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Forces in a Thin Cosine(nθ) Helical Wiggler.*

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Abstract

We wish to calculate the Lorentz body force associated with pure multipole helical magnetic fields (i.e, proportional to cos($n\theta$)) whose strength varies purely as a Fourier sinusoidal series of the longitudinal coordinate z (say proportional to $\cos\frac{(2m-1)\pi z}{L}$, where L denotes the *half-period* of the wiggler *field* and m=1,2,3 ...). We also wish to apply such forces to the current sheet, and solve for the stress distribution required to maintain such a coil in equilibrium. In the calculations of Lorentz forces we include the self field contribution as well as possible contributions arising from additional nested helical windings. We shall demonstrate that in cases where the current is situated on a surface of discontinuity at r=R (i.e. J=f(θ,z)) and the Lorentz body force is integrated on that surface, a closed form solution for the stress distribution can be obtained and such a solution includes contributions from possible nested multipole magnets. Finally we demonstrate that in the limiting 2D case where the field strength does not vary with z (period 2L tends to infinity) the stress reduces to known 2D expressions.

On a developed surface of the current sheet we place a set of coordinates, ξ in the current flow direction and η normal to that flow (or normal to the pole). The stress in the current direction can only be associated with a superimposed mechanical stress T_ξ (per unit length) such as winding tension. The solution to the stress distribution employing such a coordinate system is,

$$n \neq i \quad ; \quad P'_\eta = -\frac{1}{\mu_0 R}\sum_i \frac{i}{n^2-i^2} G_{n,m}G_{i,m}\frac{I'_i(ip_mR)}{K'_n(np_mR)}\left[i\cos n\eta\cos i\eta + n\sin n\eta \sin i\eta - n\sin\frac{\pi i}{2n}\right]$$

$$n = i \quad ; \quad P'_\eta = \frac{1}{2\mu_0 R} G^2_{n,m}\frac{I'_n(np_mR)}{K'_n(np_mR)}\cos^2 n\eta$$

$n \neq i$

$$\begin{aligned} P''_\rho = & \frac{\left[1+(p_mR)^2\right]}{2\mu_0 R^2}\sum_i i\frac{G_{n,m}G_{i,m}}{p_mR}\frac{\left(I_i(ip_mR)K_i(ip_mR)\right)'}{K'_n(np_mR)K'_i(ip_mR)}\cos n\eta\cos i\eta \\ & - \frac{1}{\mu_0 R^2\left[1+(p_mR)^2\right]}\sum_i \frac{i}{n^2-i^2}G_{n,m}G_{i,m}\frac{I'_i(ip_mR)}{K'_n(np_mR)}\left[i\cos n\eta\cos i\eta + n\sin n\eta\sin i\eta - n\sin\frac{\pi i}{2n}\right] \\ & + \frac{(p_mR)^2}{R\left[1+(p_mR)^2\right]}\sum_i T_{\xi_{i,m}} \end{aligned}$$

$$\begin{aligned} n = i \quad ; \quad P''_\rho = & \frac{\left[1+(p_mR)^2\right]}{2\mu_0 R^2} n \frac{G^2_{n,m}}{p_mR}\frac{\left(I_n(np_mR)K_n(np_mR)\right)'}{K'_n(np_mR)K'_n(np_mR)}\cos^2 n\eta \\ & - \frac{1}{2\mu_0 R^2\left[1+(p_mR)^2\right]}G^2_{n,m}\frac{I'_n(np_mR)}{K'_n(np_mR)}\cos^2 n\eta \\ & + \frac{(p_mR)^2}{R\left[1+(p_mR)^2\right]}T_{\xi_{n,m}} \end{aligned}$$

Introduction

We commence with deriving the Lorentz forces on a surface of discontinuity from previously derived expressions of the fields and currents (Appendix A). Maintaining the Lorentz body forces in equilibrium on an infinitesimal surface area, result in set of differential equations that once solved, give the stress expressions associated with a given current density.

It may proven to be useful and prudent to reduce the complexity of the multipole wiggler geometry by first transforming all fields and current densities to a coordinate system that is aligned with the direction of the current flow. A Frenet—Serret rotating unit vector coordinate system may serve such a purpose and will reduce the 3 components of the Lorentz forces to 2. We proceed in obtaining such a transformation through the use of differential geometry (Appendix C).[†] Following a solution to the force equations we continue with and example of a combined function (nested) dipole and quadrupole. The expressions for the self force and the mutual force on each magnet have been explicitly obtained.

Finally, by reducing the periodicity to zero we obtain the stress associated with combined function of long (2D) multipole magnets.

Lorentz Force on a Surface of Discontinuity

The Lorentz force density on a thin surface of discontinuity[c] (per unit area s) may be expressed as given by

$$(\nabla \cdot P)_i = -\vec{J_s} \times < \vec{B} > \qquad 1$$

where $< \vec{B} >$ denotes the average magnetic field on the surface $< \vec{B} > = \frac{\vec{B_1}+\vec{B_2}}{2}$, $\vec{J_s}$ corresponds to the surface current density and P is the stress tensor. Previously[d] we expressed the magnetic field components both inside and outside a current sheet (Appendix A), for an ideal current density that is proportional to cosine nθ. We shall evaluate $< B >$ and $\vec{J_s}$ on the surface at r=R and proceed to calculate the magnetic forces acting on such a surface. To simplify the analysis we have not included in this paper contributions from a highly permeable iron yoke.

Based on the field expressions inside and outside the current sheet we write

$$< \vec{B} > |_{r=R} = \begin{Bmatrix} -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n^{'} \sin(n\theta - \omega_m z) \\ -\frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{nG_{n,m}}{R}\frac{(I_n K_n)^{'}}{K_n^{'}} \cos(n\theta - \omega_m z) \\ \frac{1}{2}\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{(I_n K_n)^{'}}{K_n^{'}} \cos(n\theta - \omega_m z) \end{Bmatrix}$$

and

$$\vec{J_s}|_{r=R} = \frac{1}{\mu_0}\begin{Bmatrix} 0 \\ -\sum_{n=1}\sum_{m=1} \frac{G_{n,m}}{R}\frac{1}{K_n^{'}} \cos(n\theta - \omega_m z) \\ -\sum_{n=1}\sum_{m=1} \frac{nG_{n,m}}{\omega_m R^2}\frac{1}{K_n^{'}} \cos(n\theta - \omega_m z) \end{Bmatrix}$$

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† I am grateful to Ed Lee (HIFAR) for his helpful comments.

c Utility of the Maxwell Stress Tensor for Computing Magnetic Forces — L.Jackson Laslett, Lawerence Berkeley Laboratory, report ERAN-160, August 24 1971.

d Magnetic Field Components in a Sinusoidally Varying Helical Wiggler, LBL-35928, SC-MAG-464, July 1994.

where,

$$\omega_m = \frac{(2m-1)\pi}{L} \qquad and \qquad G_{n,m} = n! R^n \left(\frac{2}{\omega_m R}\right)^n B_{n,m}$$

n=1,2,3.... corresponds to a dipole, quadrupole etc, m=1,2,3...., corresponds to a given periodicity where L is the *field* half period. We consider the term $(\omega_m R)$ to be the argument of all Modified Bessel functions I_n and K_n, and all derivatives of such functions taken to be with respect to that argument.

We note that the pair of current components satisfy the conservation condition $\nabla \cdot \vec{J_s} = \frac{\partial J_z}{\partial z} + \frac{1}{R}\frac{\partial J_\theta}{\partial \theta} = 0$ as required and that for a given n,m the ratio of the current density components is fixed (although n,m dependent) independent of the coordinates

$$\left(\frac{\vec{J_z}}{\vec{J_\theta}}\right)_{n,m} = \frac{n}{\omega_m R}$$

and therefore the space curve generated by the pole n,m is a circular helix with a fixed axial to azimuthal ratio (Fig. 1),

$$\tan \alpha_{n,m} = \frac{2Ln}{2\pi R(2m-1)} = \frac{n}{\omega_m R}$$

We shall require that for all nested helical windings other than n,m the angle α will not change. That is

$$\tan \alpha_{n,m} = \frac{n}{\omega_m R} = \tan \alpha_{i,j} = \frac{i}{\omega_j R}$$

and therefore for such cases the ratio of all current densities to remain constant.

Before we proceed we need to make a clear distinction between the PERIODICITY OF THE FIELD 2L, and the PERIODICITY OF THE WIRE $2l$. Whereas the periodicity of the field depends on the magnet type n, we choose to hold the WIRE periodicity fixed for all nesting coils. Such a specific choice maintains the same direction of all current "wires" including nested coils of different types and therefore permits additional solutions resulting from interactions among all such nested windings. As an example we note that the periodicity of a dipole field (n=1) is identical to its wire period, but in a quadrupole (n=2) the field period is half of the wire period. If we express the wire period as $2l$, we can write $l = nL$, where L corresponds to the field periodicity associated with a 2n pole magnet. If we define a periodicity with respect to the wire as p_m we may write :

$$\boxed{\omega_m = \frac{(2m-1)\pi}{L} = n\frac{(2m-1)\pi}{l} = np_m}$$

and the condition for the helix incline is, (independent of n),

$$\tan \alpha = \frac{1}{p_m R} \quad ; \qquad m = j$$

We shall transform all fields and currents to a coordinate system (ρ,η,ξ) where $\hat{e}_\eta, \hat{e}_\xi$ lay in the developed plan of a cylinder of radius R and $\hat{e}_\rho$ is normal to it. In such a developed view the direction of the current flow is constant, pointing in the $\hat{e}_\xi$ direction. The direction normal to the flow in that plane $\hat{e}_\eta$, points towards the pole and is normal to the midplane (Fig. 1).

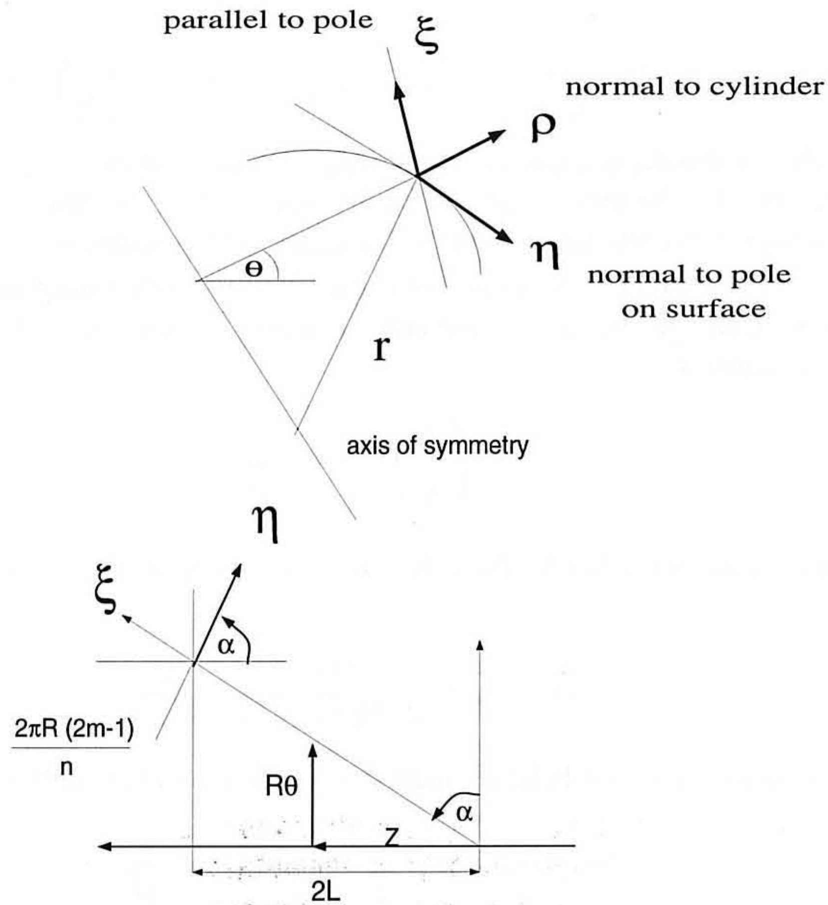

Figure 1 Frenet—Serret coordinate system along the helix path — top, and a developed view of the cylinder with ρ pointing out of the paper — bottom.

We introduce some additional relations and substitutions ,

$$\sin\alpha = \frac{1}{\sqrt{1+(p_m R)^2}} \quad ; \quad \cos\alpha = \frac{p_m R}{\sqrt{1+(p_m R)^2}}$$

and

$$\begin{aligned}
\eta &= \theta - p_m z \\
\xi &= \theta R p_m R + z \\
s_\eta &= R\theta\sin\alpha - z\cos\alpha = \frac{R\eta}{\sqrt{1+(p_m R)^2}} \\
s_\xi &= R\theta\cos\alpha + z\sin\alpha = \frac{\xi}{\sqrt{1+(p_m R)^2}}
\end{aligned}$$

Expressing the field and current in the (ρ,η,ξ) coordinate system,

$$<\vec{B}>|_{r=R} = \left\{\begin{array}{l} B_\rho \\ B_\theta\sin\alpha - B_z\cos\alpha \\ B_\theta\cos\alpha + B_z\sin\alpha \end{array}\right\} = \left\{\begin{array}{l} -\sum_{i=1}\sum_{m=1} i p_m G_{i,m} I_i^{'} \sin i\eta \\ -\frac{1}{2}\sum_{i=1}\sum_{m=1}\sqrt{1+(p_m R)^2}\frac{iG_{i,m}}{R}\frac{(I_i K_i)^{'}}{K_i^{'}}\cos i\eta \\ 0 \end{array}\right\}$$

and

$$\vec{J}_s|_{r=R} = \frac{1}{\mu_0}\left\{\begin{matrix} J_\rho \\ J_\eta \\ J_\xi \end{matrix}\right\} = \frac{1}{\mu_0}\left\{\begin{matrix} 0 \\ 0 \\ -\sum_{n=1}\sum_{m=1}\sqrt{1+(p_m R)^2}\,\frac{G_{n,m}}{p_m R^2}\frac{1}{K_n'}\cos n\eta \end{matrix}\right\}$$

where,

$$p_m = \frac{(2m-1)\pi}{l} \qquad and \qquad G_{n,m} = n!R^n\left(\frac{2}{np_m R}\right)^n B_{n,m}$$

Lorentz Force

Most electromagnets — dipoles, quads etc are built as single function magnets, in cases where combined function magnets are needed several single function magnets are superimposed. This is a direct result from the fact that we know how to wind single function magnets and lack the knowledge of winding multi function magnets in a single physical configuration (not superimposed). Therefore we focus on magnet n period m that carries a current density J and sum up all field contributions arising from magnets i period m with the same wire period *2l*. We express the contributions to the forces in two parts, one arising from a self field and the other arising from the cross interaction between all other fields and the same single function magnet n,m. At a later point we shall transform the stress components on magnet n,m from the $\hat{e}_\eta, \hat{e}_\xi$ direction, to the global coordinates ρ,θ,z. That stress may be of interest in cases where an imposed structural requirements is needed. The stress in terms of ρ, η, ξ within each individual coil n,m may be of more interest when a coil prestress is needed (such as in superconducting magnets).

The Lorentz force in a Frenet—Serret coordinate system result in two force components — f_ρ and f_η.

The Lorentz force on coil n,m is

$$\vec{f}_{n,m} = \vec{J}_{n,m} \times \vec{B} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\eta & \hat{e}_\xi \\ 0 & 0 & J_{\xi_{n,m}} \\ < B_\rho > & < B_\eta > & 0 \end{vmatrix}$$

$$\vec{f}_{n,m} = \left[-J_{\xi_{n,m}} < B_\eta >\right]\hat{e}_\rho + \left[J_{\xi_{n,m}} < B_\rho >\right]\hat{e}_\eta$$

Explicitly the force components are,

$$\boxed{\begin{aligned} f_{\eta_{n,m}} &= \frac{\sqrt{1+(p_m R)^2}}{\mu_0 R^2}\sum_i iG_{n,m}G_{i,m}\frac{I_i'(ip_m R)}{K_n'(np_m R)}\cos n\eta \sin i\eta \\ f_{\rho_{n,m}} &= -\frac{\left[1+(p_m R)^2\right]}{2\mu_0 R^2}\sum_i i\frac{G_{n,m}G_{i,m}}{p_m R}\frac{\left(I_i(ip_m R)K_i(ip_m R)\right)'}{K_n'(np_m R)K_i'(ip_m R)}\cos n\eta \cos i\eta \end{aligned}}$$

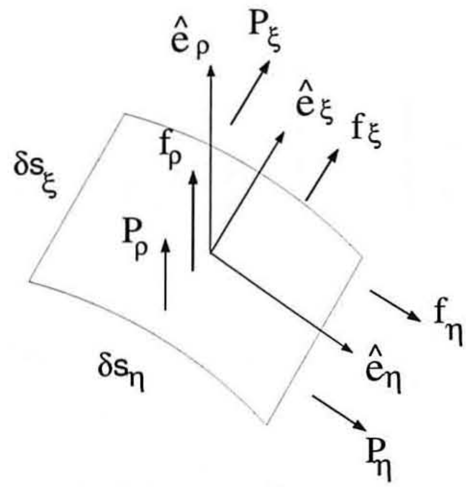

Figure 2 Forces on a current sheet.

Equilibrium — method I

The force equilibrium on a current surface element $\delta s_\eta \delta s_\xi$ requires that (Fig. 2)

$$P_\rho \hat{e}_\rho + f_\rho \delta s_\eta \delta s_\xi \hat{e}_\rho + d(P_\eta \hat{e}_\eta) + f_\eta \delta s_\eta \delta s_\xi \hat{e}_\eta + d\big(P_\xi \hat{e}_\xi\big) + f_\xi \delta s_\eta \delta s_\xi \hat{e}_\xi = 0$$

where the f's are the local Lorentz body forces and the P's are the stresses required to maintain equilibrium. ξ and η are in the direction of the principle axis.

We note that the change in the total force P in the η direction with respect to ξ is constant along ξ :

$$P_\eta^{'} = \frac{P_\eta}{\delta s_\xi} = f(\eta)$$

and similarly

$$P_\xi^{'} = \frac{P_\xi}{\delta s_\eta} = f(\xi)$$

$$P_\rho^{''} = \frac{P_\rho}{\delta s_\xi \delta s_\eta}$$

therefore, after dividing by the element area, the equilibrium equation can be written as :

$$P_\rho^{''} \hat{e}_\rho + f_\rho \hat{e}_\rho + \frac{d\Big(P_\eta^{'} \hat{e}_\eta\Big)}{ds_\eta} + f_\eta \hat{e}_\eta + \frac{d\Big(P_\xi^{'} \hat{e}_\xi\Big)}{ds_\xi} + f_\xi \hat{e}_\xi = 0$$

or

$$\boxed{P_\rho^{''} \hat{e}_\rho + f_\rho \hat{e}_\rho + \frac{dP_\eta^{'}}{ds_\eta} \hat{e}_\eta + P_\eta^{'} \frac{d\hat{e}_\eta}{ds_\eta} + f_\eta \hat{e}_\eta + \frac{dP_\xi^{'}}{ds_\xi} \hat{e}_\xi + P_\xi^{'} \frac{d\hat{e}_\xi}{ds_\xi} + f_\xi \hat{e}_\xi = 0}$$

Where P' is a force per unit length and P" is a force per unit area.

Introducing the geometric relations developed in Appendix C,

$$P_\rho''\hat{e}_\rho + f_\rho\hat{e}_\rho + \frac{dP_\eta'}{ds_\eta}\hat{e}_\eta - P_\eta'\frac{1}{R\left[1+(p_mR)^2\right]}\hat{e}_\rho + f_\eta\hat{e}_\eta + \frac{dP_\xi'}{ds_\xi}\hat{e}_\xi - P_\xi'\frac{(p_mR)^2}{R\left[1+(p_mR)^2\right]}\hat{e}_\rho + f_\xi\hat{e}_\xi = 0$$

If we assume that the force in the ξ direction per unit length is constant (e.g constant tension in a wire) and since the Lorentz force in the ξ direction is 0 (f_ξ=0) we may write:

$$P_\xi' = T_{\xi_{n,m}} = constant$$
$$\frac{dP_\xi'}{ds_\xi} = 0$$

Finally the differential equations for the stress components are:

$$\boxed{\begin{aligned} dP_\eta' &= -f_\eta ds_\eta \\ P_\rho'' &= -f_\rho + P_\eta'\frac{1}{R\left[1+(p_mR)^2\right]} + T_{\xi_{n,m}}\frac{(p_mR)^2}{R\left[1+(p_mR)^2\right]} = -f_\rho + \frac{P_\eta'}{(p_mR)\rho_\eta} + T_{\xi_{n,m}}\frac{1}{\rho_\xi} \end{aligned}}$$

Equilibrium — method II

This method is based on the equality between the divergence of the stress tensor P and the restoring Lorentz forces f,

$$(\nabla\cdot P)_i = -f_i$$

where,

$$(\nabla\cdot P)_i = \sum_j\left\{\frac{\partial P_{j,i}}{h_j\partial x_j} + \sum_l\left(\Gamma^j_{i,l}P_{j,l} + \Gamma^j_{j,l}P_{l,i}\right)\right\}$$
$$\Gamma^i_{j,k} = \frac{1}{h_jh_k}\left(\frac{\partial h_j}{\partial x_k}\delta^i_j - \frac{\partial h_k}{\partial x_j}\delta^i_k\right)$$

In the major coordinate system (ρ,η,ξ) we assume there is no shear stress — $P_{i,j} = 0$ for all $i \neq j$. Therefore, the divergence can be written,

$$(\nabla\cdot P)_i = \frac{\partial P_{i,i}}{h_i\partial x_i} + \sum_j\left(\Gamma^j_{i,j}P_{j,j} + \Gamma^j_{j,i}P_{i,i}\right)$$
$$\Gamma^j_{i,j} = \left\{\begin{matrix} 0 & i=j \\ -\frac{1}{h_ih_j}\frac{\partial h_j}{\partial x_i} & i\neq j \end{matrix}\right\}$$
$$\Gamma^j_{j,i} = \left\{\begin{matrix} 0 & i=j \\ \frac{1}{h_ih_j}\frac{\partial h_j}{\partial x_i} & i\neq j \end{matrix}\right\}$$

and finally,

$$(\nabla \cdot P)_i = \frac{1}{h_i}\left[\frac{\partial P_{i,i}}{\partial x_i} + \sum_{j\neq i}\frac{\partial \ln h_j}{\partial x_i}(P_{i,i} - P_{j,j})\right]$$

The components of the metric tensor are :

$$h_\rho = 1$$
$$h_\eta = \left|\frac{\partial r}{\partial \eta}\right| = \sqrt{\rho^2\left(\frac{\partial \theta}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2} = \frac{\rho}{\sqrt{1 + (p_m\rho)^2}}$$
$$h_\xi = \left|\frac{\partial r}{\partial \xi}\right| = \sqrt{\rho^2\left(\frac{\partial \theta}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2} = \frac{1}{\sqrt{1 + (p_m\rho)^2}}$$

On the "current sheet" we shall define P' as force per unit length (instead of per unit area). Since $h_j = f(x_1)$ and $\delta x_1 = \delta\rho \to 0$ we write$P_{i,i} = \lim_{\delta x_1 \to 0} \frac{P'_{i,i}}{\delta x_1}$, $i \neq 1$, therefore

$$(\nabla \cdot P)_1 \delta x_1 = \frac{1}{h_1}\left[P_{1,1} - \frac{\partial \ln h_2}{\partial x_1}P'_{2,2} - \frac{\partial \ln h_3}{\partial x_1}P'_{3,3}\right]$$
$$(\nabla \cdot P)_2 \delta x_1 = \frac{1}{h_2}\frac{\partial P'_{2,2}}{\partial x_2}$$
$$(\nabla \cdot P)_3 \delta x_1 = \frac{1}{h_3}\frac{\partial P'_{3,3}}{\partial x_3}$$

If we now substitute the derivatives,

$$\frac{\partial \ln h_2}{\partial x_1}\bigg|_{x1=R} = \frac{1}{R\left[1 + (p_m R)^2\right]}$$
$$\frac{\partial \ln h_3}{\partial x_1}\bigg|_{x1=R} = -\frac{(p_m R)^2}{R\left[1 + (p_m R)^2\right]}$$

we get,

$$P_{1,1} = -f_{x1} + \frac{1}{R\left[1 + (p_m R)^2\right]}P'_{2,2} - \frac{(p_m R)^2}{R\left[1 + (p_m R)^2\right]}P'_{3,3}$$
$$\frac{1}{h_2}\frac{\partial P'_{2,2}}{\partial x_2} = -f_{x2}$$
$$\frac{\partial P'_{3,3}}{\partial x_3} = 0$$

With $x_1=\rho=R$, $x_2=\eta$, $x_3=\xi$ and $P_{3,3}^{'} = constant = -T_{\xi_{n,m}}$ — where $T_{\xi_{n,m}}$ is the tension, or the force in the flow direction per unit length — we get a set of differential equations identical to that in part I,

$$P_\rho^{''} = -f_\rho + \frac{1}{R\left[1+(p_m R)^2\right]} S_\eta^{'} - \frac{(p_m R)^2}{R\left[1+(p_m R)^2\right]} T_{\xi_{n,m}}$$

$$dP_\eta^{'} = -\frac{R}{\sqrt{1+(p_m R)^2}} f_\eta d\eta$$

$$\frac{\partial T_{\xi_{n,m}}}{\partial \xi} = 0$$

Note that a specific choice was made in both sections in the way the current sheet is placed. We assume that an initial tension is applied to the windings while they are placed around the mandrel or bore tube. Under such circumstances an initial outward (radial) stress is exerted by the bore tube on the windings. We may wish to consider a case where a bore tube is placed outside the current sheet. Such a case permits placing the winding under compression, requiring a sign inversion of $P_{3,3}^{'} = T_{\xi_{n,m}}$. The case of no tension or compression is trivial $P_{3,3}^{'} = T_{\xi_{n,m}} = 0$.

Solution to $P_\eta^{'}$ and $P_\rho^{''}$

Proceeding with the solution to the stress equations we shall first solve the hoop stress in the η direction and then solve for the radial stress.

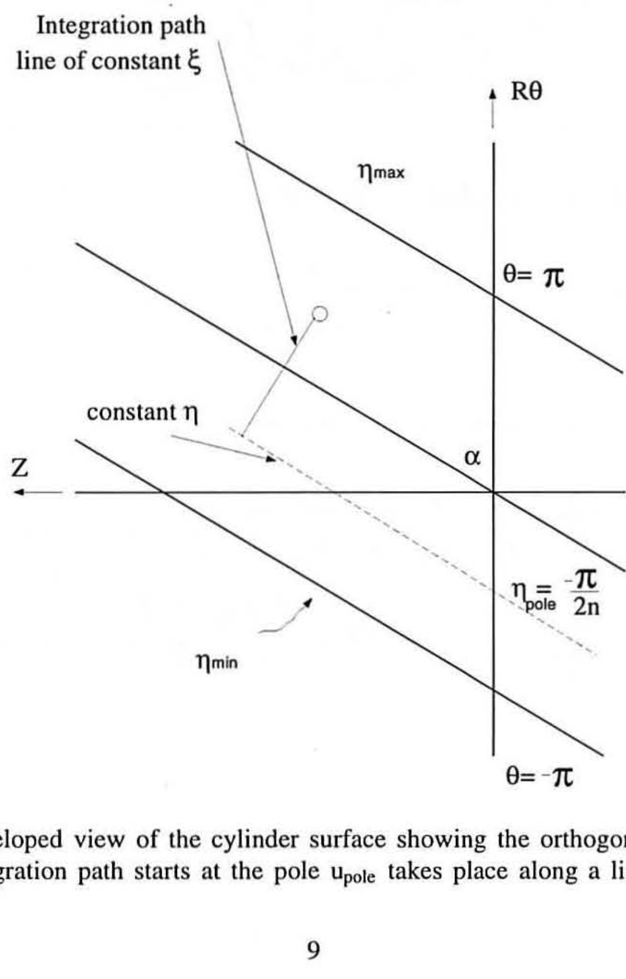

Figure 3 A developed view of the cylinder surface showing the orthogonal coordinates ξ and η. The integration path starts at the pole u_{pole} takes place along a line of constant ξ.

The general differential equation for $P_\eta^{'}$ is :

$$\frac{dP_\eta^{'}}{ds_\eta} = -f_\eta$$
$$= -\frac{\sqrt{1+(p_m R)^2}}{\mu_0 R^2} \sum_i i G_{n,m} G_{i,m} \frac{I_i^{'}(ip_m R)}{K_n^{'}(np_m R)} \cos n\eta \sin i\eta$$

With $ds_\eta = \frac{R}{\sqrt{1+(p_m R)^2}} d\eta$, the integration above is carried out along a constant ξ, commencing at the pole $\eta_{pole} = -\frac{\pi}{2n}$ (Fig. 3).

$$P_\eta^{'} = -\frac{1}{\mu_0 R} \sum_i i G_{n,m} G_{i,m} \frac{I_i^{'}(ip_m R)}{K_n^{'}(np_m R)} \int_{-\frac{\pi}{2n}}^{\eta} \cos n\eta^{'} \sin i\eta^{'} d\eta^{'}$$

For the case $n \neq i$,

$$\boxed{P_\eta^{'} = -\frac{1}{\mu_0 R} \sum_i \frac{i}{n^2 - i^2} G_{n,m} G_{i,m} \frac{I_i^{'}(ip_m R)}{K_n^{'}(np_m R)} \left[i \cos n\eta \cos i\eta + n \sin n\eta \sin i\eta - n \sin \frac{\pi i}{2n} \right]}$$

and for i=n,

$$\boxed{P_\eta^{'} = \frac{1}{2\mu_0 R} G_{n,m}^2 \frac{I_n^{'}(np_m R)}{K_n^{'}(np_m R)} \cos^2 n\eta}$$

We can now solve for the local radial pressure $P_\rho^{''}$

$$P_\rho^{''} = -f_\rho + P_\eta^{'} \frac{1}{R\left[1+(p_m R)^2\right]} + T_{\xi_{n,m}} \frac{(p_m R)^2}{R\left[1+(p_m R)^2\right]}$$

$$\boxed{\begin{aligned} P_\rho^{''} = & \frac{\left[1+(p_m R)^2\right]}{2\mu_0 R^2} \sum_i i \frac{G_{n,m} G_{i,m}}{p_m R} \frac{(I_i(ip_m R) K_i(ip_m R))^{'}}{K_n^{'}(np_m R) K_i^{'}(ip_m R)} \cos n\eta \cos i\eta \\ & - \frac{1}{\mu_0 R^2 \left[1+(p_m R)^2\right]} \sum_i \frac{i}{n^2 - i^2} G_{n,m} G_{i,m} \frac{I_i^{'}(ip_m R)}{K_n^{'}(np_m R)} \left[i \cos n\eta \cos i\eta + n \sin n\eta \sin i\eta - n \sin \frac{\pi i}{2n} \right] \\ & + \frac{(p_m R)^2}{R\left[1+(p_m R)^2\right]} T_{\xi_{i,m}} \end{aligned}}$$

and for the special case n=i :

$$
\boxed{
\begin{aligned}
P_\rho'' = & \frac{\left[1+(p_m R)^2\right]}{2\mu_0 R^2} n \frac{G_{n,m}^2}{p_m R} \frac{\left(I_n(np_m R)K_n(np_m R)\right)'}{K_n'(np_m R)K_n'(np_m R)} \cos^2 n\eta \\
& + \frac{1}{2\mu_0 R^2\left[1+(p_m R)^2\right]} G_{n,m}'^2 \frac{I_n'(np_m R)}{K_n'(np_m R)} \cos^2 n\eta \\
& + \frac{(p_m R)^2}{R\left[1+(p_m R)^2\right]} T_{\xi_{n,m}}
\end{aligned}
}
$$

Transforming the stress tensor to the global coordinates ρ,θ,z

Since ξ and η are principal axis, the shear stress $\tau_{\xi,\eta} = 0$ and the relations[e] for the stress components in ρ,θ,z are :

$$
\begin{aligned}
P_\theta' &= P_\eta' \sin^2\alpha + P_\xi' \cos^2\alpha \\
P_z' &= P_\eta' \cos^2\alpha + P_\xi' \sin^2\alpha \\
\tau_{z\theta}' &= -\frac{P_\eta' - P_\xi'}{2} \sin 2\alpha \\
P_\rho' &= P_\rho'
\end{aligned}
$$

We shall make use, as before, of

$$
\sin\alpha = \frac{1}{\sqrt{1+(p_m R)^2}} \quad ; \quad \cos\alpha = \frac{p_m R}{\sqrt{1+(p_m R)^2}} \quad ; \quad \sin 2\alpha = \frac{2p_m R}{1+(p_m R)^2}
$$

[e] Theory of Elasticity — Timoshenko and Goodier, pp. 185–186.

So that the stress for the case $n \neq i$ is,

$$
\begin{aligned}
P_\theta' &= \begin{aligned}[t] & -\frac{1}{\mu_0 R}\frac{1}{\left[1+(p_m R)^2\right]}\sum_i \frac{iG_{n,m}G_{i,m}}{n^2-i^2}\frac{I_i'(ip_m R)}{K_n'(np_m R)}\left[i\cos n\eta\cos i\eta + n\sin n\eta\sin i\eta - n\sin\frac{\pi i}{2n}\right] \\ & +\frac{(p_m R)^2}{1+(p_m R)^2}\sum_i T_{\xi_{i,m}} \end{aligned} \\
P_z' &= \begin{aligned}[t] & -\frac{1}{\mu_0 R}\frac{(p_m R)^2}{\left[1+(p_m R)^2\right]}\sum_i \frac{iG_{n,m}G_{i,m}}{n^2-i^2}\frac{I_i'(ip_m R)}{K_n'(np_m R)}\left[i\cos n\eta\cos i\eta + n\sin n\eta\sin i\eta - n\sin\frac{\pi i}{2n}\right] \\ & +\frac{1}{1+(p_m R)^2}\sum_i T_{\xi_{i,m}} \end{aligned} \\
\tau_{z,\theta} &= \begin{aligned}[t] & \frac{1}{\mu_0 R}\frac{p_m R}{\left[1+(p_m R)^2\right]}\sum_i \frac{iG_{n,m}G_{i,m}}{n^2-i^2}\frac{I_i'(ip_m R)}{K_n'(np_m R)}\left[i\cos n\eta\cos i\eta + n\sin n\eta\sin i\eta - n\sin\frac{\pi i}{2n}\right] \\ & +\frac{p_m R}{1+(p_m R)^2}\sum_i T_{\xi_{i,m}} \end{aligned} \\
P_\rho'' &= \sum_i P_\rho''
\end{aligned}
$$

and for n=i

$$
\begin{aligned}
P_\theta' &= \frac{1}{2\mu_0 R}\frac{1}{\left[1+(p_m R)^2\right]}G_{n,m}^2\frac{I_n'(np_m R)}{K_n'(np_m R)}\cos^2 n\eta + \frac{(p_m R)^2}{1+(p_m R)^2}T_{\xi_{n,m}} \\
P_z' &= \frac{1}{2\mu_0 R}\frac{(p_m R)^2}{\left[1+(p_m R)^2\right]}G_{n,m}^2\frac{I_n'(np_m R)}{K_n'(np_m R)}\cos^2 n\eta + \frac{1}{1+(p_m R)^2}T_{\xi_{n,m}} \\
\tau_{z,\theta} &= -\frac{1}{2\mu_0 R}\frac{p_m R}{\left[1+(p_m R)^2\right]}G_{n,m}^2\frac{I_n'(np_m R)}{K_n'(np_m R)}\cos^2 n\eta + \frac{p_m R}{1+(p_m R)^2}T_{\xi_{n,m}} \\
P_\rho'' &= P_{\rho_{n,m}}''
\end{aligned}
$$

We note that in the 2d limiting case :

$$
\begin{aligned}
P_\theta' &= P_{\eta,2d}' \\
P_z' &= T_{\xi_n} \\
\tau_{\theta,z} &= 0
\end{aligned}
$$

Example 1 — Combined helical Dipole and Quad

We illustrate and apply the solution to a combined helical dipole and quad, having both a single and identical wire period p_1.

Helical dipole n=1 and quad i=2

The stress on the dipole will arise from the self field n=1 plus a contribution from the quadrupole i=2 coil (Fig. 4). We shall substitute n=1 and i=2 into the stress solution and assume that all Bessel functions and derivatives are with respect to their corresponding argument,

$$p_1 = \frac{\pi}{l}$$

$$G_{1,1} = R\frac{2B_{1,1}}{p_1 R} \text{ where } B_{1,1} \text{ is the dipole} \quad , \quad G_{2,1} = \frac{R^2 2B_{2,1}}{(p_1 R)^2} \text{ where } 2B_{2,1} \text{ is the gradient}$$

$$P_\eta' = \frac{1}{2\mu_0 R} G_{1,1}^2 \frac{I_1'(p_1 R)}{K_1'(p_1 R)} \cos^2\eta + \frac{4}{3\mu_0 R} G_{1,1} G_{2,1} \frac{I_2'(2p_1 R)}{K_1'(p_1 R)} \cos^3\eta$$

$$\begin{aligned} P_\rho'' = {} & \frac{\left[1+(p_1 R)^2\right]}{2\mu_0 R^2} \frac{G_{1,1}^2}{p_1 R} \frac{\left(I_1(p_1 R) K_1(p_1 R)\right)'}{K_1'(p_1 R) K_1'(p_1 R)} \cos^2\eta \\ & + 2\frac{\left[1+(p_1 R)^2\right]}{2\mu_0 R^2} \frac{G_{1,1} G_{2,1}}{p_1 R} \frac{\left(I_2(2p_1 R) K_2(2p_1 R)\right)'}{K_1'(p_1 R) K_2'(2p_1 R)} \cos\eta \cos 2\eta \\ & + \frac{1}{2\mu_0 R^2 \left[1+(p_1 R)^2\right]} G_{1,1}^2 \frac{I_1'(p_1 R)}{K_1'(p_1 R)} \cos^2\eta \\ & + \frac{1}{\mu_0 R^2 \left[1+(p_1 R)^2\right]} \frac{4}{3} G_{1,1} G_{2,1} \frac{I_2'(2p_1 R)}{K_1'(p_1 R)} \cos^3\eta \\ & + \frac{(p_1 R)^2}{R\left[1+(p_1 R)^2\right]} T_{\xi_{1,1}} \end{aligned}$$

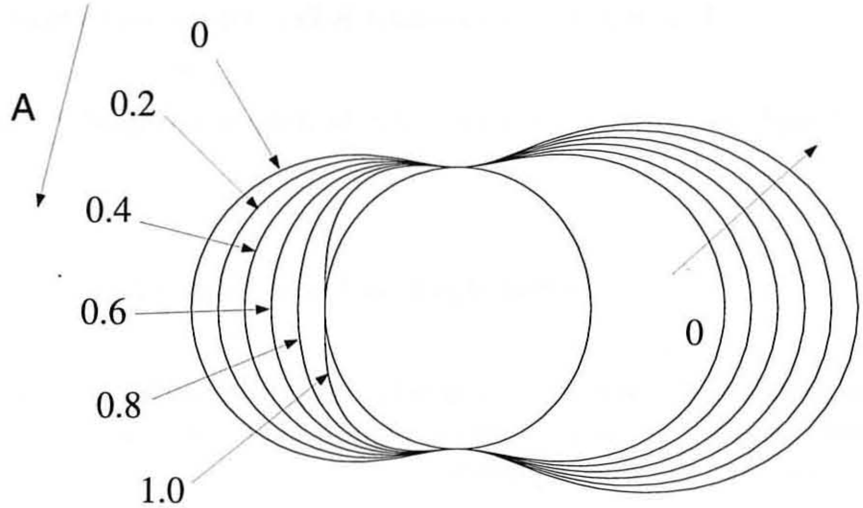

Figure 4 A polar plot of the magnitude of P_η' for n=1 and i=2
$P_\eta' = \cos^2\eta + A\cos^3\eta \quad ; \quad \eta = \theta - \frac{\pi}{l}$, A=0 corresponds to a single function dipole (no quad)

Helical quadrupole n=2 and dipole i=1

Similarly the stress on the quad arise from the self field n=2, with a contribution from the dipole i=1.

$$p_1 = \frac{\pi}{l}$$

$$G_{1,1} = R\frac{2B_{1,1}}{p_1 R} \text{ where } \mathrm{B}_{1,1} \text{ is the dipole} \quad , \quad G_{2,1} = \frac{R^2 2B_{2,1}}{(p_1 R)^2} \text{ where } 2\mathrm{B}_{2,1} \text{ is the gradient}$$

$$P_\eta' = \frac{1}{2\mu_0 R} G_{2,1}^2 \frac{I_2'(2p_1R)}{K_2'(2p_1R)} \cos^2 2\eta + \frac{1}{3\mu_0 R} G_{2,1}G_{1,1}\frac{I_1'(p_1R)}{K_2'(2p_1R)}\left(2\cos^3\eta - 3\cos\eta + \sqrt{2}\right)$$

$$\begin{aligned}
P_\rho'' = {} & 2\frac{\left[1+(p_1R)^2\right]}{2\mu_0 R^2}\frac{G_{2,1}^2}{p_1 R}\frac{\left(I_2(2p_1R)K_2(2p_1R)\right)'}{K_2'(2p_1R)K_2'(2p_1R)}\cos^2 2\eta \\
& + \frac{\left[1+(p_1R)^2\right]}{2\mu_0 R^2}\frac{G_{2,1}G_{1,1}}{p_1 R}\frac{\left(I_1(p_1R)K_1(p_1R)\right)'}{K_2'(2p_1R)K_1'(p_1R)}\cos 2\eta\cos\eta \\
& + \frac{1}{2\mu_0 R^2\left[1+(p_1R)^2\right]}G_{2,1}^2\frac{I_2'(2p_1R)}{K_2'(2p_1R)}\cos^2 2\eta \\
& + \frac{1}{3\mu_0 R^2\left[1+(p_1R)^2\right]}G_{2,1}G_{1,1}\frac{I_1'(p_1R)}{K_2'(2p_1R)}\left(2\cos^2\eta - 3\cos\eta + \sqrt{2}\right) \\
& + \frac{(p_1R)^2}{R\left[1+(p_1R)^2\right]}T_{\xi_{2,1}}
\end{aligned}$$

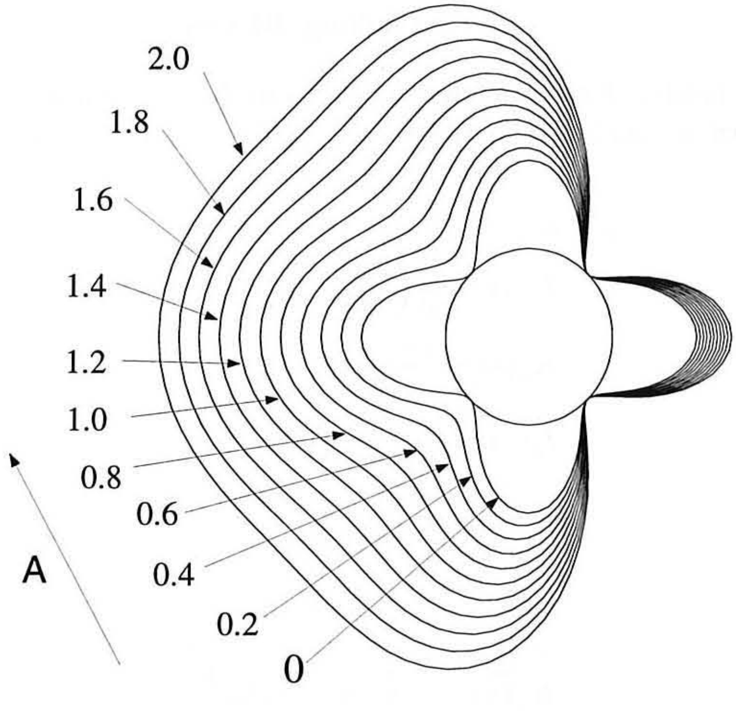

Figure 5 A polar plot of the magnitude of $P_\eta^{'}$ for n=2 and i=1
$P_\eta^{'} = \cos^2\eta + A\left[2\cos^3\eta - 3\cos\eta + \sqrt{2}\right]$, B=0 corresponds to a single function quad (no dipole)

Example 2

In a previous report[f] we have proposed a possible design of a thick superconducting helical dipole wiggler (n=1 only) that has a short sample central field of 1.92 T and 2.48 T corresponding to two different types of NbTi superconductors. Other parameters associated with that design were: an inner diameter of 6.5 mm that has an equivalent radius of R=3.836 mm corresponding to a thin coil approximation and a 27 mm period ($2l$) . Therefore, with $s = \frac{\pi R}{l} = 0.892678$ we calculate : $I_1(s) = 0.49$, $I_1^{'}(s) = 0.654$, $K_1(s) = 0.735$, $K_1^{'}(s) = -1.3$ and get:

$$\frac{1}{2\mu_0 R}\frac{G_{n,m}^2 I_n^{'}}{K_n^{'}} = \frac{2RB_{1,1}^2}{\mu_0}\frac{1}{s^2}\frac{I_1^{'}}{K_1^{'}} = -3854.28B_{11}^2\left(\frac{N}{m}\right)$$

leading to the following results :

$$P_\eta^{'} = -3854.28B_{1,1}^2\cos^2\eta \ \ (N/m)$$

$$P_\rho^{''} = -3854.28\frac{B_{1,1}^2}{R}\left[(1+s^2)\frac{(I_1K_1)^{'}}{sI_1^{'}K_1^{'}} + \frac{1}{1+s^2}\right]\cos^2\eta = -9.3101E5B_{11}^2\cos^2\eta \ \ (N/m^2)$$

The maximum force is therefore applied at the midplane (η=0), where for $B_{1,1}$=1.92 T:

$$P_\eta^{'} = -1.42084E4 \ \ (N/m) = -\ 81\ (lb/inch)$$

$$P_\rho^{''} = -\ 3.432\ (MPascal) \ \ = -\ 498\ (psi)$$

and for $B_{1,1}$=2.48 T:

$$P_\eta^{'} = -2.370536E4 \ \ (N/m) = -135\ (lb/inch)$$

$$P_\rho^{''} = -\ 5.72608\ (MPascal) \ \ \ = -830\ (psi)$$

[f] A Superconducting Helical Undulator for Short Wavelength FELs. — S.Caspi, SC-MAG-475, LBID-2052, Lawrence Berkeley Lab., September 19, 1994.

Limiting 2D case

We can reduce the results of general force equations to the more familiar 2D case by extending the period $2L \to \infty$. We note that for such a limit when $s = \omega_m R = np_m R \to 0$ (as well as $s = \omega_j R = ip_j R \to 0$) and make use of

$$
\begin{aligned}
& s \longrightarrow 0 \\
& I_n(s) \sim \frac{1}{n!}\left(\frac{s}{2}\right)^n \\
& K_n(s) \sim \frac{(n-1)!}{2}\left(\frac{s}{2}\right)^{-n} \\
& I_n'(s) \sim \frac{1}{2(n-1)!}\left(\frac{s}{2}\right)^{n-1} \\
& K_n'(s) \sim -\frac{n!}{4}\left(\frac{s}{2}\right)^{-(n+1)} \\
& I_n'(s)K_n'(s) \sim -\frac{n}{2s^2} \\
& \frac{I_n'(s)}{K_n'(s)} \sim -\frac{2}{n!(n-1)!}\left(\frac{s}{2}\right)^{2n} \\
& lims_{s\to 0}\frac{G_{n,m}^2 I_n'(s)}{K_n'(s)} \to -2nR^{2n}B_n^2 \\
& lims_{s\to 0}\frac{G_{n,m}G_{i,j}I_i'(s_j)}{K_n'(s_n)} \to -2nR^{n+i}B_nB_i \\
& lims_{s\to 0}[I_nK_n]' \to 0
\end{aligned}
$$

reducing the stress in a helix to the 2D expressions :

$$
\boxed{
\begin{aligned}
P_\theta' &= lim_{s\to 0}P_\eta' = -\frac{nR^{2n-1}}{\mu_0}\left\{B_n^2\cos^2 n\theta + \sum_{i\neq n}\frac{2iR^{i-n}B_nB_i}{i^2-n^2}\left[i\cos i\theta\cos n\theta + n\sin i\theta\sin n\theta - n\sin\frac{\pi i}{2n}\right]\right\} \\
P_\rho'' &= lim_{s\to 0}P_\rho' = \frac{P_\eta'}{R} \\
&= -\frac{nR^{2(n-1)}}{\mu_0}\left\{B_n^2\cos^2 n\theta + \sum_{i\neq n}\frac{2iR^{i-n}B_nB_i}{i^2-n^2}\left[i\cos i\theta\cos n\theta + n\sin i\theta\sin n\theta - n\sin\frac{\pi i}{2n}\right]\right\}
\end{aligned}
}
$$

From the linear current density relation

$$B_n = \frac{\mu_0 J_{0z}}{2nR^{n-1}}$$

we may write the expression for the field B_{max} corresponding to the field just inside the windings at radius r=R

$$B_{max,n} = nB_nR^{n-1} = \frac{\mu_0 J_{0z}}{2}$$

Note that :

$$nB_n = \begin{Bmatrix} 1B_1 = B_1 \text{ dipole field} \\ 2B_2 = G \text{ quad gradient} \\ 3B_3 = S \text{ Sextupole} \\ \ldots \end{Bmatrix}$$

therefore in a dipole magnet B_{max} is the dipole field, in a quadrupole B_{max}=Gradient*R=$2B_2$*R, and in a sextupole B_{max}=Sextupole*R^2=$3B_3$*R^2 etc. The above 2D stress equations can be expressed in terms of the maximum field or in terms of current density :

$$\boxed{\begin{aligned} P_\theta^{'} &= -\frac{R}{\mu_0}\left\{\frac{B^2_{max,n}}{n}\cos^2 n\theta + \sum_{i\neq n}\frac{2B_{max,n}B_{max,i}}{i^2-n^2}\left[i\cos i\theta\cos n\theta + n\sin i\theta\sin n\theta - n\sin\frac{\pi i}{2n}\right]\right\} \\ P_\rho^{''} &= \frac{P_\eta^{'}}{R} \\ &= -\frac{1}{\mu_0}\left\{\frac{B^2_{max,n}}{n}\cos^2 n\theta + \sum_{i\neq n}\frac{2B_{max,n}B_{max,i}}{i^2-n^2}\left[i\cos i\theta\cos n\theta + n\sin i\theta\sin n\theta - n\sin\frac{\pi i}{2n}\right]\right\} \end{aligned}}$$

$$\boxed{\begin{aligned} P_\theta^{'} &= -\frac{\mu_0 J_{0z,n} R}{4}\left\{\frac{J_{0z,n}}{n}\cos^2 n\theta + \sum_{i\neq n}\frac{2J_{0z,i}}{i^2-n^2}\left[i\cos i\theta\cos n\theta + n\sin i\theta\sin n\theta - n\sin\frac{\pi i}{2n}\right]\right\} \\ P_\rho^{''} &= \frac{P_\eta^{'}}{R} \\ &= -\frac{\mu_0 J_{0z,n}}{4}\left\{\frac{J_{0z,n}}{n}\cos^2 n\theta + \sum_{i\neq n}\frac{2J_{0z,i}}{i^2-n^2}\left[i\cos i\theta\cos n\theta + n\sin i\theta\sin n\theta - n\sin\frac{\pi i}{2n}\right]\right\} \end{aligned}}$$

We note that for single function magnets the above expressions are in agreement with the 2D analysis, as it should be[g].

---

[g] Forces in a Thin Cosine nθ Winding — R.Meuser, Engineering Note M5266, November 15, 1978.

Example A. n=1 , i=2 (stress in a dipole with a superimposed quad)

$$P_\theta^{'} = \left\{ \begin{array}{l} -\dfrac{R}{\mu_0}\left(B_1^2\cos^2\theta + \dfrac{8}{3}RB_1B_2\cos^3\theta\right) \\ or \\ -\dfrac{R}{\mu_0}\left(B_{max,1}^2\cos^2\theta + \dfrac{4}{3}B_{max,1}B_{max,2}\cos^3\theta\right) \\ or \\ -\dfrac{\mu_0 R}{4}\left(J_{0,1}^2\cos^2\theta + \dfrac{4}{3}J_{0,1}J_{0,2}\cos^3\theta\right) \end{array} \right\}$$

$$P_\rho^{''} = \left\{ \begin{array}{l} -\dfrac{1}{\mu_0}\left(B_1^2\cos^2\theta + \dfrac{8}{3}RB_1B_2\cos^3\theta\right) \\ or \\ -\dfrac{1}{\mu_0}\left(B_{max,1}^2\cos^2\theta + \dfrac{4}{3}B_{max,1}B_{max,2}\cos^3\theta\right) \\ or \\ -\dfrac{\mu_0}{4}\left(J_{0,1}^2\cos^2\theta + \dfrac{4}{3}J_{0,1}J_{0,2}\cos^3\theta\right) \end{array} \right\}$$

Example B. n=2 , i=1 (stress in a quad with a superimposed dipole)

$$P_\theta^{'} = \left\{ \begin{array}{l} -\dfrac{2R^3}{\mu_0}\left[B_2^2\cos^2 2\theta - \dfrac{2}{3}\dfrac{B_2B_1}{R}\left(\cos\theta + \sin 2\theta\sin\theta - \sqrt{2}\right)\right] \\ or \\ -\dfrac{R}{\mu_0}\left[\dfrac{B_{max,2}^2}{2}\cos^2 2\theta - \dfrac{2}{3}B_{max,2}B_{max,1}\left(\cos\theta + \sin 2\theta\sin\theta - \sqrt{2}\right)\right] \\ or \\ -\dfrac{\mu_0 R}{8}\left[J_{0,2}^2\cos^2 2\theta - \dfrac{4}{3}J_{0,2}J_{0,1}\left(\cos\theta + \sin 2\theta\sin\theta - \sqrt{2}\right)\right] \end{array} \right\}$$

$$P_\rho^{''} = \left\{ \begin{array}{l} -\dfrac{2R^2}{\mu_0}\left[B_2^2\cos^2 2\theta - \dfrac{2}{3}\dfrac{B_2B_1}{R}\left(\cos\theta + \sin 2\theta\sin\theta - \sqrt{2}\right)\right] \\ or \\ -\dfrac{1}{\mu_0}\left[\dfrac{B_{max,2}^2}{2}\cos^2 2\theta - \dfrac{2}{3}B_{max,2}B_{max,1}\left(\cos\theta + \sin 2\theta\sin\theta - \sqrt{2}\right)\right] \\ or \\ -\dfrac{\mu_0}{8}\left[J_{0,2}^2\cos^2 2\theta - \dfrac{4}{3}J_{0,2}J_{0,1}\left(\cos\theta + \sin 2\theta\sin\theta - \sqrt{2}\right)\right] \end{array} \right\}$$

Average stress or Magnetic Pressure (2D)

We can define the "magnetic pressure" associated with multipole magnets by integrating the local stress and dividing by the area of integration. With the help of :

$$\int_0^{2\pi} \cos^2 n\theta d\theta = \pi \quad ; \quad \int_0^{2\pi} \cos i\theta \cos n\theta d\theta = 0 \quad ; \quad \int_0^{2\pi} \sin i\theta \sin n\theta d\theta = 0$$

we express the average radial pressure $\overline{P_{\rho,n}}''$ as

$$\overline{P_{\rho,n}}'' = \frac{l \int_0^{2\pi} P_\rho'' R d\theta}{l 2\pi R} = -\frac{n R^{2(n-1)} B_n^2}{2\mu_0}$$
$$= -\frac{B_{max,n}^2}{2\mu_0 n}$$
$$= -\frac{\mu_0 J_{0z}^2}{8n}$$

We also note that in a single function magnet the stored energy and pressure can be written as :

$$\boxed{e_{n,2d} = \frac{\mu_0 J_{z0}^2}{4n} = \frac{B_{max,n}^2}{\mu_0 n} \qquad \bar{P}_{\rho_n,2d}'' = \frac{e_{n,2d}}{2}}$$

Units :

In MKS units :

I : amp
B : Tesla (or Weber/meter^2)
L : meter
F : newton

$\frac{N}{m} = T \cdot A \quad , \quad \frac{J}{m^3} = \frac{T \cdot A}{m}$

Useful conversions :

multiply (N/m) by 5.710174e-3 to get (lb/inch)
multiply (N) by 0.22481 to get (lb)
multiply (N/m^2) by 1.450377e—4 to get (psi)
multiply (psi) by 6.8947e-3 to get (MPascal)

Appendix A Field Components

The field components in the region interior to the windings r<R are :

$$B_r = -\frac{\partial V}{\partial r} = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n'(\omega_m r)\sin(n\theta - \omega_m z)$$
$$B_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{1}{r} I_n(\omega_m r)\cos(n\theta - \omega_m z)$$
$$B_z = -\frac{\partial V}{\partial z} = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n(\omega_m r)\cos(n\theta - \omega_m z)$$

The field components in the region exterior to the windings r>R are :

$$B_r = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I_n'(\omega_m R)}{K_n'(\omega_m R)} K_n'(\omega_m r)\sin(n\theta - \omega_m z)$$
$$B_\theta = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{I_n'(\omega_m R)}{K_n'(\omega_m R)}\frac{1}{r} K_n(\omega_m r)\cos(n\theta - \omega_m z)$$
$$B_z = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I_n'(\omega_m R)}{K_n'(\omega_m R)} K_n(\omega_m r)\cos(n\theta - \omega_m z)$$

Appendix B Lorentz Force on a Surface of Discontinuity

The Lorentz force density on a thin surface of discontinuity[h] (per unit area s) may be expressed as given by

$$\frac{d\vec{F}}{dS} = \vec{J}_s \times < \vec{B} > \tag{1}$$

where $< \vec{B} >$ denotes the average magnetic field at the surface $< \vec{B} >= \frac{\vec{B}_1 + \vec{B}_2}{2}$ and $\vec{J}_s$ corresponds to the surface current density. With $\hat{n}$ corresponding to a unit vector normal to the surface, the current density may be expressed as

$$\vec{J}_s = -\left(\vec{J}_s \times \hat{n}\right) \times \hat{n}$$

and since

$$\vec{J}_s \times \hat{n} = \frac{1}{\mu_0} \delta \vec{B}$$

where

$$\delta \vec{B} = \vec{B}_2 - \vec{B}_1$$

the force density (Eq. 1) may be written in terms of field

$$\frac{d\vec{F}}{dS} = \frac{1}{\mu_0}\left(\hat{n} \times \delta \vec{B}\right) \times < \vec{B} >$$

and reduce, with the aid of the vector identity ($(A \times B) \times C = B(A \cdot C) - A(B \cdot C)$), to

$$\frac{d\vec{F}}{dS} = \frac{1}{\mu_0}\left[\delta \vec{B}\left(\hat{n} \cdot < \vec{B} >\right) - \hat{n}\left(\delta \vec{B} \cdot < \vec{B} >\right)\right]$$

Specifically, in cylindrical coordinates with a surface of discontinuity at r=R we write

$$\vec{f} = \frac{d\vec{F}}{dS} = \frac{1}{\mu_0}[-(< B_\theta > \delta B_\theta + < B_z > \delta B_z)\hat{e}_r + < B_r > \delta B_\theta \hat{e}_\theta + < B_r > \delta B_z \hat{e}_z]|_{r=R} \tag{2}$$

We note that with B_r continuous at r=R we may write $B_r =< B_r >$ and $\delta B < B >= \frac{B_2^2 - B_1^2}{2}$ so that the Lorentz force is

$$\vec{f} = \frac{d\vec{F}}{dS} = \frac{1}{\mu_0}\left[-\frac{1}{2}\left(\vec{B_2^2} - \vec{B_1^2}\right)\hat{e}_r + B_r \delta B_\theta \hat{e}_\theta + B_r \delta B_z \hat{e}_z\right]|_{r=R}$$

where $\vec{B_i}^2 = \sum_{i=1}^{3} B_i^2$.

[h] Utility of the Maxwell Stress Tensor for Computing Magnetic Forces — L.Jackson Laslett, Lawerence Berkeley Laboratory, report ERAN-160, August 24 1971.

For a helical magnet the combined fields can be written as

$$
< \vec{B} > |_{r=R} = \begin{Bmatrix} -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n^{'} \sin\left(n\theta - \omega_m z\right) \\ -\frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{nG_{n,m}}{R}\frac{\left(I_n K_n\right)^{'}}{K_n^{'}} \cos\left(n\theta - \omega_m z\right) \\ \frac{1}{2}\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{\left(I_n K_n\right)^{'}}{K_n^{'}} \cos\left(n\theta - \omega_m z\right) \end{Bmatrix}
$$

and the difference of such fields (with the help of the Wronskian $I_n K_n^{'} - I_n^{'} K_n = -\frac{1}{(\omega_m R)}$) as

$$
\delta\vec{B}|_{r=R} = \begin{Bmatrix} 0 \\ -\sum_{n=1}\sum_{m=1} \frac{nG_{n,m}}{R}\frac{1}{(\omega_m R)}\frac{1}{K_n^{'}} \cos\left(n\theta - \omega_m z\right) \\ -\sum_{n=1}\sum_{m=1} \frac{G_{n,m}}{R}\frac{1}{K_n^{'}} \cos\left(n\theta - \omega_m z\right) \end{Bmatrix}
$$

where

$$
\omega_m = \frac{(2m-1)\pi}{L} \qquad and \qquad G_{n,m} = n!R^n\left(\frac{2}{\omega_m R}\right)^n B_{n,m}
$$

The resulting current density is therefore

$$
\vec{J_s}|_{r=R} = \frac{1}{\mu_0}\begin{Bmatrix} 0 \\ -\sum_{n=1}\sum_{m=1} \frac{G_{n,m}}{R}\frac{1}{K_n^{'}} \cos\left(n\theta - \omega_m z\right) \\ -\sum_{n=1}\sum_{m=1} \frac{nG_{n,m}}{\omega_m R^2}\frac{1}{K_n^{'}} \cos\left(n\theta - \omega_m z\right) \end{Bmatrix}
$$

We consider the term $(\omega_n R)$ to be the argument of all Modified Bessel functions I_n and K_n, and all derivatives of such functions taken to be with respect to that argument.

Appendix C Frenet—Serret coordinate system

To derive the unit vectors associated with the helix geometry (Fig. 1) we need to introduce the radius vector $\vec{r}$:

$$\tan \alpha = \frac{n}{\omega_m R} = \frac{1}{p_m R}$$

so that

$$z = R\theta \tan \alpha = \frac{\theta}{p_m}$$

therefore the radius vector along the helix is :

$$\vec{r}(\theta) = R \cos \theta \hat{\mathrm{i}} + R \sin \theta \hat{\mathrm{j}} + \frac{\theta}{p_m} \hat{\mathrm{k}}$$

From

$$\frac{d\vec{r}}{d\theta} = -R \sin \theta \hat{\mathrm{i}} + R \cos \theta \hat{\mathrm{j}} + \frac{1}{p_m} \hat{\mathrm{k}}$$

$$\left| \frac{d\vec{r}}{d\theta} \right| = R\sqrt{1 + \tan^2 \alpha} = \frac{ds_\xi}{d\theta}$$

The unit vector in the helix (flow) direction is

$$\hat{\mathrm{e}}_\xi = \frac{d\vec{r}}{ds_\xi} = \frac{\frac{d\vec{r}}{d\theta}}{\frac{ds_\xi}{d\theta}} = \frac{1}{\sqrt{1 + \tan^2 \alpha}} \left(-\sin \theta \hat{\mathrm{i}} + \cos \theta \hat{\mathrm{j}} + \frac{1}{p_m R} \hat{\mathrm{k}} \right)$$

and since $\hat{\mathrm{e}}_\theta = -\sin \theta \hat{\mathrm{i}} + \cos \theta \hat{\mathrm{j}}$

$$\hat{\mathrm{e}}_\xi = \frac{1}{\sqrt{1 + \tan^2 \alpha}} (\hat{\mathrm{e}}_\theta + \tan \alpha \hat{\mathrm{e}}_z)$$

We show that for a helix the change in the unit vector $\hat{\mathrm{e}}_\xi$ with s_ξ,

$$\frac{d\hat{\mathrm{e}}_\xi}{ds_\xi} = K \hat{\mathrm{e}}_\rho$$

will result in a constant radius of curvature K :

$$\frac{d\hat{\mathrm{e}}_\xi}{d\theta} = \frac{1}{\sqrt{1 + \tan^2 \alpha}} \left(-\cos \theta \hat{\mathrm{i}} - \sin \theta \hat{\mathrm{j}} \right) = -\frac{1}{\sqrt{1 + \tan^2 \alpha}} \hat{\mathrm{e}}_\rho$$

$$\frac{d\hat{\mathrm{e}}_\xi}{ds_\xi} = \frac{\frac{d\hat{\mathrm{e}}_\xi}{d\theta}}{\frac{ds_\xi}{d\theta}} = -\frac{1}{R(1 + \tan^2 \alpha)} \left(\cos \theta \hat{\mathrm{i}} + \sin \theta \hat{\mathrm{j}} \right)$$

therefore

$$|K||\hat{\mathrm{e}}_\rho| = K = \left| \frac{d\hat{\mathrm{e}}_\xi}{ds_\xi} \right| = \frac{1}{R(1 + \tan^2 \alpha)} = \frac{1}{\rho_\xi}$$

The unit vector in the η direction is

$$\hat{\mathrm{e}}_\eta = \hat{\mathrm{e}}_\xi \times \hat{\mathrm{e}}_\rho = \frac{1}{\sqrt{1 + \tan^2 \alpha}} \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ -\sin \theta & \cos \theta & \tan \alpha \\ \cos \theta & \sin \theta & 0 \end{vmatrix} = \frac{1}{\sqrt{1 + \tan^2 \alpha}} \left(-\sin \theta \tan \alpha \hat{\mathrm{i}} + \cos \theta \tan \alpha \hat{\mathrm{j}} - \hat{\mathrm{k}} \right)$$

or

$$\hat{e}_\eta = \frac{1}{\sqrt{1+\tan^2\alpha}}(\tan\alpha\hat{e}_\theta - \hat{e}_z)$$

Since

$$\frac{d\hat{e}_\eta}{d\theta} = -\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}\left(\cos\theta\hat{\mathrm{i}} + \sin\theta\hat{\mathrm{j}}\right) = -\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}\hat{e}_\rho$$

the radius of curvature along the helix (η=constant) upon which $\hat{e}_\eta$ varies is constant as well :

$$\frac{d\hat{e}_\eta}{ds_\xi} = \frac{\frac{d\hat{e}_\eta}{d\theta}}{\frac{ds_\xi}{d\theta}} = \frac{-\tan\alpha}{R(1+\tan^2\alpha)}\left(\cos\theta\hat{\mathrm{i}} + \sin\theta\hat{\mathrm{j}}\right)$$

therefore

$$\frac{d\hat{e}_\eta}{ds_\xi} = -\frac{1}{\rho_\eta}\hat{e}_\rho$$

where

$$\rho_\eta = \frac{R(1+\tan^2\alpha)}{\tan\alpha}$$

and one may note that

$$\frac{\rho_\xi}{\rho_\eta} = \tan\alpha$$

We have now two sets of coordinate :

$$\begin{aligned}
\hat{e}_\rho &= \hat{e}_\rho \\
\hat{e}_\eta &= \frac{1}{\sqrt{1+\tan^2\alpha}}(\tan\alpha\hat{e}_\theta - \hat{e}_z) \\
\hat{e}_\xi &= \frac{1}{\sqrt{1+\tan^2\alpha}}(\hat{e}_\theta + \tan\alpha\hat{e}_z)
\end{aligned}$$

and

$$\begin{aligned}
\hat{e}_r &= \hat{e}_\rho \\
\hat{e}_\theta &= \frac{1}{\sqrt{1+\tan^2\alpha}}(\tan\alpha\hat{e}_\eta + \hat{e}_\xi) \\
\hat{e}_z &= \frac{1}{\sqrt{1+\tan^2\alpha}}(-\hat{e}_\eta + \tan\alpha\hat{e}_\xi)
\end{aligned}$$

Additional useful relations,

$$\begin{aligned}
\frac{d\hat{e}_\eta}{d\theta} &= -\frac{1}{\sqrt{1+(P_mR)^2}}\hat{e}_\rho \\
\frac{ds_\eta}{d\theta}|_\xi &= R\sqrt{1+(P_mR)^2} \\
\frac{d\hat{e}_\eta}{ds_\eta} &= \frac{d\hat{e}_\eta}{d\theta}\frac{d\theta}{ds_\eta} = -\frac{1}{R\left[1+(P_mR)^2\right]}\hat{e}_\rho \\
\frac{d\hat{e}_\xi}{ds_\xi} &= -\frac{1}{\rho_\xi}\hat{e}_\rho = -\frac{(P_mR)^2}{R\left[1+(P_mR)^2\right]}\hat{e}_\rho
\end{aligned}$$

Current Density

We have previously expressed the current density as

$$\vec{J} = J_\theta \hat{e}_\theta + J_z \hat{e}_z$$

Since $\left(\frac{J_z}{J_\theta}\right)_{n,m} = \tan\alpha$ we may write in the new coordinate system

$$\vec{J}_{n,m} = J_{z_{n,m}} \frac{\sqrt{1+\tan^2\alpha}}{\tan\alpha} \hat{e}_\xi$$

which is consistent with the statement that the current flow is solely in the ξ direction.

As a side issue we note that the current density

$$\vec{J}_{n,m} = \vec{J}_{\xi_{n,m}} = -\frac{G_{n,m}\sqrt{1+\left(\frac{n}{\omega_m R}\right)^2}}{\mu_0 R K_n'} \cos(n\theta - \omega_m z)$$

preserves the "cos-nθ" current density distribution and therefor we may write

$$J_{\xi_{n,m}} = J_{0\xi}\cos(n\theta - \omega_m z)$$

$$J_{0\xi} = -\frac{G_{n,m}\sqrt{1+\left(\frac{n}{\omega_m R}\right)^2}}{\mu_0 R K_n'}$$

and introducing the expression for $G_{n,m}$ we write

$$J_{0\xi} = -\frac{n! R^{n-1}\sqrt{n^2+(\omega_m R)^2}}{2\mu_0 K_n'}\left(\frac{2}{\omega_m R}\right)^{n+1} B_{n,m}$$

or

$$B_{n,m} = -\frac{2\mu_0 K_n' J_{0\xi}}{n! R^{n-1}\sqrt{n^2+(\omega_m R)^2}}\left(\frac{\omega_m R}{2}\right)^{n+1}$$

We wish to normalize the above expression with respect to the corresponding 2D case ($\omega \to 0$)

$$B_{n2D} = \frac{\mu_0 J_{0z}}{2nR^{n-1}}$$

so that :

$$\boxed{\frac{B_{n,m}}{B_{n2D}} = -\frac{4nK_n'}{n!\sqrt{n^2+(\omega_m R)^2}}\left(\frac{\omega_m R}{2}\right)^{n+1}\left(\frac{J_{0\xi}}{J_{0z}}\right)}$$

By comparing a helical magnet and a straight magnet under the condition that both magnets carry the same total current (independent of periodicity), such that

$$I_\xi = \int_{\eta_1}^{\eta_2} J_\xi ds_\eta = J_{0\xi} \int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}} \cos n\theta \frac{R \tan\alpha}{\sqrt{1+\tan^2\alpha}} d\theta = J_{0\xi} \frac{2R}{\sqrt{n^2 + (\omega_m R)^2}}$$

$$I_z = lim_{\omega_m R \to 0} \int_{\eta_1}^{\eta_2} J_\xi ds_\eta = J_{0z} \frac{2R}{n}$$

we may write

$$\frac{J_{0\xi}}{J_{0z}} = \frac{\sqrt{n^2 + (\omega_m R)^2}}{n}$$

By substituting the above relation into the field expression we get :

$$\boxed{\frac{B_{n,m}}{B_{n2D}} = -\frac{4K_n'}{n!}\left(\frac{\omega_m R}{2}\right)^{n+1}}$$

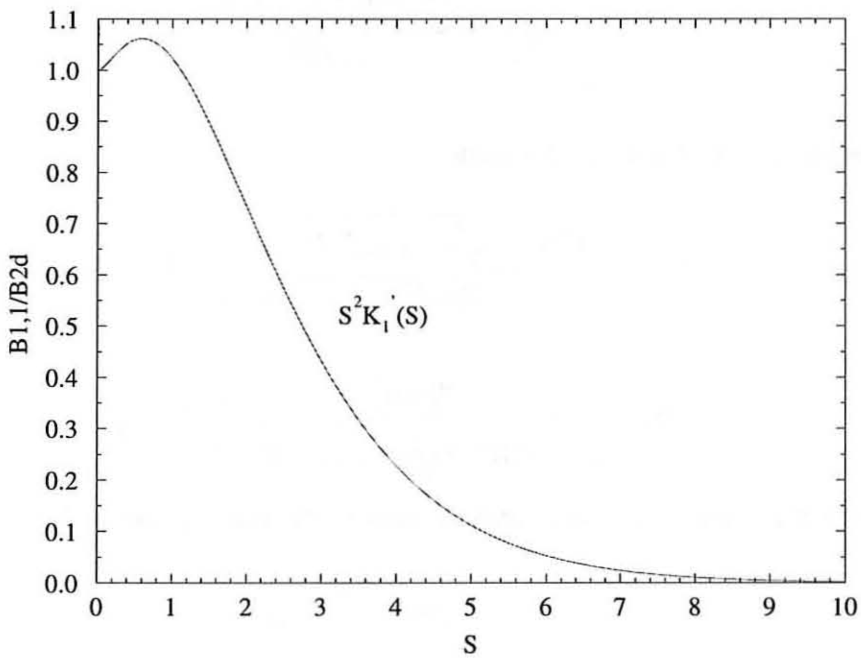

Figure 6 The normalized field as a function of $s = \omega R$ for a dipole magnet, n=1.