotropic poroelasticity

If the overall porous medium is anisotropic — either due to some preferential alignment of the constituent particles or due to externally imposed stress (such as a gravity field and weight of overburden, for example) — we consider the orthorhombic anisotropic version of the poroelastic equations:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & -\beta_1 \\ s_{12} & s_{22} & s_{23} & -\beta_2 \\ s_{13} & s_{23} & s_{33} & -\beta_3 \\ -\beta_1 & -\beta_2 & -\beta_3 & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \quad (1)$$

(Note: Throughout most of the paper, the δ's that are sometimes used to emphasize smallness of the stresses and strains will be suppressed, as this extra notation is truly redundant when they are all being treated as small (and therefore resulting in linear effects) as we do here, for small deviations from an initial rest state. Nevertheless, we will occasionally revert to this notation when it seems important to emphasize smallness of certain variables, and/or linearity of the system being studied.)

The e_{ii} (no summation over repeated indices) are strains in the $i = 1, 2, 3$ directions. The

σ_{ii} are the corresponding stresses, assumed to be positive in tension. The fluid pressure is p_f, which is positive in compression. The increment of fluid content is ζ, and is often defined via:

$$\zeta \equiv \frac{\delta(\phi V) - \delta V_f}{V} \simeq \phi \left(\frac{\delta V_\phi}{V_\phi} - \frac{\delta V_f}{V_f} \right), \tag{2}$$

where $V = V_\phi/\phi \simeq V_f/\phi$ is the pertinent local volume (within a layer in present circumstances) of the initially fully fluid-saturated porous layer at the first instant of consideration, $V_\phi = \phi V$ is the corresponding pore volume, with ϕ being the fluid-saturated porosity of the same volume. V_f is the volume occupied by the pore-fluid, so that $V_f = \phi V$ before any new deformations begin. (As mentioned already, δ's indicate small changes in the quantities immediately following them.) For "drained" systems, there would ideally be a reservoir of the same fluid just outside the volume V that can either supply more fluid or absorb any excreted fluid as needed during the nonstationary phase of the poroelastic process. The amount of pore fluid (*i.e.*, the number of fluid molecules) can therefore either increase or decrease from that of the initial amount of pore fluid; at the same time, the pore volume can also be changing, but — in general — not necessarily at exactly the same rate as the pore fluid itself. The one exception to these statements is when the surface pores of the layer volume V are sealed, in which case the layer is "undrained" and $\zeta \equiv 0$, identically. In such circumstances, it is still possible that both V_f and $V_\phi = \phi V$ are changing; but, because of the imposed undrained boundary conditions, they are necessarily changing at the same rate. The drained compliances are $s_{ij} = s_{ij}^d$, with or without the d superscript.

Undrained compliances (not yet shown) are symbolized by s_{ij}^u.

Coefficients

$$\beta_i = s_{i1} + s_{i2} + s_{i3} - 1/3K_R^g, \tag{3}$$

where K_R^g is again the Reuss average modulus of the grains. The drained Reuss average bulk modulus is defined by

$$\frac{1}{K_R^d} = \sum_{ij=1,2,3} s_{ij}^d. \tag{4}$$

For the Reuss average [13] undrained bulk modulus K_R^u, we have drained compliances replaced by undrained compliances in a formula analogous to (4). A similar definition of the effective grain modulus K_R^g is:

$$\frac{1}{K_R^g} = \sum_{i,j=1,2,3} s_{ij}^g. \tag{5}$$

with grain compliances replacing drained compliances as discussed earlier by Berryman [9]. The alternative Voigt [14] average (also see [15]) of the stiffnesses plays no role in the present work. And, finally, $\gamma = \sum_{i=1,2,3} \beta_i / BK_R^d$, where B is the second Skempton [16] coefficient, which will be defined carefully later.

The shear terms due to twisting motions (*i.e.*, strains e_{23}, e_{31}, e_{12} and stresses σ_{23}, σ_{31}, σ_{12}) are excluded from this poroelastic discussion since they typically do not couple to the modes of interest for anisotropic systems having orthotropic symmetry, or any more symmetric system such as those being either transversely isotropic (*i.e.*, hexagonal) or isotropic. We have also assumed that we know the true axes of symmetry, and make use of them in our formulation of the problem. Note that the s_{ij}'s are the elements of the compliance matrix $\mathbf{S}$ and are all independent of the fluid, and therefore would be the same if the medium were treated as elastic (*i.e.*, by ignoring the fluid pressure, or assuming that the fluid saturant is air – or vacuum). In keeping with the earlier discussions, we typically call these compliances the drained compliances and the corresponding matrix the drained compliance matrix $\mathbf{S}^d$. The fluids do not contribute to the stored mechanical energy if they are free to drain into a surrounding reservoir containing the same type of fluid. In contrast, the undrained compliance matrix $\mathbf{S}^u$ presupposes that the fluid is trapped (unable to drain from the system into an adjacent reservoir) and therefore contributes in a significant and measurable way to the compliance and stiffness ($\mathbf{C}^u = [\mathbf{S}^u]^{-1}$), as well as to the stored mechanical energy of the undrained system.

Although the significance of the formula is somewhat different now, we find again that

$$\beta_1 + \beta_2 + \beta_3 = \frac{1}{K_R^d} - \frac{1}{K_R^g} = \frac{\alpha_R}{K_R^d} \tag{6}$$

if we also define (as we did for the isotropic case) a Reuss effective stress coefficient:

$$\alpha_R \equiv 1 - K_R^d / K_R^g. \tag{7}$$

Furthermore, we have

$$\gamma = \frac{\beta_1 + \beta_2 + \beta_3}{B} = \frac{\alpha_R}{K_R^d} + \phi \left(\frac{1}{K_f} - \frac{1}{K_R^\phi} \right), \tag{8}$$

since we have the rigorous result in this notation [3,16] that Skempton's B coefficient is given by

$$B \equiv \frac{1 - K_R^d / K_R^u}{1 - K_R^d / K_R^g} = \frac{\alpha_R / K_R^d}{\alpha_R / K_R^d + \phi(1/K_f - 1/K_R^\phi)}. \tag{9}$$

Note that both (8) and (9) contain dependence on the distinct pore bulk modulus K_R^ϕ that comes into play when the pores are heterogeneous [3], regardless of whether the system is isotropic or anisotropic. We emphasize that all these formulas are rigorous statements based on the earlier anisotropic analyses. The appearance of both the Reuss average quantities K_R^d and α_R is not an approximation, but merely a choice of notation made to provide both emphasis and some additional clarity.

2.2 Determining off-diagonal coefficients β_i

We will now provide several results for the β_i coefficients, and then follow the results with a general proof of their correctness.

In many useful and important cases, the coefficients β_i are determined by

$$\beta_i = s_{i1}^d + s_{i2}^d + s_{i3}^d - \frac{1}{3K_R^g}. \tag{10}$$

Again, K_R^g is the Reuss average of the grain modulus, since the local grain modulus is not necessarily assumed uniform here as discussed previously. Equation (10) holds as written for homogeneous grains, such that $K_R^g = K^g$.

It also holds true for the case when K_R^g is determined instead [19] by an average over isotropic grains such that

$$\frac{1}{K_R^g} \equiv \sum_{m=1,\ldots,n} \frac{v_m}{K_m}, \tag{11}$$

where v_m is the volume fraction (out of all the solid material present, so that $\sum_m v_m = 1$). However, when the grains themselves are anisotropic, we need to allow again for this possibility, and this can be accomplished by defining three directional grain bulk moduli determined by:

$$\frac{1}{3\overline{K}_i^g} \equiv s_{i1}^g + s_{i2}^g + s_{i3}^g = s_{1i}^g + s_{2i}^g + s_{3i}^g, \tag{12}$$

for $i = 1, 2, 3$. The second equality follows because the compliance matrix is always symmetric. We call these quantities in (12) the "partial grain-compliance sums," and the $\overline{K}_i^g$ are the directional grain bulk moduli. Note that the factors of three have again been correctly accounted for because

$$\sum_{i=1,2,3} \frac{1}{3\overline{K}_i^g} = \frac{1}{K_R^g}, \tag{13}$$

in agreement with (5).

We can further simplify and symmetrize our notation somewhat by introducing a similar concept for the drained constants, so that

$$\frac{1}{3\overline{K}_i^d} \equiv s_{i1}^d + s_{i2}^d + s_{i3}^d = s_{1i}^d + s_{2i}^d + s_{3i}^d, \tag{14}$$

for $i = 1, 2, 3$. Then, the formula for (10) is replaced by

$$\beta_i = \frac{1}{3\overline{K}_i^d} - \frac{1}{3\overline{K}_i^g}. \tag{15}$$

If the three contributions represented by (12) for $i = 1, 2, 3$ happen to be equal, then clearly each equals one-third of the sum (13).

The preceding results are for perfectly aligned grains. If the grains are instead perfectly randomly oriented, then it is clear that the formulas in (10) hold as before, but now K_R^g is determined instead by (5).

All of these statements about the β_i's are easily proven by considering the simple experimental situation when $\sigma_{11} = \sigma_{22} = \sigma_{33} \equiv -p_c = -p_f$. Because then, from (1), we have (no i summation convention):

$$-e_{ii} = \frac{1}{3\overline{K}_i^d} p_c + \beta_i(-p_f) = (s_{i1}^g + s_{i2}^g + s_{i3}^g)\, p_f \equiv \frac{p_f}{3\overline{K}_i^g}, \tag{16}$$

in the most general of the three cases discussed, and holding true for each value of $i = 1, 2, 3$. This result is a statement about the strain e_{ii} that would be observed in this situation, as it must be the same if these anisotropic (or inhomogeneous) grains were immersed in the fluid, while measurements were taken of the strains observed in each of the three directions $i = 1, 2, 3$, during variations of the fluid pressure p_f. We may consider this proof to be a thought experiment for determining these coefficients, in the same spirit as those proposed originally by Biot and Willis [2,17] for the isotropic and homogeneous case.

2.3 The β_i coefficients and effective stress

Making use of our previous definitions, it is easy to see that the coefficients β_i are closely related to a different sort of effective stress coefficient, for the individual principal strain coefficients:

$$e_{ii} = -\frac{1}{3\overline{K}_i^d}(p_c - D_i p_f), \qquad \text{for} \qquad i = 1, 2, 3, \tag{17}$$

where

$$D_i = 3\overline{K}_i^d \beta_i = 1 - \frac{\overline{K}_i^d}{\overline{K}_i^g}, \qquad \text{for} \qquad i = 1, 2, 3, \tag{18}$$

and $-p_c = \sigma_{11} = \sigma_{22} = \sigma_{33}$ in the case of uniform applied confining pressure p_c. Then clearly, the D_i's are completely analogous to the usual Biot (or Biot-Willis [2,17]) coefficient $\alpha_R = 1 - K_R^d/K_R^g$ commonly defined for isotropic poroelasticity.

2.4 Coefficient γ

The relationship of coefficient γ to the other coefficients is easily established because we have already discussed the main issue, which involves determining the role of the various other constants contained in Skempton's coefficient B [16]. This result is

$$B = \left(\frac{1}{K_R^d} - \frac{1}{K_R^g}\right)\left[\left(\frac{1}{K_R^d} - \frac{1}{K_R^g}\right) + \phi\left(\frac{1}{K_f} - \frac{1}{K_R^\phi}\right)\right]^{-1} \tag{19}$$

Again, from (1), we find that

$$-\zeta = 0 = -\left(\beta_1 + \beta_2 + \beta_3\right)\sigma_c - \gamma p_f, \tag{20}$$

for undrained boundary conditions. Thus, we find that

$$\frac{p_f}{p_c} \equiv B = \frac{\beta_1 + \beta_2 + \beta_3}{\gamma}, \tag{21}$$

where $p_c = -\sigma_c$ is the confining pressure. Therefore, the scalar coefficient γ is determined immediately and given by

$$\gamma = \frac{\beta_1 + \beta_2 + \beta_3}{B} = \frac{\alpha_R/K_R^d}{B} = \alpha_R/K_R^d + \phi\left(\frac{1}{K_f} - \frac{1}{K_R^\phi}\right). \tag{22}$$

Alternatively, we could say that

$$B = \frac{\alpha_R}{\gamma K_R^d}, \tag{23}$$

which gives a definition of Skempton B in terms of other constants.

We have now determined the physical/mechanical significance of all the coefficients in the poroelastic matrix (1). These results are as general as possible without considering poroelastic symmetries that have less than orthotropic symmetry, while also taking advantage of our assumption that we do typically know the three directions of the principal axes of symmetry.

2.5 Inverting poroelastic compliance

Being in compliance form, the matrix in (1) has extremely simple poroelastic behavior in the sense that all the fluid mechanical effects appear only in the single coefficient γ. We can simplify the notation a little more by lumping some coefficients together, combining the 3×3 submatrix in the upper left corner of the matrix in (1) as $\mathbf{S}$, and defining the column vector $\mathbf{b}$ by

$$\mathbf{b}^T \equiv (\beta_1, \beta_2, \beta_3). \tag{24}$$

The resulting 4×4 matrix and its inverse are now related by:

$$\begin{pmatrix} \mathbf{S} & -\mathbf{b} \\ -\mathbf{b}^T & \gamma \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{q}^T & z \end{pmatrix}^{-1}, \tag{25}$$

where the elements of the inverse matrix can be shown to be written in terms of drained stiffness matrix $\mathbf{C}^d = \mathbf{C} = \mathbf{S}^{-1}$ by introducing three components: (a) scalar $z = \left[\gamma - \mathbf{b}^T\mathbf{C}\mathbf{b}\right]^{-1}$, (b) column vector $\mathbf{q} = z\mathbf{C}\mathbf{b}$, and (c) undrained 3×3 stiffness matrix (*i.e.*, the pertinent one connecting the principal strains to principal stresses) is given by $\mathbf{A} = \mathbf{C} + z\mathbf{C}\mathbf{b}\mathbf{b}^T\mathbf{C} = \mathbf{C}^d + z^{-1}\mathbf{q}\mathbf{q}^T \equiv \mathbf{C}^u$, since $\mathbf{C}^d$ is drained stiffness and $\mathbf{A} = \mathbf{C}^u$ is clearly undrained stiffness by construction. This result is the same as that of Gassmann [1] for anisotropic porous media.

Note the important fact that the observed decoupling of the fluid effects occurs only in the compliance form (1) of the equations, and never in the stiffness (inverse) form for the poroelasticity equations.

From these results, it is not hard to show that

$$\mathbf{S}^d = \mathbf{S}^u + \gamma^{-1}\mathbf{b}\mathbf{b}^T. \tag{26}$$

This result emphasizes the fact that the drained compliance matrix can be found directly from knowledge of the inverse of undrained stiffness, and the still unknown, but sometimes relatively easy to estimate, values of γ, and the three distinct orthotropic β_i coefficients, for $i = 1, 2, 3$.

There are clearly many measurements required to determine all these various poroelastic coefficients. Furthermore, the strategy for finding the coefficients depends on available data sets, and whether the porous media of interest are constructed from a homogeneous or heterogeneous set of solid materials, and whether the individual grains are isotropic or anisotropic. It also makes some difference if the pores are approximately round (for granular media) or flat (for fractured media). All these issues have been discussed previously at some length [9], and this discussion will not be repeated here.

The remainder of the paper will concentrate on making use of these general poroelastic equations in situations where at least two and possibly many distinct layers of porous materials (individually obeying the equations of this type) are under stress (either quasi-static or dynamic as would occur in a wave propagation scenario). As we shall see, the layered poroelastic equations behave somewhat differently from layered elastic equations because there are two distinct additional pore-fluid boundary conditions (drained and undrained) that can occur depending on the details of the excitation itself.

3. Layer-Averaging Results for All Drained or All Undrained Boundaries

The two most common boundary conditions to consider in poroelastic media are the drained and undrained conditions. Drained conditions imply that the fluid pressure change is zero, while the increment of fluid content in the individual layers may be considered arbitrary. Of course, the total amount of fluid present needs to be properly conserved in the analysis we present, but the usual idea for drained conditions is that the poroelastic system is immersed in an infinite reservoir of like-fluid so that pore fluid is free to move in and out of the region of interest. For our present considerations, this situation implies that the layer increments ζ can take arbitrary (but typically small) values, but the fluid pressure is constrained to be a constant value p_f everywhere. So changes in p_f always vanish for drained conditions.

Undrained boundary conditions place the hard constraint on the fluid increment ζ, requiring no flow at the boundaries, so $\zeta = 0$ at all boundaries. These conditions ensure that the fluid pressure p_f does change, since as the boundaries move in or out the pressure on the confined fluid is increasing or decreasing.

Both of these conditions must be approximations to conditions in a generally realistic earth model. We can easily imagine situations where some boundaries between layers (the vertical direction) are plugged, so undrained boundary conditions $\zeta_z \equiv 0$ might be correct while neighboring layers (horizontal direction) might be open to fluid flow (so perhaps ζ_x and/or $\zeta_y \neq 0$ at the x and y boundaries). We will neverthless limit the analysis to that for either *all drained* conditions or *all undrained* conditions. All undrained conditions are also appropriate, as mentioned previously, regardless of the physical boundary conditions if the probe changing the physical variables is a passing high frequency acoustic or seismic wave train or pulse.

3.1 General analysis for layered poroelastic systems

We will now formulate the layered porelastic earth problem in a way so that both of these standard boundary conditions can be imposed, as needed in any particular modeling problem.

We assume throughout the remainder of the paper that the porous layers are stacked vertically (along the 3- or z-axis), and for this geometry it is easy to see that the three horizontal strains e_{11}, e_{22}, and e_{12} must be continuous if the layers are in solid-welded contact. Furthermore, the vertical stress σ_{33}, and rotational stresses involving the vertical direction σ_{13} and σ_{23} must also be continuous. These conditions follow from an assumption of welded solid-solid contact between layers. If contact is not welded, then the system can have much more complicated behaviors than we are considering here.

Appendix A summarizes the Backus [20] and/or Schoenberg-Muir [21] approach to elastic layer averaging. The method we present here is a slight generalization of this approach, taking the presence of the pore fluid into account. For the drained situation, the influence of the fluid on the system mechanics is minimal (as we shall see). But we should nevertheless have this result available to compare it with the more interesting case of the undrained layers.

Although the shear moduli normally associated with the twisting shear components e_{23}, e_{31}, and e_{12} usually do not interact with the pore-fluid itself in systems as symmetric or more symmetric than orthotropic, we nevertheless need to carry these terms along in the poroelastic formulation for layered systems because of possible boundary effects due to welded contact at interfaces. To accomplish this goal, we will generalize the form of equation (79) from Appendix A. In compliance form, the equations will relate the strains

$$E_T \equiv \begin{pmatrix} e_{11} \\ e_{22} \\ e_{12} \end{pmatrix}, \quad \text{and} \quad E_N \equiv \begin{pmatrix} e_{33} \\ e_{32} \\ e_{31} \end{pmatrix}, \tag{27}$$

and fluid increment ζ to the stresses

$$\Pi_T \equiv \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \text{and} \quad \Pi_N \equiv \begin{pmatrix} \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \end{pmatrix}, \tag{28}$$

and the fluid pressure change p_f.

The required general relationship is:

$$\begin{pmatrix} E_T \\ -\zeta \\ E_N \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{TT} & -\mathbf{g}_{12} & \mathbf{S}_{TN} \\ -\mathbf{g}_{12}^T & \gamma & -\mathbf{g}_3^T \\ \mathbf{S}_{NT} & -\mathbf{g}_3 & \mathbf{S}_{NN} \end{pmatrix} \begin{pmatrix} \Pi_T \\ -p_f \\ \Pi_N \end{pmatrix}, \tag{29}$$

where, for example, in the orthotropic media considered here we have

$$\mathbf{S}_{TT} \equiv \begin{pmatrix} s_{11} & s_{12} & s_{16} \\ s_{21} & s_{22} & s_{26} \\ s_{61} & s_{62} & s_{66} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \\ s_{21} & s_{22} & \\ & & s_{66} \end{pmatrix}, \tag{30}$$

$$\mathbf{S}_{NN} \equiv \begin{pmatrix} s_{33} & s_{34} & s_{35} \\ s_{43} & s_{44} & s_{45} \\ s_{53} & s_{54} & s_{55} \end{pmatrix} = \begin{pmatrix} s_{33} & & \\ & s_{44} & \\ & & s_{55} \end{pmatrix}, \tag{31}$$

and

$$\mathbf{S}_{NT} \equiv \begin{pmatrix} s_{31} & s_{32} & s_{36} \\ s_{41} & s_{42} & s_{46} \\ s_{51} & s_{52} & s_{56} \end{pmatrix} = \begin{pmatrix} s_{31} & s_{32} & \\ & 0 & \\ & & 0 \end{pmatrix}, \tag{32}$$

with $\mathbf{S}_{TN} = \mathbf{S}_{NT}^T$ (the T superscript indicates the matrix transpose). Here all these expressions for elastic compliance refer specifically to drained compliances $s_{ij} = s_{ij}^d$, for all $i, j = 1, \ldots, 6$ within each poroelastic anisotropic layer.

All the poroelastic contributions to (29) are determined by γ, $\mathbf{g}_{12}$, and $\mathbf{g}_3$. The scalar γ within the 7×7 matrix in (29) was defined earlier in (8), and is the only term in the 7×7 matrix that includes fluid effects directly through fluid bulk modulus K_f. The remaining pair of vectors contained within the 7×7 matrix in (29) is defined by:

$$\mathbf{g}_{12}^T = (\beta_1, \beta_2, 0) \tag{33}$$

and

$$\mathbf{g}_3^T = (\beta_3, 0, 0), \tag{34}$$

where the β's were defined previously following (1).

We now consider two examples of special uses of the general equation (29) for different choices of boundary conditions. These two physical circumstances being considered are distinct end-members. For relatively high-frequency wave propagation, it is appropriate to consider that the fluids do not have time to equilibrate during the time of wave passage. Therefore fluid pressures can be different in distinct layers. The fluid particles do not have time to move very far during wave passage time, so the fluid increment is also $\zeta = 0$ essentially everywhere. This situation is called the "undrained" condition. An alternative condition considers the fully drained condition, in which the fluid particles have as much time as they need to achieve fluid-pressure equilibration, so that p_f = constant at longer times. These two limiting situations may be connected physically via Darcy's law, which provides the mechanism to move fluid particles, and ultimately to guarantee that the fluid pressure reaches an equilibrium state. Bringing Darcy's law actively into play in the equations would result in Biot-style equations which are beyond our current quasi-static scope. So we limit further discussion to these end-member conditions.

3.2 Drained scenario ($p_f \equiv 0$)

Now, recall that, in the drained scenario, changes in pore-fluid pressure are assumed to be zero (or at least negligibly small), so $p_f \equiv 0$ in these equations. Accounting for this condition, the results should (and do) recover the Backus [20] and Schoenberg-Muir [21] results for the elastic parts of the system (found in Appendix A) exactly. Also, we find the additional (expected) result for the poroelastic case that the average fluid increment is:

$$\langle \zeta \rangle = \langle \beta_1 \sigma_{11} \rangle + \langle \beta_2 \sigma_{22} \rangle + \langle \beta_3 \rangle \sigma_{33}, \tag{35}$$

if σ_{33} is nearly constant. The result shown in (35) is easy to reconcile with the definitions of the β's, and the meaning of averaging operator $\langle \cdot \rangle$ across all layers. When p_f vanishes everywhere, the final results for the averaging and the various stresses and strains are identical to the results in Appendix A. For the drained scenario, the only difference is the addition of equation (35).

3.3 Undrained scenario ($\zeta \equiv 0$)

Now consider that the fluid pressure might vary across the stack of layers (as should be expected to happen either because of hydrostatic overburden, or due to fluid injection or extraction at certain chosen depths). Then we can treat this case as well, assuming undrained circumstances, by averaging the fluid pressure itself via $\langle p_f \rangle$. For this undrained scenario, the fluid pressure in each undrained layer is free to vary compared to all the others; so there is no constancy of p_f across layers. The averaging condition resulting from the formulation for such a reservoir according to (29) is:

$$\langle p_f \rangle = -\left\langle \frac{1}{\gamma} (\beta_1 \sigma_{11} + \beta_2 \sigma_{22}) \right\rangle + \left\langle \frac{\beta_3 \sigma_{33}}{\gamma} \right\rangle. \tag{36}$$

Proper choice of the range of depth for averaging will clearly depend on the details of each reservoir, and the type of physical probe being used. For example, either quarter- or half-wavelength for seismic waves (otherwise the average over a full cycle is always zero for wave propagation problems), when used as the probe, would be typical choices of the averaging depth in this case.

While the preceding part of the averaging for undrained boundary conditions was straightforward, we still need to check what happens when averaging the remainder of the equations. We show the work in Appendix B leading to the general undrained result (98), and just quote the final result here – being valid for each undrained layer in the overall system:

$$\begin{pmatrix} E_T \\ E_N \end{pmatrix} = \begin{pmatrix} \mathbf{S}^u_{TT} & \mathbf{S}^u_{TN} \\ \mathbf{S}^u_{NT} & \mathbf{S}^u_{NN} \end{pmatrix} \begin{pmatrix} \Pi_T \\ \Pi_N \end{pmatrix}, \tag{37}$$

where

$$\mathbf{S}^u_{TT} \equiv \begin{pmatrix} s^u_{11} & s^u_{12} & \\ s^u_{21} & s^u_{22} & \\ & & s_{66} \end{pmatrix}, \tag{38}$$

$$\mathbf{S}^u_{NN} \equiv \begin{pmatrix} s^u_{33} & & \\ & s_{44} & \\ & & s_{55} \end{pmatrix}, \tag{39}$$

and

$$\mathbf{S}^u_{NT} \equiv \begin{pmatrix} s^u_{31} & s^u_{32} & \\ & 0 & \\ & & 0 \end{pmatrix}, \tag{40}$$

while $\mathbf{S}^u_{TN} = (\mathbf{S}^u_{NT})^T$. Once these definitions are used for the undrained matrices, the layer analysis for the system follows exactly the same steps as in Appendix A. Note that we arrived at these results in another (step-by-step) way in Appendix B independently in order to verify that this is the right answer for the undrained problem. Fortunately, the right answer is also the same as the intuitive answer.

4. Application to Effective Stress for Fluid Permeability of Granular Systems or Tubular Pores

Following Reference [24], Darcy's constant k for the fluid permeability has dimensions of length squared, so a uniform shrinking or swelling of an isotropic porous medium changes the value of the isotropic permeability by a factor proportional to $V^{2/3}$ (volume to the two-thirds power, since volume has dimensions of length cubed). For anisotropic permeability of the orthotropic porous media under consideration, we need to make some assumptions about the strain dependence of the principal permeability components in three dimensions: k_{11}, k_{22}, and k_{33}. First we assume that these three components are in fact the eigenvalues of the permeability tensor, and that the axes are aligned with axes of the orthotropic system itself. These assumptions can be modified as needed, but for first considerations, they should be adequate for our purposes.

Many models of fluid permeability are in use, including those in References [24-27]. Here we will emphasize the formula (together with analogous ones for directions 2 and 3):

$$k_{11} = \frac{\phi_1^2}{2s_1^2 F_1}, \tag{41}$$

where ϕ_1 is an apparent (averaged over the volume) porosity as seen in the x_1-direction. That is to say, the porous surface area per unit surface area may be found by viewing a cross-section of the material that is orthogonal to the x_1-direction. Similarly, s_1 is the apparent surface area per unit volume (also averaged over the volume), again for pores when viewed in cross sections. Both of these values can be determined to high accuracy by the use of digital image processing methods [28] on cross-sections of rocks. The remaining term is the pertinent formation factor F_1 for electrial conduction in brine-filled porous media. This value is not so easy to determine from images, but can nevertheless be estimated using one of the well-known forms of Archie's law [29], such as

$$F_1 = \phi_1^{-m_1}, \tag{42}$$

where m_1 is an appropriate Archie cementation exponent associated with electrical current conduction in the x_1-direction. Typical values of exponent m_1 [24] lie in the range $1 < m_1 \leq 2$.

If the composite material model we wish to model contains several significantly different types of poroelastic materials, then we may also need to consider additional formation factors associated with the composite structure itself, as was done in Reference [24] for two-component porous media. We will ignore this issue for now, as analysis of the layer structure under consideration suggests that, for many cases of interest, the pertinent distributional formation factors could be close unity. If this is not true in a particular application, then the methods developed in [28] can be generalized fairly easily to account for such additional complications. For now, we assume these particular effects are not of primary importance.

There is also potential for mismatching/offsetting of pores at the boundaries between layers [46], and these effects can further reduce the effective overall permeabilities of these systems. However, such effects are fairly easy to take into account whenever it is known that they are present, so we shall not treat such issues specifically here.

Now it is clear that the pertinent porosities and formation factors (also closely related to the same porosity values) are unitless measures of areas perpendicular to the three main flow directions. That means the strains that need to be considered are also the ones perpendicular to those directions. So for example, we must have

$$\begin{aligned} k_{11} &\simeq \frac{\phi^{2+m_1}}{2[s_{11}^{(0)}]^2}\left(1+e_{22}\right)\left(1+e_{33}\right), \\ k_{22} &\simeq \frac{\phi^{2+m_2}}{2[s_{22}^{(0)}]^2}\left(1+e_{33}\right)\left(1+e_{11}\right), \\ k_{33} &\simeq \frac{\phi^{2+m_3}}{2[s_{33}^{(0)}]^2}\left(1+e_{11}\right)\left(1+e_{22}\right), \end{aligned} \tag{43}$$

for the diagonal permeabilities of such porous systems. Motivation for such statements comes from an equivalent result for the orthotropic/anisotropic system of the form $s^{-2} \propto V^{2/3}$ for the specific surface area dependence on volume in the isotropic case [24].

Our permeability result then takes the form:

$$\frac{\delta k_{11}}{k_{11}} = (2+m_1)\frac{\delta\phi}{\phi} + \delta e_{22} + \delta e_{33} = (2+m_1)\frac{\delta\phi}{\phi} + \delta e - \delta e_{11}. \tag{44}$$

The δ's are shown explicitly to emphasize that these are all presumed to be relatively small changes in each of these respective quantities. The second equality follows from the definition of total strain $e = e_{11} + e_{22} + e_{33}$, and provides emphasis to a general symmetry of the dependencies: the permeability changes depend explicitly on strains in those directions perpendicular to the flow. So changes in k_{11} depend on the total strain minus the strain in the x_1-direction of flow, and analogously for the other two permeability eigenvalues.

The author has shown in earlier work [24] that the porosity for an isotropic system satisfies the following effective stress rule:

$$-\frac{\delta\phi}{\phi} = \left(\frac{\alpha_R - \phi}{\phi K_R^d}\right)(\delta p_c - \chi\delta p_f), \tag{45}$$

where the effective stress coefficient is χ is given by

$$\chi = \left(\frac{\omega - \phi}{\alpha_R - \phi}\right)\alpha_R, \tag{46}$$

where, for isotropic systems, we have:

$$\omega = \frac{1}{B} - \frac{K_p}{K_f}. \tag{47}$$

Again, B is Skempton's second coefficient from (19), and $K_p = \phi K_R^d/\alpha_R$ is the bulk modulus of pore volume.

Another result for the effective stress coefficient for these types of porous materials is discussed in Appendix C.

5. Quasi-statics and Layer-Averaged Permeability

Our treatment has purposely concentrated on the parts of the analysis that are quasi-static. This choice is not as limiting as it might seem at first sight, because it includes both drained and undrained behavior, and also higher frequency wave propagation problems that can be treated as undrained whenever we want to analyze only the wave speeds, but not the wave attenuation aspects of these problems. (For field applications, attenuation is also harder to measure reliably than wave speeds.) Modelling wave attenuation in poroelastic media necessarily requires knowledge of fluid permeability, and therefore spatial gradients of fluid pressure as well as time derivatives of the increments of fluid content. The results presented here are nevertheless of direct interest in those more complicated problems, as we also need to know these quasi-static results, especially in the low frequency limit of the full frequency-dependent problems.

It is useful to note that Schoenberg [42] studies the problem of layered permeable systems, using analysis methods very similar to the ones used here. However, Schoenberg's paper does not address the issues of poroelasticity and effective stress for changes in the permeability, but only the effects of the layering itself on the overall fluid-flow behavior and effective permeability.

The averaging scheme used by Schoenberg [42] for permeability is completely analogous to the one described here in Appendices A and B for elasticity and poroelasticity. If the permeability in each layer can be written in the form:

$$\mathbf{k} = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix}, \tag{48}$$

and, if $\mathbf{w}$ is the flow rate of the moving fluid particles while ρ is the fluid density such that $\rho\mathbf{w}$ is the local fluid momentum per unit volume, then

$$\mathbf{w} = -\frac{1}{\rho\nu}\mathbf{k}\cdot\nabla p_f, \tag{49}$$

or equivalently

$$\begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = -\frac{1}{\rho\nu}\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix}\begin{pmatrix} p_{f,x} \\ p_{f,y} \\ p_{f,z} \end{pmatrix} \tag{50}$$

is Darcy's law for the local behavior in any one layer, with ν being the uniform viscosity of the pore fluid.

Now (again following Schoenberg), we define

$$\mathbf{R}_{TT} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}, \tag{51}$$

$$\mathbf{r}_{TN} = \begin{pmatrix} k_{13} \\ k_{23} \end{pmatrix} = \mathbf{r}_{NT}^T, \tag{52}$$

and

$$r_{NN} = k_{33}. \tag{53}$$

Then, we can rewrite the permeability $\mathbf{k}$ for one layer of the overall layered systems as:

$$\mathbf{k} = \begin{pmatrix} \mathbf{R}_{TT} & \mathbf{r}_{TN} \\ \mathbf{r}_{NT} & r_{NN} \end{pmatrix}, \tag{54}$$

in terms of 2×2 matrix $\mathbf{R}$, vector $\mathbf{r}$, and scalar r. Next, averaging the results for the permeability, we find:

$$r^*_{NN} = \left\langle r^{-1}_{NN} \right\rangle^{-1}, \tag{55}$$

$$\mathbf{r}^*_{NT} = r^*_{NN} \left\langle r^{-1}_{NN} \mathbf{r}_{NT} \right\rangle = \mathbf{r}^{*T}_{TN}, \tag{56}$$

and

$$\mathbf{R}^*_{TT} = \langle \mathbf{R}_{TT} \rangle - \left\langle \mathbf{r}_{TN} r^{-1}_{NN} \mathbf{r}_{NT} \right\rangle + \mathbf{r}^*_{TN} \left(r^*_{NN} \right)^{-1} \mathbf{r}^*_{NT}. \tag{57}$$

So finally, the averaged permeability is given by:

$$\mathbf{k}^* = \begin{pmatrix} \mathbf{R}^*_{TT} & \mathbf{r}^*_{TN} \\ \mathbf{r}^*_{NT} & r^*_{NN} \end{pmatrix}. \tag{58}$$

Schoenberg [42] also describes how to do the analogous layered calculations for the inverse of permeability (*i.e.*, the *impermeability*). Defining the form of the impermeability analogously via

$$\mathbf{k}^{-1} \equiv \begin{pmatrix} \mathbf{X}_{TT} & \mathbf{x}_{TN} \\ \mathbf{x}_{NT} & x_{NN} \end{pmatrix}, \tag{59}$$

then the corresponding result for the averaged impermeability is

$$(\mathbf{k}^*)^{-1} \equiv \begin{pmatrix} \mathbf{X}^*_{TT} & \mathbf{x}^*_{TN} \\ \mathbf{x}^*_{NT} & x^*_{NN} \end{pmatrix}, \tag{60}$$

where

$$\mathbf{X}^*_{TT} = \left\langle \mathbf{X}^{-1}_{TT} \right\rangle^{-1}, \tag{61}$$

$$\mathbf{x}^*_{TN} = \mathbf{X}^*_{TT} \left\langle \mathbf{X}^{-1}_{TT} \mathbf{x}_{TN} \right\rangle = \left(\mathbf{x}^*_{NT} \right)^T, \tag{62}$$

and

$$x^*_{NN} = \mathbf{x}^*_{NT} \left(\mathbf{X}^*_{TT} \right)^{-1} \mathbf{x}^*_{TN}. \tag{63}$$

[Note that there is an obvious typographical error in Schoenberg's equation (9) since his term corresponding to x^*_{NN} is not dimensionally correct.]

When these formulas are compared to the ones obtained here (and also by others) for the average compliances, we see a close formal similarity between averaged compliance and averaged impermeability results. So it is not surprising that there must also be a formal similarity between the averaged stiffness and the averaged permeability results for such layered materials. This simple fact is useful nevertheless, as it provides another means of checking that results have been correctly derived.

6. Analysis of Fracture Permeability: Formulation and Effective Stress Behavior

6.1 Horizontal fractures

Chen and Bai [43] and Chen *et al.* [44] present a careful and clear analysis of the permeability tensor for fractured systems, including some discussion of stress dependence, which we will carry forward in our current applications. Two key parameters for simple, flat fractures (which was the focus of [43] and [44] and will also be our focus in this section) are the aperture b and the spacing between fractures d. If these values are not uniform in the material, then these values should be replaced by their average values. When the only porosity present is the fracture porosity ϕ_f, it is easy to see then that $\phi_f = b/d$ — assuming only that the individual fractures are connected across the entire region of study, and that they do not intersect or overlap. Now, writing permeability $\mathbf{k}$ as a matrix for horizontal fractures, we have

$$\mathbf{k} \equiv \frac{\phi_f^3 d^2 (1 + 2e_{33})}{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (64)$$

In xyz-coordinates, this formula says that the anisotropic permeability is exactly zero in the vertical or z-direction, and has equal values of $k_0 \equiv \phi_f^3 d^2/12$ in both the x- and y-directions, with a small correction due to strain e_{33} in the z-direction, whenever this effect might be significant enough to consider. Permeability $\mathbf{k}$ has dimensions of length squared, and these dimensions are supplied by the d^2 factor for the squared-spacing, and this factor can change due to changes in strain along the z-axis. Thus, the factor due to strain changes in the

z-direction is $(1+e_{33})^2 \simeq (1+2e_{33})$, and this determines the magnitude of any extension or contraction occurring along the z-axis. The factor ϕ_f^3 also provides one means of emphasizing the experimentally well-known cube-law for fluid flow in fractured media [45]. This choice of representation also provides an efficient means of determining the effect of pressure on the fracture permeability, since (as will next be argued) the only quantity to change significantly when the fluid pressure changes will be the fracture porosity itself.

Porosity is defined as the ratio of pore volume over total volume, so $\phi = V_\phi/V$ and

$$\delta\phi = \frac{\delta V_\phi}{V} - \frac{V_\phi \delta V}{V^2} = \phi\left(\frac{\delta V_\phi}{V_\phi} - \frac{\delta V}{V}\right). \tag{65}$$

From our earlier analysis, we have:

$$-\frac{\delta V}{V} = \frac{\delta p_d}{K_R^d} + \frac{\delta p_f}{K_R^g} = \frac{1}{K_R^d}\left(\delta p_c - \alpha_R \delta p_f\right), \tag{66}$$

and

$$-\frac{\delta V_\phi}{V_\phi} = \frac{\delta p_d}{K_p} + \frac{\delta p_f}{K_R^\phi} = \frac{1}{K_p}\left(\delta p_c - \xi \delta p_f\right), \tag{67}$$

where

$$K_p \equiv \frac{\phi K_R^d}{\alpha_R}, \quad \text{and} \quad \xi = 1 - K_p/K_R^\phi, \tag{68}$$

where (67) may be treated as the defining equation for K_R^ϕ, and $\alpha_R \equiv 1 - K_R^d/K_R^g$ is the usual Biot-Willis [2] parameter. The pertinent equation for the strain component e_{33} is determined by (1) and (3). When the confining pressure and fluid pressure have comparable values, this strain component depends mostly on the compliance factor $1/K_R^g$, which will normally be one of the smallest compliances in the system since it depends only on the solid grain behavior. This fact suggests that it should be a good approximation in many cases to neglect the contribution to the effective stress coming from this term in (64).

Thus, we conclude that the effective stress behavior for permeability due to horizontal fractures is determined largely by the changes in fracture porosity ϕ_f itself, and therefore we have:

$$\delta\mathbf{k} = \delta\phi_f \frac{\phi_f^2 d^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{69}$$

where $\delta\phi_f$ was determined by (65) through (68).

6.2 Nonhorizontal fractures

While we generally expect fractures to be more or less planar, there is no reason to think that they will always be horizontal. We can treat the more general problem of nonhorizontal fractures by considering rotations of the matrix equation for $\mathbf{k}$ given in (64).

We will consider the three choices of rotation matrices:

$$R_x = \begin{pmatrix} 1 & & \\ & \cos\psi & \sin\psi \\ & -\sin\psi & \cos\psi \end{pmatrix}, R_y = \begin{pmatrix} \cos\chi & & \sin\chi \\ & 1 & \\ -\sin\chi & & \cos\chi \end{pmatrix}, R_z = \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}, \quad (70)$$

which are, respectively, appropriate rotation matrices for rotations about the x-axis, y-axis, and z-axis.

Since rotations about the z-axis if performed first actually do not change the permeability matrix in (64), and since rotations about the z-axis after rotations in x and y only produce an effective change in the definitions of x and y, we will ignore the z-rotations, and only consider the possibility of doing first x-rotations, and then y-rotations. The result is:

$$R_yR_x\mathbf{k}R_x^TR_y^T = k_0 \begin{pmatrix} 1-\sin^2\chi\cos^2\psi & -\sin\chi\cos\psi\sin\psi & -\sin\chi\cos^2\psi\cos\chi \\ -\sin\chi\cos\psi\sin\psi & 1-\sin^2\psi & -\sin\psi\cos\psi\cos\chi \\ -\sin\chi\cos^2\psi\cos\chi & -\sin\psi\cos\psi\cos\chi & 1-\cos^2\psi\cos^2\chi \end{pmatrix}. \quad (71)$$

Checks on the correctness of this rotation matrix include: (*i*) the sum of the diagonal should be $2k_0$, (*ii*) the matrix should be symmetric, and (*iii*) the determinant should equal zero. All three of these necessary constraints are satisfied, as is easily checked.

To conform with the results of Chen *et al.* [43, 44], we should also note that, if the direction cosines are given by c_x, c_y, and c_z (where $c_x^2+c_y^2+c_z^2=1$), this matrix should have the general form:

$$k_0 \begin{pmatrix} 1-c_x^2 & -c_xc_y & -c_zc_x \\ -c_xc_y & 1-c_y^2 & -c_yc_z \\ -c_zc_x & -c_yc_z & 1-c_z^2 \end{pmatrix}. \quad (72)$$

Equation (71) differs from (72) only because we did not allow for any rotations about the z-axis in the previous case. Note that the rotated permeability matrix in (72) also has zero determinant and trace equal to $2k_0$, as expected.

6.3 Effective stress for fracture permeability

The preceding results show that the effective stress for changes in permeability depends only the scalar factor k_0. This factor also depends principally on the pressure dependence of the cube of fracture porosity ϕ_f. Recall that, as a general rule, we also expect $K_R^d < (1-\phi_f)K_R^g$ (for example, from Hashin-Shtrikman bounds [24, 31]), which then implies that $\phi_f < \alpha_R$. So we find that magnitude of the permeability k_0 varies as:

$$\frac{\delta k_0}{3k_0} \propto \frac{\delta\phi_f}{\phi_f} = -\frac{\alpha_R - \phi_f}{\phi_f K_R^d}(\delta p_c - \delta p_f) - \left(\frac{1}{K_R^\phi} - \frac{1}{K_R^g}\right)\delta p_f = -\frac{\alpha_R - \phi_f}{\phi_f K_R^d}(\delta p_c - \kappa_f \delta p_f), \quad (73)$$

where the main contribution to the effective stress coefficient for fracture permeability is:

$$\kappa_f = \alpha_R \frac{(\xi - \phi_f)}{(\alpha_R - \phi_f)}, \quad (74)$$

where ξ was defined in (68). We cannot say much more than this in general, but in many cases it will be true (for clean pores and only a single grain-type present) that the pore bulk modulus $K_R^\phi \simeq K_R^g$, *i.e.*, it is well approximated in such cases by the unique solid grain modulus. When this is true (*i.e.*, for simple granular systems without soft grain coatings, or fractures with clean surfaces), we can show explicitly that $\xi - \phi_f \simeq 1 - \phi_f/\alpha_R$. When this holds true, we find from (73) or (74) that

$$\kappa_f \equiv 1. \quad (75)$$

Thus, the fracture effective-stress coefficient κ_f for some simple systems may be approximately equal to unity, and that means the effect of fluid pressure for opening fractures is about as strong as it can get.

The general statement (74) can also be rewritten as

$$\kappa_f = 1 + \frac{\phi_f K_R^d}{\alpha_R - \phi_f}\left(\frac{1}{K_R^g} - \frac{1}{K_R^\phi}\right), \quad (76)$$

since $\xi - \phi_f = 1 - (\phi_f/\alpha_R)[\alpha_R + (K_R^d/K_R^\phi)]$. The result (76) shows again that, if $K_R^\phi \simeq K_R^g$, then $\kappa_f \simeq 1$.

Furthermore, when $K_R^\phi < K_R^g$, then $\kappa_f < 1$. When $K_R^\phi > K_R^g$, then $\kappa_f > 1$. Thus, soft fracture-surface coatings imply the permeability can be increased more easily by increasing p_f, while hard fracture-surface coatings imply the permeability cannot be increased so easily by increasing p_f, since the effective-stress coefficient κ_f is then less than unity.

The language being used in the preceding discussion is intended to be intuitive rather than rigorous. The rigorous interpretation should be based instead on the fact that pore bulk modulus K_R^ϕ is a general concept, not limited to coatings on the surfaces of pores. In particular, if the medium has heterogeneous grains, and the consolidated system is then fractured, all the types of grains can potentially contribute to the behavior described here, and the use of the "surface-coating" language is then certainly inappropriate. The formulas for κ_f are nevertheless general, and do not depend on these various possible grain arrangement scenarios. Choices of physical interpretations need to be model specific, whereas the formulas presented are general within the limitations already set in the study (mainly orthotropy). While the values of the constants such as K_R^ϕ are indeed model dependent, the formulas giving the effective stress coefficients do not change.

7. Summary and Conclusions

The thrust of the paper has focused on applications of layered porous meterials containing fluids. The first results show how to model individual poroelastic layers consistently and correctly when these layers are both poroelastic and anisotropic. As long as the anisotropy is at least as symmetric as orthotropy (*i.e.*, including isotropic, cubic, transversely isotropic or hexagonal, tetragonal, and orthotropic, but not including less symmetric cases such as trigonal, monoclinic, and triclinic crystal symmetries) the methods can be applied as long as one of the axes of symmetry is aligned with the layering direction. (For the less symmetric cases, these methods can also be generalized, but that work will appear elsewhere.) The resulting method is very analogous to earlier work by Backus [20], Schoenberg and Muir [21], and others. The assumed poroelastic boundary conditions were limited to either the drained (fluid free to move) or the undrained (fluid trapped) conditions, as these are the ones most easily handled in the proposed framework. These cases are also the only poroelastic cases that are usually treated analytically.

We then presented a discussion of the permeability variation in such porous layers, and in particular showed how the fluid permeability itself depends on changes in both confining and pore pressure, leading to conclusions about poroelastic effective stress. Two main types of permeability were considered: (1) intergranular or tubular pores, and (2) fractures. We treated the intergranular/tubular pores as a simple network-style generalization of an earlier isotropic analysis of effective stress for permeability of such systems. We followed this analysis by considering the case of more general anisotropic permeability, and especially permeability due to oriented fractures. The permeability analysis for this case was actually more general than the corresponding mechanical analysis for orthotropic systems, since there are fewer degrees of freedom involved in the permeability analysis. Nevertheless, it is clear that the two parts of the analysis can easily be brought into sync by limiting the fracture orientations so that the two systems (mechanical and fluid flow) have at least one of the symmetry directions of the permeability in common with that of the orthotropic poromechanical system studied. Indeed the presence of the fractures clearly influences the poroelastic mechanics, as well as the fluid flow properties, of these systems. So we would naturally expect a fracture system having orthorhombic mechanical symmetry to result in similar symmetries for the fluid flow as well as the poromechanical properties. The elastic parts of this analysis have been studied by many researchers including Sayers and Kachanov [47], and we will not pursue this aspect of the analysis further here.

One point that should be emphasized however is that, while it is not necessarily easy to infer directly and correctly the existence of oriented fractures from the observed mechanical behavior of these systems, it is nevertheless true that — if fractures are known to be present via the fluid-flow measurements — then it is certain that the mechanical behavior of the system will depend in predictable ways on the orientations of those fractures through the drained compliances introduced as the starting point of this paper.

Acknowledgments

Work performed under the auspices of the U.S. Department of Energy, at the Lawrence Berkeley National Laboratory under Contract No. DE-AC02-05CH11231. Support was provided specifically by the Geosciences Research Program of the DOE Office of Basic Energy Sciences, Division of Chemical Sciences, Geosciences and Biosciences. Additional support was provided by the Enhanced Geothermal Systems Demonstration Project on Induced Seismicity. All support of this research is gratefully acknowledged.

APPENDIX A: SCHOENBERG-MUIR METHOD

The quasi-static elasticity equations are often written in compliance form using the Voigt 6×6 matrix notation as:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} \equiv \mathbf{S} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}, \tag{77}$$

where $\mathbf{S}$ is the symmetric 6×6 compliance matrix. The numbers 1,2,3 always indicate Cartesian axes (say: x,y,z respectively). The z-direction is usually chosen as the layering direction, which could be oriented any direction in the earth. But, in many geological and geophysical applications, the 3-axis (or z-axis) is also taken to be the vertical direction, and we conform to this convention here. The principal stresses are σ_{11}, σ_{22}, σ_{33}, in the directions 1,2,3, respectively. Similarly, the principal strains are e_{11}, e_{22}, e_{33}. The stresses σ_{23}, σ_{31}, σ_{12} are the torsional shear stresses, associated with rotation-based strains around the 1, 2, or 3 axes, respectively. The corresponding torsional strains are e_{23}, e_{31}, and e_{12}, where the torsional motion is again a rotational straining motion around the 1, 2, or 3 axes. The compliance matrix is symmetric, so $s_{ij} = s_{ji}$, and this fact could have been used when displaying the matrix. The axis pairs in the subscripts 11, 22, 33, 23, 31, and 12 for stresses and strains, are often labelled (again following the conventions of Voigt) as 1,2,3,4,5,6, respectively.

The important contribution made by Backus [20] (also see Postma [22]) is the observation that, in a layered system, there are certain strains e_{ij} and stresses σ_{ij} that are necessarily continuous across boundaries between layers, while the others are not necessarily continuous. We have been implicitly (and now explicitly by calling this fact out) assuming that the interfaces between layers are in welded contact, which means practically that the in-plane strains are always continuous: so if axis 3 (or z) is the symmetry axis (as is most often chosen for the layering problem), we have e_{11}, $e_{12} = e_{21}$, and e_{22} are all continuous. Similarly, in welded contact, we must have continuity of all the stresses involving the 3 (or z) direction: therefore σ_{33}, $\sigma_{13} = \sigma_{31}$, and $\sigma_{23} = \sigma_{32}$ must all be continuous.

Then, following Backus [20] and/or Schoenberg and Muir [21], but — for present purposes considering instead the compliance (inverse of stiffness) matrix — we have rearranged the statement of the problem so that:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{12} \\ e_{33} \\ e_{32} \\ e_{31} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{16} & s_{13} & s_{14} & s_{15} \\ s_{21} & s_{22} & s_{26} & s_{23} & s_{24} & s_{25} \\ s_{61} & s_{62} & s_{66} & s_{63} & s_{64} & s_{65} \\ s_{31} & s_{32} & s_{36} & s_{33} & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{46} & s_{43} & s_{44} & s_{45} \\ s_{51} & s_{52} & s_{56} & s_{35} & s_{54} & s_{55} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \end{pmatrix} . \tag{78}$$

Note that this equation, although similar to (77) is quite different because of the rearrangement of the matrix elements and the reordering of the strains and stresses. The chosen expression in (78) is general for all elastic media. In the main text we restrict our discussion to orthotropic media. Assuming then that we are using the correct axes as the symmetry axes in the presentation, all off-diagonal compliances having subscripts 4, 5, or 6 in (77) vanish identically. The diagonal shear compliances s_{44}, etc., generally do not vanish however.

Expression of (78) can be made more compact by writing it as:

$$\begin{pmatrix} E_T \\ E_N \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{TT} & \mathbf{S}_{TN} \\ \mathbf{S}_{NT} & \mathbf{S}_{NN} \end{pmatrix} \begin{pmatrix} \Pi_T \\ \Pi_N \end{pmatrix} , \tag{79}$$

where

$$\mathbf{S}_{TT} \equiv \begin{pmatrix} s_{11} & s_{12} & s_{16} \\ s_{21} & s_{22} & s_{26} \\ s_{61} & s_{62} & s_{66} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \\ s_{21} & s_{22} & \\ & & s_{66} \end{pmatrix} , \tag{80}$$

$$\mathbf{S}_{NN} \equiv \begin{pmatrix} s_{33} & s_{34} & s_{35} \\ s_{43} & s_{44} & s_{45} \\ s_{53} & s_{54} & s_{55} \end{pmatrix} = \begin{pmatrix} s_{33} & & \\ & s_{44} & \\ & & s_{55} \end{pmatrix} , \tag{81}$$

and

$$\mathbf{S}_{NT} \equiv \begin{pmatrix} s_{31} & s_{32} & s_{36} \\ s_{41} & s_{42} & s_{46} \\ s_{51} & s_{52} & s_{56} \end{pmatrix} = \begin{pmatrix} s_{31} & s_{32} & \\ & 0 & \\ & & 0 \end{pmatrix} , \tag{82}$$

with $\mathbf{S}_{TN} = \mathbf{S}_{NT}^T$ (with T superscript indicating the matrix transpose). Also we have

$$E_T \equiv \begin{pmatrix} e_{11} \\ e_{22} \\ e_{12} \end{pmatrix}, \quad \text{and} \quad E_N \equiv \begin{pmatrix} e_{33} \\ e_{32} \\ e_{31} \end{pmatrix}, \tag{83}$$

and

$$\Pi_T \equiv \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \text{and} \quad \Pi_N \equiv \begin{pmatrix} \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \end{pmatrix}. \tag{84}$$

It is important to distinguish between "slow" and "fast" variables in this analysis, since this distinction makes it clear when and how averaging should be performed. The "slow" variables, *i.e.*, those that are continuous across the (here assumed horizontal) boundaries and also essentially constant for (the present) quasi-static applications, are those contained in E_T and Π_N. So, after averaging $\langle\cdot\rangle$ along the layering direction, we should have:

$$\begin{pmatrix} E_T \\ \langle E_N \rangle \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{TT}^* & \mathbf{S}_{TN}^* \\ \mathbf{S}_{NT}^* & \mathbf{S}_{NN}^* \end{pmatrix} \begin{pmatrix} \langle \Pi_T \rangle \\ \Pi_N \end{pmatrix}, \tag{85}$$

where $\mathbf{S}_{TN}^* = (\mathbf{S}_{NT}^*)^T$, and all the starred quantities are the *nontrivial* average compliances we seek. They are defined in terms of layer-average quantities where the symbol $\langle\cdot\rangle$ indicates a simple volume average of all the layers. By this notation we mean that a quantity Q that takes on different values in different layers has the layer average $\langle Q \rangle \equiv x_a Q_a + x_b Q_b + \ldots$. The definition is general and applies to an arbitrary number of different layers where the fraction of the total volume occupied by layer a is x_a, etc. Total fractional volume is $x_a + x_b + \ldots \equiv 1$.

Of the three final results, the two easiest ones to compute are:

$$\mathbf{S}_{TT}^* = \left\langle \mathbf{S}_{TT}^{-1} \right\rangle^{-1}, \tag{86}$$

$$\mathbf{S}_{TN}^* = (\mathbf{S}_{NT}^*)^T = \left\langle \mathbf{S}_{TT}^{-1} \right\rangle^{-1} \left\langle \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle = \mathbf{S}_{TT}^* \left\langle \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle, \tag{87}$$

where $\langle\cdot\rangle$ is the layer average of some quantity. These results follow from this equation:

$$\left\langle \mathbf{S}_{TT}^{-1} \right\rangle E_T = \langle \Pi_T \rangle + \left\langle \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle \Pi_N, \tag{88}$$

which followed immediately from the formula

$$E_T = \mathbf{S}_{TT} \Pi_T + \mathbf{S}_{TN} \Pi_N, \tag{89}$$

multiplying through first by the inverse of $\mathbf{S}_{TT}$, and then performing the layer average. [Note that $\mathbf{S}_{TT}$ and $\mathbf{S}_{NN}$ are both normally square and invertible matrices, whereas for most systems the off-diagonal matrix $\mathbf{S}_{NT}$ is not invertible. But, this fact does not cause problems in the analysis, because we do not need to invert $\mathbf{S}_{NT}$ in order to solve the averaging problem at hand.] These averages are meaningful because when the matrix equations presented are multiplied out, we never have any cross products of two quantities that are both unknown. [From this view point, Eq. (88) is an equation for $\langle \Pi_T \rangle$, just as the unaveraged version of (88) is an equation for Π_T in each layer.] So simple layer averaging suffices (thereby providing the main motivation and value of this method). Multiplying (88) through by $\left\langle \mathbf{S}_{TT}^{-1} \right\rangle^{-1}$ then gives the results (86) and (87).

The remaining result is more tedious to compute, since it requires several intermediate steps in its derivation. But the final result is given by the formula:

$$\mathbf{S}_{NN}^* = \langle \mathbf{S}_{NN} \rangle - \left\langle \mathbf{S}_{NT} \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle + \mathbf{S}_{NT}^* \left(\mathbf{S}_{TT}^* \right)^{-1} \mathbf{S}_{TN}^*. \tag{90}$$

To provide some clues to the derivation, again consider:

$$\Pi_T = \mathbf{S}_{TT}^{-1} E_T - \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \Pi_N, \tag{91}$$

which is just a rearrangement of (89). The point is that $\langle \Pi_T \rangle$ is then given immediately in terms of the quantities E_T and Π_N, which are both "slow" variables and therefore essentially constant. An intermediate result that helps to explain the form of this relation (90) is:

$$\mathbf{S}_{NT}^* \left(\mathbf{S}_{TT}^* \right)^{-1} \mathbf{S}_{TN}^* = \left\langle \mathbf{S}_{NT} \mathbf{S}_{TT}^{-1} \right\rangle \left\langle \mathbf{S}_{TT}^{-1} \right\rangle^{-1} \left\langle \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle = \left\langle \mathbf{S}_{NT} \mathbf{S}_{TT}^{-1} \right\rangle \mathbf{S}_{TN}^*. \tag{92}$$

Substituting for Π_T from (91) into

$$E_N = \mathbf{S}_{NT} \Pi_T + \mathbf{S}_{NN} \Pi_N, \tag{93}$$

and then averaging, we find that

$$\langle E_N \rangle = \left\langle \mathbf{S}_{NT} \mathbf{S}_{TT}^{-1} \right\rangle E_T + \left\langle \mathbf{S}_{NN} - \mathbf{S}_{NT} \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle \Pi_N. \tag{94}$$

This expression completely determines all the remaining coefficients. After some more algebra, the formula giving the final result is:

$$\begin{aligned} \langle E_N \rangle &= \left\langle \mathbf{S}_{NT} \mathbf{S}_{TT}^{-1} \right\rangle \left\langle \mathbf{S}_{TT}^{-1} \right\rangle^{-1} \left[\langle \Pi_T \rangle + \left\langle \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle \Pi_N \right] \\ &+ \left[\langle \mathbf{S}_{NN} \rangle - \left\langle \mathbf{S}_{NT} \mathbf{S}_{TT}^{-1} \mathbf{S}_{TN} \right\rangle \right] \Pi_N \\ &= \mathbf{S}_{NT}^* \langle \Pi_T \rangle + \mathbf{S}_{NN}^* \Pi_N. \end{aligned} \tag{95}$$

Equation (95) contains all the information needed to produce the third and final result found in (90).

Another check on these formulas is to compare them directly to those found by Schoenberg and Muir [21]. However, direct comparison is not so easy, since their analysis focuses on the stiffness version of the equations. Our treatment makes use of the compliance version instead. Since the symmetries of the two forms of the equations are nevertheless nearly identical, cross-checks and comparisons will be left to the motivated reader.

APPENDIX B: POROELASTIC FORMULAS FOR UNDRAINED BOUNDARY CONDITIONS IN LAYERED SYSTEMS

Using equation (29) as our starting point, we now consider the boundary condition $\zeta = 0$ for undrained layers (meaning that the fluid is actually physically trapped in the layer, or the physical process is so fast – such as high frequency wave propagation – that the fluid inertia prevents rapid movement of fluid particles over non-infinitesimal distances). Depending on the application scenario, this boundary condition might be applied to all layers, or only to just one or a few layers at a time.

We consider first a single layer having the undrained boundary condition. For this case, the condition from Eq. (29) becomes

$$0 = \mathbf{g}_{12}^T \Pi_T + \gamma p_f + \mathbf{g}_3^T \Pi_N, \tag{96}$$

within the layer. Next, the equation can be solved to express the fluid pressure p_f strain dependence in each undrained layer (the layer labels are suppressed here for simplicity) as

$$p_f = -\frac{1}{\gamma} \left(\mathbf{g}_{12}^T \Pi_T + \mathbf{g}_3^T \Pi_N \right). \tag{97}$$

Then, substituting this condition back into the expressions for E_T and E_N from (29), we find that

$$\begin{pmatrix} E_T \\ E_N \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{TT} - \gamma^{-1} \mathbf{g}_{12} \mathbf{g}_{12}^T & \mathbf{S}_{TN} - \gamma^{-1} \mathbf{g}_{12} \mathbf{g}_3^T \\ \mathbf{S}_{NT} - \gamma^{-1} \mathbf{g}_3 \mathbf{g}_{12}^T & \mathbf{S}_{NN} - \gamma^{-1} \mathbf{g}_3 \mathbf{g}_3^T \end{pmatrix} \begin{pmatrix} \Pi_T \\ \Pi_N \end{pmatrix}. \tag{98}$$

To understand the significance of (98), we next find it is straightforward to show that each of these composite matrix elements corresponds exactly to the undrained version of the Schoenberg-Muir matrices. So that,

$$\mathbf{S}_{TT}^u \equiv \mathbf{S}_{TT} - \gamma^{-1} \mathbf{g}_{12} \mathbf{g}_{12}^T, \tag{99}$$

$$\mathbf{S}^u_{NN} \equiv \mathbf{S}_{NN} - \gamma^{-1}\mathbf{g}_3\mathbf{g}_3^T, \tag{100}$$

and

$$\mathbf{S}^u_{TN} \equiv \mathbf{S}_{TN} - \gamma^{-1}\mathbf{g}_{12}\mathbf{g}_3^T = \left(\mathbf{S}^u_{NT}\right)^T. \tag{101}$$

All these expressions follow directly from the form of (98).

Thus, we arrive methodically at a result that might have been anticipated, which is that the undrained layers respond according to the usual undrained conditions in each individual layer. The part of the result that is new concerns the forms of the undrained matrices $\mathbf{S}^u_{TT}$, $\mathbf{S}^u_{NT} = \left(\mathbf{S}^u_{TN}\right)^T$, and $\mathbf{S}^u_{NN}$, now in the modified Schoenberg-Muir formalism.

This analogy can be pushed somewhat further to include the effective values for the undrained moduli $\mathbf{S}^{u*}_{TT}$, $\mathbf{S}^{u*}_{NT} = \left(\mathbf{S}^{u*}_{TN}\right)^T$, and $\mathbf{S}^{u*}_{NN}$, with formulas entirely analogous to (86), (87), and (90), and undrained constants replacing drained constants everywhere. Since there is nothing subtle about this step, we leave these details again to the interested reader.

APPENDIX C: EFFECTIVE STRESS FOR PERMEABILITY OF ISOTROPIC POROELASTIC SYSTEMS WITH GRANULAR STRUCTURE AND/OR TUBULAR PORES

Berryman [24] shows that the effective stress response of fluid permeability k in isotropic poroelastic systems, having either granular structure or tubular pores, is given by:

$$\frac{\delta k}{k} = -\left[\frac{2}{3} + n\left(\frac{\alpha_R - \phi}{\phi}\right)\right]\frac{1}{K^d_R}\left(\delta p_c - \kappa\delta p_f\right), \tag{102}$$

where the pertinent effective stress coefficient is

$$\kappa = 1 - \frac{2\phi(1-\alpha_R)}{2\phi + 3n(\alpha_R - \phi)}. \tag{103}$$

The numerical constant n is model dependent, but often has a value $n \simeq 4$. The porosity is ϕ. The poroelastic factor $\alpha_R = 1 - K^d_R/K^g_R$ is the usual Biot or Biot-Willis coefficient [2]. The subscripts R are redundant in these expressions, since Reuss and Voigt averages are the same for isotropic systems; but we show them here nevertheless to emphasize their connection to results in the anisotropic problem. Also note that $(\alpha_R - \phi) \geq 0$ in general. The result (103) follows from the commonly used formula for isotropic permeability in this class of systems in terms of porosity ϕ, specific surface area s, and formation factor F, which is given by

$$k \simeq \frac{\phi^2}{2s^2 F}, \tag{104}$$

and which formula is consistent with the work of many researchers, including Paterson [25] and Walsh and Brace [26]. The formation factor is often estimated in the form $F \simeq \phi^{-m}$, in which case the constant $n \simeq 2 + m$. Since k has the dimensions of length squared, it scales with volume V like $V^{2/3}$.

A useful approximation [30-35] to K_R^d for isotropic systems composed of a single isotropic grain-type having bulk modulus K_R^g and shear modulus μ^g is:

$$K_R^d \simeq \frac{(1-\phi)K_R^g}{1 + 3K_R^g \phi/4\mu^g} \simeq \frac{(1-\phi)K_R^g}{1 + c\phi}, \tag{105}$$

where the dimensionless factor $c \simeq 3K_R^g/4\mu^g$ is sometimes called a consolidation parameter; in the absence of definitive information concerning elastic frame constants, c can also be used as a fitting parameter. A typical range of values for this parameter for sandstones is $2 \leq c \leq 20$. Lower values of c correspond to stronger states of consolidation, while higher values correspond to weaker states of consolidation.

Substituting this expression (105) into the formula for the effective stress coefficient κ, we find:

$$\kappa \simeq 1 - \frac{1-\phi}{1 + c[(3n/2)(1-\phi) + \phi]}. \tag{106}$$

REFERENCES

[1] F. Gassmann, "Über die Elastizität poröser Medien," *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich* **96** (1951) 1–23.

[2] M. A. Biot, D. G. Willis, The elastic coefficients of the theory of consolidation, Journal of Applied Mechanics **24** (1957) 594–601.

[3] R. J. S. Brown, J. Korringa, On the dependence of the elastic properties of a porous rock on the compressiblity of the pore fluid, Geophysics **40** (1975) 608–616.

[4] J. R. Rice, M. P. Cleary, Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents, Reviews of Geophysics and Space Physics **14** (1976) 227–241.

[5] L. Thigpen, J. G. Berryman, Mechanics of porous elastic materials containing multiphase fluid, International Journal of Engineering Science **23** (1985) 1203.

[6] R. W. Zimmerman, Compressibility of Sandstones, Elsevier, Amsterdam, 1991, Chapter 6.

[7] A. H.-D. Cheng, Material coefficients of anisotropic poroelasticity, International Journal of Rock Mechanics **34** (1997) 199–205.

[8] H. F. Wang, Theory of Linear Poroelasticity with Applications to Geomechanics and Hydrogeology, Princeton University Press, Princeton, NJ, 2000.

[9] J. G. Berryman, Poroelastic measurement schemes resulting in complete data sets for granular and other anisotropic porous media, International Journal of Engineering Science **48** (2010) 446–459.

[10] S. R. Pride, J. G. Berryman, Linear dynamics of double-porosity dual-permeability materials. I. Governing equations and acoustic attenuation," Physical Review E **68** (2003) 036693.

[11] S. R. Pride, J. G. Berryman, Linear dynamics of double-porosity dual-permeability materials. II. Fluid transport equations," Physical Review E **68** (2003) 036694.

[12] S. R. Pride, J. G. Berryman, J. M. Harris, Seismic attenuation due to wave-induced flow, Journal of Geophysical Research **109** (2004) B01201.

[13] A. Reuss, "Berechung der Fliessgrenze von Mischkristallen," *Z. Angew. Math. Mech.* **9** (1929) 55.

[14] W. Voigt, *Lehrbuch der Kristallphysik*, Teubner, Leipzig, 1928, p. 962.

[15] R. Hill, The elastic behaviour of crystalline aggregate, Proceedings of the Physical Society of London **A65** (1952) 349–354.

[16] A. W. Skempton, The pore-pressure coefficients A and B, Géotechnique **4** (1954) 143–147.

[17] R. D. Stoll, Acoustic waves in saturated sediments, in *Physics of Sound in Marine Sediments*, edited by L. Hampton, Plenum, New York, 1974, pp. 19–39.

[18] J. G. Berryman, Transversely isotropic poroelasticity arising from thin isotropic layers, in *Mathematics of Multiscale Materials*, edited by K. M. Golden, G. R. Grimmett, R. D. James, G. W. Milton, and P. N. Sen,Springer, New York, 1998, pp. 37–50.

[19] A. W. Wood, A Textbook of Sound, Bell, London, 1955, p. 360.

[20] G. E. Backus, Long-wave elastic anisotropy produced by horizontal layering, Journal of Geophysical Research **67** (1962) 4427–4440.

[21] M. Schoenberg, F. Muir, A calculus for finely layered anisotropic media, Geophysics **54** (1989) 581–589.

[22] G. W. Postma, Wave propagation in a stratified medium, Geophysics **20** (1955) 780–806.

[23] M. A. Biot, Mechanics of deformation and acoustic propagation in porous media, J. Appl. Phys. **33** (1962) 1482–1498.

[24] J. G. Berryman, Effective stress for transport properties of inhomogeneous porous rock, J. Geophys. Res. **97** (1992) 17409–17424.

[25] M. S. Paterson, The equivalent channel model for permeability and resistivity in fluid-saturated rocks – A reappraisal, Mech. Mater. **2** (1983) 345–352.

[26] J. B. Walsh, W. F. Brace, The effect of pressure on porosity and the transport properties of rocks, J. Geophys. Res. **89** (1984) 9425–9431.

[27] N. S. Martys, S. Torquato, D. P. Bentz, Universal scaling of fluid permeability for sphere packings, Phys. Rev. E **50** (1994) 402–408.

[28] J. G. Berryman, Measurement of spatial correlation functions using image processing techniques, J. Appl. Phys. **57** (1985) 2374–2384.

[29] G. E. Archie, The electrical resistivity log as an aid in determining some reservoir characteristics, Trans. Am. Inst. Min. Metall. Pet. Eng. **146** (1942) 54–62.

[30] D. P. Hasselman, On the porosity dependence of mechanical strength of brittle polycys-talline refractory materials, J. Am. Ceran. Soc. **45** (1962) 452–453.

[31] Z. Hashin, Elastic moduli of heterogeneous materials, J. Appl. Mech. **29** (1962) 143–150.

[32] J. Korringa, R. J. S. Brown, D. D. Thompson, R. J. Runge, Self-consistent imbedding and the ellipsoidal model for porous rocks, J. Geophys. Res. **84** (1979) 5591–5598.

[33] S. R. Pride, E. Tromeur, J. G. Berryman, Biot slow-wave effects in stratified rock, Geophysics **67** (2002) 271–281.

[34] S. R. Pride, Relationships between seismic and hydrological properties, in *Hydrogeophysics*, edited by Y. Rubin and S. S. Hubbard, Springer (2005) pp. 253–290.

[35] X.-S. Wang, X.-W. Jiang, L. Wan, G. Song, Q. Xia, Evaluation of depth-dependent porosity and bulk modulus of a shear using permeability-depth trends, Int. J. Rock Mech. Min. Sci. **46** (2009) 1175–1181.

[36] S. P. Neuman, Trends, prospects, and challenges in quantifying flow and transport through fractured rocks, Hydrogeol. J. **13** (2005) 124–147.

[37] L. J. Pyrak-Nolte, J. P. Morris, Single fractures under normal stress: The relation between fracture specific stiffness and fluid flow, Int. J. Rock Mech. Min. Sci. **37** (2000) 245–262.

[38] S.-H. Ji, K.-K. Lee, Y.-C. Park, Effects of correlation length on the hydraulic parameters of a fracture network, Transport in Porous Media **55** (2004) 153–168.

[39] X.-W. Jiang, L. Wan, X.-S. Wang, S.-H. Liang, B. X. Hu, Estimation of fracture normal stiffness using a transmissity-depth correlation, Int. J. Rock Mech. Min. Sci **46** (2009) 51–58.

[40] H. H. Liu, J. Rutqvist, A new coal-permeability model: Internal swelling stress and fracture-matrix interaction, Transport in Porous Media **82** (2010) 157–171.

[41] H. H. Liu, J. Rutqvist, J. G. Berryman, On the relationship between stress and elastic strain for porous and fractured rock, Int. J. Rock Mech. Min. Sci. **46** (2009) 792–802.

[42] M. Schoenberg, Layered permeable systems, Geophys. Prospecting **39** (1991) 219–240.

[43] M. Chen, M. Bai, Modeling stress-dependent permeability for anisotropic fractured porous rocks, International Journal of Rock Mechancis and Mining Sciences **35** (1998) 1113–1119.

[44] M. Chen, M. Bai, J.-C. Roegiers, Permeability tensors of anisotropic fracture networks, Mathematical Geology **31** (1999) 355–373.

[45] P. A. Witherspoon, J. S. Y. Wang, K. Imai, J. E. Gale, Validity of cubic law for fluid flow in a deformable rock fracture, Water Resources Research **16** (1980) 1016–1024.

[46] H. Deresiewicz, R. Skalak, On uniqueness in dynamic poroelasticity, Bulletin of the Seismological Society of America **53** (1963) 783–788.

[47] C. M. Sayers, M. Kachanov, A simple technique for finding effective elastic constants of cracked solids for arbitrary crack orientation statistics, International Journal of Solids and Structures **27** (1991) 671–680.

[48] T. C. T. Ting, *Anisotropic Elasticity: Theory and Applications*, Oxford University Press, New York, 1996, pp. 53–59.