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QUANTUM THEORY OF THE GENERALIZED WAVE EQUATIONS, I.*

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* Research supported by the U. S. Atomic Energy Commission.

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ABSTRACT

We have made a systematic analysis of the quantum theory of the infinite component fields that transform under the combined representations of SL(2,C) (Majorana) ⊗ Dirac. A complete set of solutions of the wave-equation includes solutions with time-like and space-like momentum. We have explicitly calculated the mass-spectra for the time-like and space-like cases. Our method makes use of the decomposition of the product representation into reducible representations of the 'little' groups SU(2) and SU(1,1). Finally, the quantization of the generalized fields have been presented.

I. INTRODUCTION

Recent investigations on the infinite component field equations and their algebraic formalisms have added a lot to our understanding of strong-interaction dynamics[1,2,3,4]. Models based on these equations have many interesting consequences of direct experimental interest. Attempts have been made to obtain solutions for the algebra of local current densities[5]. Unlike finite component field equation case, one is able to treat here infinitely many mass and spin states satisfying the same wave-equation. Quantum systems described by such equations, indeed posses 'internal structure'[3]. However, the theory is plagued with the socalled 'diseases'. These speculative and malign pathologies are rather irrelevant! The existence of the reduntant or unphysical space-like solutions finds its way in describing an entirely new kind of phenomena of radiation involving 'faster-than-light particles[6]. A systematic formulation of the quantum field theory compatible with substitution law and with right spin-statistics relations have also been furnished[1].

The concept of infinite component field equations is not of recent origin. In the thirties, Majorana discovered a type of wave equation describing the infinite component spinor fields and tensor fields respectively[7]. These field equations possess both discrete and continuous solutions (the light-like solutions can be treated as a limiting case of the space-like solutions). Detailed analysis of the quantum theory of these Majorana fields have been studied elsewhere[1,2]. We just want to make a passing remark that these field equations have solutions for the masses, which are quite unrealistic in hadron physics. The masses vary inversely as the spin. Subsequently, attempts have been made to avert this situation. We will analyze an interesting field equation first proposed by Abers, Grodosky and Norton and subsequently studied by others in the context of obtaining solutions for the current algebra at infinite momentum.[5,3] We will confine ourselves in formulating a systematic quantization scheme for this generalised-field equation.

The contents of the paper are arranged as follows:

In Section II, we describe the field equation. The fields transform as infinite component column vectors under the product representation of SL(2,C) (Majorana) $\otimes$ D (Dirac). The algebraic properties of SL(2,C)$\otimes$ D representations under various subgroups of interest are discussed in Section III. We classify the field equation under each 'little group' and then have summarized the corresponding mass-spin spectra in Section IV. In Section V, we display the complete set of solutions of the field equation. Finally, we have furnished the quantization scheme for these infinite-component generalized fields.

II. THE WAVE-EQUATION

Let
$$\mathcal{L} = \int d^4x \;\; \bar{\psi}(x)\left[i\gamma_\mu \partial^\mu - m_0 - \tfrac{1}{2} m_1 \sigma_{\mu\nu} \Gamma^{\mu\nu} \right] \psi(x), \quad \text{(II.1)}$$
be the Lagrangian for a theory of a set $\{ \psi_{\sigma n}(x) \}$ of fields. The generators of the Poincaré transformations are:

$$P_\mu , \quad J_{\mu\nu} = \tfrac{1}{2}\sigma_{\mu\nu} + \Gamma_{\mu\nu} + \Gamma^{x}_{\mu\nu} \quad \text{(II.2)}$$

where, $\Gamma_{\mu\nu}$ and $\frac{1}{2}\sigma_{\mu\nu}$ respectively generate the infinite dimensional representation and finite dimensional (non-unitary) Dirac representation of SL(2,C); $\Gamma^{x}_{\mu\nu}$ is the orbital part of $J_{\mu\nu}$ given by

$$\Gamma^{x}_{\mu\nu} = i\left(x_\mu \partial^\nu - x_\nu \partial^\mu \right)$$

In the rest system, $J_{\mu\nu}$ provides us the total angular momentum of the quantum system.

The fields $\{\psi_{\sigma n}(x)\}$ are labelled by two indices: the Greek index σ and the Latin index n characterize respectively the infinite dimensional Majorana representation and the finite-dimentional (non-unitary) Dirac representation. Thus the transformation property of the fields $\{\psi_{\sigma n}(x)\}$ is given by

$$\psi_{\sigma n}(x) \rightarrow \psi'_{\sigma n}(x) = U(\Lambda)_{\sigma}^{\sigma'} \; V(\Lambda)_{n}^{n'} \; \psi_{\sigma' n'}[\Lambda^{-1}(x-a)] \tag{II.3}$$

The equation of motion follows from the given Lagrangian (II.1):

$$(i\gamma_\mu \partial^\mu - M)\psi(x) = 0 \tag{II.4}$$

where M is now a Lorentz invariant mass-matrix given by

$$M = m_0 + \frac{1}{2} m_1 \sigma_{\mu\nu} \Gamma^{\mu\nu} \tag{II.5}$$

Note that M is now a matrix and is no longer a constant and hence does not commute with γ_μ . However, it commutes with p_μ , a condition necessary to describe free-particle motion. In the limit $m_1=0$, the wave-equation (II.4) reduces to the ordinary Dirac equation. We will come to this point in detail in Section IV.

III. PROPERTIES OF THE SL(2,C)(MAJORANA) ⊗ DIRAC REPRESENTATIONS

In this section, we will first briefly recapulate the mathematical properties of the two Majorana representations, the Dirac representation and then display in detail the representation of the product space namely $X = \mathcal{H}_{SL(2,C)} \oplus \mathcal{H}_D$ under various sub-groups of interest

a) The Majorana Representations:

The generators of the homogeneous Lorentz group $\Gamma_{\mu\nu}$ satisfy the commutation relations

$$[\Gamma_{\mu\nu}, \Gamma_{\rho\sigma}] = i[g_{\nu\rho}\Gamma_{\mu\sigma} - g_{\mu\sigma}\Gamma_{\nu\rho} + g_{\nu\sigma}\Gamma_{\rho\mu} - g_{\rho\mu}\Gamma_{\nu\sigma}], \tag{III.1}$$

where $\mu, \nu, \rho, \sigma = 0,1,2,3$

$g_{00} = -g_{kk} = 1 \quad , \quad k = 1,2,3$

$g_{\mu\nu} = 0 \quad , \quad \mu \neq \nu$

To obtain the Majorana representations, we introduce as usual the operators a_α , a_α^+ (α = 1,2,) which satisfy the Bose-commutation relations:

$$[a_\alpha, a_\beta] = [a_\alpha^+, a_\beta^+] = 0$$

$$[a_\alpha, a_\beta^+] = \delta_{\alpha\beta} \quad , \quad \alpha,\beta = 1,2 \tag{III.2}$$

explicitly, we can express the generators $\Gamma_{\mu\nu}$ in terms of a_α and a_α^+ as follows:

$$\Gamma_{ij} = \epsilon_{ijk}\Sigma_k = \frac{1}{2}\,\epsilon_{ijk}\, a^+\sigma_k a$$

$$\Gamma_{i0} = \Lambda_i = \frac{1}{4}\left(a^+\sigma_i c a^+ - a c \sigma_i a\right) \tag{III.3}$$

$C = i\sigma_2$, σ_i's are the usual Pauli matrices The two Casimir operators of the Lorentz group C_0 and C_1 are given by

$$C_0 = \frac{1}{2}\Gamma_{\mu\nu}\Gamma^{\mu\nu} = \underline{\Sigma}^2 - \underline{\Lambda}^2 \tag{III.4a}$$

$$= \frac{1}{2}a^+a(\frac{1}{2}a^+a+1) - \left\{\frac{1}{2}a^+a(\frac{1}{2}a^+a+1)+3/4\right\}$$

$$= -3/4$$

and $$C_1 = \frac{1}{4}\,\epsilon^{\alpha\beta\gamma\delta}\,\Gamma_{\alpha\beta}\Gamma_{\gamma\delta} = \underline{\Sigma}\cdot\underline{\Lambda} \equiv 0 \tag{III.4b}$$

so, we find that (j_0,ν) or equivalently $(-j_0,-\nu)$ label the unitary irreducible representations of SL(2,C) as,

$$C_0 = j_0^2 + \nu^2 - 1 = -3/4$$

$$C_1 = -i\, j_0\nu = 0 \tag{III.5}$$

From (III.5), we find the solutions for j_0 and ν:

$$j_0 = \frac{1}{2}\,,\quad \nu = 0\,, \tag{III.6a}$$

$$j_0 = 0\,,\quad \nu = 1/2 \tag{III.6b}$$

Equation (III.6a) and (III.6b) respectively characterize the principal series and supplementary series representations of SL(2,C). In (III.6a), the ranges of Σ are given by

$$\Sigma = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \text{ ad infinitum;}$$

and from (III.6b), values of Σ are

$$\Sigma = 0, 1, 2, \ldots \text{ ad infinitum}$$

In either case, $\Sigma = j_o + k$, $k = 0, 1, 2, 3, \ldots$

$$\Sigma_3' = -\Sigma, \Sigma- +1, \ldots, +\Sigma$$

Thus, we have obtained the Majorana representations for the infinite component Fermi-fields and Bose-fields.

b) The Dirac Representation

The generators of the Dirac representation satisfy the commutation relation

$$\left[\frac{1}{2}\sigma_{\mu\nu}, \frac{1}{2}\sigma_{\rho\tau}\right] = \frac{i}{2}\left[g_{\nu\rho}\sigma_{\mu\tau} - g_{\mu\tau}\sigma_{\nu\rho} + g_{\nu\tau}\sigma_{\rho\mu} - g_{\rho\mu}\sigma_{\nu\tau}\right] \tag{III.7}$$

To obtain the representation of D, we look for the ranges of j_o and ν, which characterize the proper Lorentz group. The representation is finite if,[9]

(i) j_o and ν are simultaneously half-integral or integral; and

(ii) $|\nu| > |j_o|$

The ranges of spin values are given by

$$j = jo + n$$
$$= |jo| \ldots \ldots |\nu| - 1$$

The finite component Dirac fields belong to the coupled representation (if parity is admitted), $(\frac{1}{2}, \nu) + (-\frac{1}{2}, \nu), \nu = \pm\frac{3}{2}$ (real!)

c) The Properties of SL(2,C) (Majorana) $\otimes$ D Representations[9]:

Define $$S_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} + \Gamma_{\mu\nu} \tag{III.8}$$

then, $$J_{\mu\nu} = S_{\mu\nu} + \Gamma^{x}_{\mu\nu} \tag{III.9}$$

we will consider the case,

$$\left[\Gamma_{\mu\nu}, \tfrac{1}{2}\sigma_{\mu\nu}\right] = 0 \tag{III.10}$$

the generators $S_{\mu\nu}$ satisfy the commutation relation

$$\left[S_{\mu\nu}, S_{\rho\sigma}\right] = i\left[g_{\nu\rho}S_{\mu\sigma} - g_{\mu\sigma}S_{\nu\rho} + g_{\nu\sigma}S_{\rho\mu} - g_{\rho\mu}S_{\nu\sigma}\right], \tag{III.11}$$

To obtain the representations of $G = SL(2,C) \otimes D$, we proceed as follows:

Let us define,

$$\begin{aligned} \underline{J} &= \underline{\Sigma} + \tfrac{1}{2}\underline{\sigma} \\ \underline{K} &= \underline{\Lambda} + \tfrac{1}{2}\underline{\tau} = \underline{\Lambda} + \tfrac{i}{2}\underline{\alpha} \end{aligned} \tag{III.12}$$

where, we have identified

$$\begin{aligned} \Gamma_{ij} &= \epsilon_{ijk}\Sigma_k \\ \Gamma_{i0} &= \Lambda_i \\ \tfrac{1}{2}\sigma_{ij} &= \tfrac{1}{2}\epsilon_{ijk}\sigma_k \\ \tfrac{1}{2}\sigma_{i0} &= \tfrac{1}{2}\tau_i = \tfrac{1}{2}i\alpha_i \; ; \quad i,j,k = 1,2,3 . \end{aligned} \tag{III.13}$$

Further,

$$\begin{aligned} [\Sigma_i, \Sigma_j] &= i\epsilon_{ijk}\Sigma_k \\ [\Sigma_i, \Lambda_j] &= i\epsilon_{ijk}\Lambda_k \\ [\Lambda_i, \Lambda_j] &= -i\epsilon_{ijk}\Sigma_k \end{aligned} \tag{III.14}$$

and

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$$

$$\alpha_i^2 = \sigma_i^2 = 1 \qquad \text{(no summation over i)}$$

$$\left[\tfrac{1}{2}\tau_i, \tfrac{1}{2}\tau_j\right] = -i\epsilon_{ijk}(\sigma_k/2), \quad \left[\tfrac{1}{2}\tau_i, \tfrac{1}{2}\sigma_j\right] = i\epsilon_{ijk}(\tau_k/2), \tag{III.15}$$

The two Casimir operators Q_o and Q_1 are given by

$$Q_o = \underline{J}^2 - \underline{K}^2$$

and $$Q_1 = \underline{J} \cdot \underline{K} \qquad \text{(III.16)}$$

Now, $$Q_o = \underline{J}^2 - \underline{K}^2 = (\underline{\Sigma} + \tfrac{1}{2}\underline{\sigma})^2 - (\underline{\Lambda} + \tfrac{1}{2} i \underline{\alpha})^2$$

or, $$Q_0 = (\underline{\Sigma}^2 - \underline{\Lambda}^2) + 3/2 + (\underline{\sigma} \cdot \underline{\Sigma} - i \underline{\alpha} \cdot \underline{\Lambda})$$

$$= -3/4 + 3/2 + (\underline{\sigma} \cdot \underline{\Sigma} - i \underline{\alpha} \cdot \underline{\Lambda})$$

[∵ For Majorana representations $C_0 = \underline{\Sigma}^2 - \underline{\Lambda}^2 = -3/4$;

For Dirac Repres., $C_o = 3/2$]

or, $$(Q_o - 3/4) = (\underline{\sigma} \cdot \underline{\Sigma} - i \underline{\alpha} \cdot \underline{\Lambda}) \qquad \text{(III.17)}$$

squaring both sides of (III.17), we solve for Q_o:

$$(Q_o - 3/4)^2 = \frac{3}{4} - 2 Q_o$$

or $$(Q_o + \tfrac{3}{4})(Q_o - \tfrac{1}{4}) = 0$$

ie, $$Q_o = -3/4 \qquad \text{(III.18a)}$$

$$Q_o = \frac{1}{4} \qquad \text{(III.18b)}$$

similarly

$$Q_1 = \underline{J} \cdot \underline{K} = (\underline{\Sigma} + \tfrac{1}{2}\underline{\sigma})(\underline{\Lambda} + \tfrac{i}{2}\underline{\alpha})$$

$$= -\frac{i}{2} \gamma_5 \left(Q_0 + 3/4\right) \qquad \text{(III.19)}$$

In obtaining (III.19), we have made use of the property $C_1 = \underline{\Sigma} \cdot \underline{\Lambda} = 0$.

Thus from (III.18) and (III.19) we obtain

i) $Q_1 = 0$, for $Q_o = -3/4$

ii) $Q_1 = -\frac{i}{2} \gamma_5$, for $Q_o = \frac{1}{4}$ (III.20)

It then follows that, since D is non-unitary, the two representations (III.20) characterizing G are also non-unitary and also reducible. Since *each* $\underline{J}^2 = (\underline{\Sigma} + \frac{1}{2}\underline{\sigma})^2$ value appears twice in the generalized fields, these two non-unitary representations (III.20) exhaust the total reduction.

Reduction of G with respect of SU(1,1):

The SU(1,1) subgroup of G may be taken to be generated by the elements J_3, K_1 and K_2. They obey the commutation rules:

$$
\begin{aligned}
[J_3, K_1] &= i K_2 \\
[J_3, K_2] &= -i K_1 \\
[K_1, K_2] &= -i J_3
\end{aligned}
\qquad \text{(III.21)}
$$

Note that $J_3 = \Sigma_3 + \sigma_3/2$, $K_1 = \Lambda_1 + \tau_1/2$, $K_2 = \Lambda_2 + \tau_2/2$.

The quadratic Casimir operator of SU(1,1) is given by,

$$
\begin{aligned}
Q &= J_3^2 - K_1^2 - K_2^2 \\
&= (\Sigma_3 + \sigma_3/2)^2 - (\Lambda_1 + i\alpha_1/2)^2 - (\Lambda_2 + i\alpha_2/2)^2 \\
&= (\Sigma_3^2 - \Lambda_1^2 - \Lambda_2^2) + 3/4 + (\sigma_3\Sigma_3 - \tau_1\Lambda_1 - \tau_2\Lambda_2) \\
&= \mathcal{C} + 3/4 + (\sigma_3\Sigma_3 - \tau_1\Lambda_1 - \tau_2\Lambda_2)
\end{aligned}
\qquad \text{(III.22)}
$$

where $\mathcal{C} = \Sigma_3^2 - \Lambda_1^2 - \Lambda_2^2$ is the quadratic Casimir operator of $SU(1,1) \subset SL(2,C)$.

Thus we have,

$$
(Q - \mathcal{C} - 3/4) = \sigma_3\Sigma_3 - \tau_1\Lambda_1 - \tau_2\Lambda_2 \qquad \text{(III.23)}.
$$

Squaring both sides of (III.23) and after a little algebraic manipulation, we obtain

$$
(Q - \mathcal{C} - 3/4)^2 = 2\mathcal{C} - Q + 3/4
$$

or, solving for $\mathcal{C}$, we have,

$$
\mathcal{C} = (Q + \tfrac{1}{4}) \pm \sqrt{Q + \tfrac{1}{4}} \qquad \text{(III.24)}
$$

or, solving for Q,

$$
Q = (\mathcal{C} + \tfrac{1}{4}) \pm (\mathcal{C} + \tfrac{1}{4})^{1/2} \qquad \text{(III.24')}
$$

Using $Q = \tilde{\mathcal{J}}(\tilde{\mathcal{J}}+1)$ and $\mathcal{C} = \tilde{\Sigma}(\tilde{\Sigma}+1)$, we obtain from Eq. (III.24) two sets of values for $\tilde{\mathcal{J}}$, namely,

$$\tilde{\mathcal{J}}^{(1)} = \tilde{\Sigma} + 1/2 , \quad \tilde{\Sigma} - 1/2$$

$$\tilde{\mathcal{J}}^{(2)} = -\tilde{\Sigma} - 1/2 , \quad -\tilde{\Sigma} - 3/2 \qquad \text{(III.25)}$$

Thus, we find for each value of Q, there are two values of $\mathcal{C}$ (Eq. III.24). Since, the Dirac representation is non-unitary, the SU(1,1) $\subset$ SL(2,C) multiplied by the Dirac spinor is a reducible non-unitary representation and precisely reduces to the above forms (Eq. III.25). This could have been formally checked from the fact that each value of $(\tilde{\mathcal{J}})^2 = (\tilde{\Sigma} + \frac{1}{2}\tilde{\sigma})^2$ appears twice in the generalized fields. To find the possible states, we recollect some of the unitary irreducible representation properties of SU(1,1). They fall into three classes:

Class (i) The continuous non-exceptional class;

a) $\frac{1}{4} \leq -\mathcal{C} < \infty , \quad \Sigma_3 = 0, \pm 1, \pm 2, \ldots$ ad infinitum

or b) $\frac{1}{4} \leq -\mathcal{C} < \infty , \quad \Sigma_3 = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$

Class (ii) Continuous exceptional class;

$$0 < -\mathcal{C} < \frac{1}{4} , \quad \Sigma_3 = 0, \pm 1, \pm 2, \ldots$$

Class(iii) Discrete class;

$$\mathcal{C} = k(k-1) , \quad k = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

$$\Sigma_3 = k, k+1, \ldots, \infty \text{ for } D_k^{(+)}$$

$$\Sigma_3 = -k, -k-1, \ldots, -\infty \text{ for } D_k^{(-)} .$$

We will see in next section that for the $(\text{mass})^2$ to be $-ve$ (space-like solution), the <u>only</u> values of $\tilde{\mathcal{J}}$ admitted are given by combining Dirac representation with the class (i) representation.

Some properties of the continuous non-exceptional non-unitary representations of $SU(1,1) \subset G$;

We know that each U.I.R. of SL(2,C) characterised by (m , ρ) contains each U.I.R. of SU(1,1) of the continuous non-exceptional class twice. Correspondingly, the representation space $\mathcal{H}\,SL(2,C)$ decomposes into $\mathcal{H}_{+m}(SU(1,1)) + \mathcal{H}_{-m}(SU(1,1))$. Thus, on restricting SL(2,C) ⓧ D to SU(1,1), we obtain two sets of reducible representations defined by (III.25) and each reducible set further contains two irreducible parts. We have to note here that the representations characterised by $\tilde{\mathcal{J}}^{(1)}$ and $\tilde{\mathcal{J}}^{(2)}$ in (III.25) are equivalent.

Thus we have obtained all the possible irreducible representations when we restrict the group SL(2,C) ⓧ D with respect to its sub-group SU(1,1).

IV. Classification of the Plane-Wave Solutions and the Mass-Spectra:

Let us consider the field-equation (II.4):

$$\left[i\,\gamma_\mu\,\partial^\mu - m_0 - \tfrac{1}{2}\,m_1\,\sigma_{\mu\nu}\,\Gamma^{\mu\nu} \right] \psi(x) = 0 \tag{II.4}$$

In the momentum representation,

$$\psi(x) = \frac{1}{(2\pi)^2} \int d^4 p \; e^{-ipx}\,\psi(p)$$

we can re-write Eq.(II.4) as

$$\left[\gamma_\mu\,p^\mu - m_0 - \tfrac{1}{2}\,m_1\,\sigma_{\mu\nu}\,\Gamma^{\mu\nu} \right] \psi(p) = 0 \tag{IV.1}$$

Depending upon p_μ time-like, space-like or light-like (i.e. $\underline{p}^2 - p_0^2 > 0$, $\underline{p}^2 - p_0^2 < 0$, or $\underline{p}^2 - p_0^2 = 0$) , we will have in general three classes of solutions for the equation (IV.1). We will discuss these solutions and their corresponding mass-spectra in the below.

Class I time-like case (slower-than-light particles):

Let us rewrite the equation (IV.1) as

$$(E - \underline{\alpha}\cdot\underline{p} - \beta M)\psi(p) = 0$$

or

$$H\psi(p) = E\psi(p) = (\underline{\alpha}\cdot\underline{p} + \beta M)\psi(p) \tag{IV.2}$$

where H is the Hamiltonian of the system. In general, the spinors $\psi(p)$ can be labelled by the eigenvalues of β , σ_3 , Σ^2 and Σ_3 or alternatively by $|\beta, J, \lambda\, \Sigma\rangle$, where β , J , λ , Σ are the eigenvalues of β , the total spin $\underline{J}^2 = (\underline{\Sigma} + \frac{1}{2}\underline{\sigma})^2$, J_3 and $\underline{\Sigma}^2$ respectively.

$$\text{ie.,} \quad \psi(p) \sim \psi(p, \Sigma, \Sigma_3) \otimes \chi(p, \sigma)$$

Thus, solving for the eigenvalues of H in Eq. (IV.2), we obtain the masses m_J. In the rest system,

$$\begin{aligned} m &= \beta M = \beta m_0 + \tfrac{1}{2} m_1 \beta\, \sigma_{\mu\nu} \Gamma^{\mu\nu} \\ &= \beta m_0 + m_1 \beta\, (\underline{\sigma}\cdot\underline{\Sigma} - \underline{\tau}\cdot\underline{\Lambda}) \end{aligned} \tag{IV.3}$$

Note that for m to be self-adjoint, we choose β , $\beta\underline{\sigma}$, $\beta\underline{\tau}$ Hermitian and for the Majorana representations, $\underline{\Sigma}$, and $\underline{\Lambda}$ are also Hermitian. Since β does not commute with m , the states $|\beta, J, \lambda, \Sigma\rangle$ are not the eigenstates of m . Nonetheless, the mass-matrix can be written as (for detailed derivation of m , vide appendix.A),

$$m = \begin{pmatrix} m_0 - m_1\,(J + 3/2) & -i m_1\,[J(J+1)]^{1/2} \\ i m_1\,[J(J+1)]^{1/2} & -[m_0 + m_1(J - 1/2)] \end{pmatrix}, \tag{IV.4}$$

or, diagonalizing the above matrix, we obtain

$$m_J = m_1 \ (J+1/2) \pm \left[(m_0-m_1)^2 + m_1^2 \ (J+1/2)^2 - \frac{1}{4} m_1^2\right]^{1/2} \tag{IV.5}$$

or

$$(m_J)^2 = 2m_1^2 \ (J+1/2)^2 + (m_0-m_1)^2 - \frac{1}{4} m_1^2$$
$$\pm 2m_1 (J+\tfrac{1}{2})\left[(m_0-m_1)^2 - \frac{1}{4} m_1^2 + m_1^2 (J+\tfrac{1}{2})^2\right]^{\frac{1}{2}} \tag{IV.6}$$

Thus, we find from expression (IV.5) that there are two values of m_J for each value of J. A detailed discussion of this mass-matrix will be presented at the end of this section.

<u>Class II: space-like solutions (faster-than-light particles)</u>

Let us choose the frame,

$$p_\mu = (\tilde{m} \ \sinh\zeta \ ; 0, 0, \ \tilde{m} \cosh\zeta) \quad ; \quad \tilde{m} > 0$$

Then, in the rest-system ($\zeta = 0$), we obtain from Equ.(IV.1)

$$(\tilde{m} - \gamma_3 M)\psi(0, \tilde{\alpha}) = 0$$

or

$$\tilde{m}\,\psi(0,\tilde{\alpha}) = \gamma_3 M \psi(0, \tilde{\alpha}) \tag{IV.7}$$

Thus, the mass-operator is given by,

$$\tilde{m} = m_0\gamma_3 + m_1\gamma_3 \ (\tilde{\underline{\sigma}}\cdot\tilde{\underline{\Sigma}} - \tilde{\underline{\tau}}\cdot\tilde{\underline{\Lambda}}) \tag{IV.8}$$

where

$$\tilde{\underline{\sigma}}\cdot\tilde{\underline{\Sigma}} = (\sigma_3\Sigma_3 - \tau_1\Lambda_1 - \tau_2\Lambda_2)$$

$$\tilde{\underline{\tau}}\cdot\tilde{\underline{\Lambda}} = (\tau_3\Lambda_3 - \sigma_1\Sigma_1 - \sigma_2\Sigma_2)$$

and

$$\tilde{\underline{\sigma}} = (\sigma_3, \tau_1, \tau_2) \quad ; \quad \tilde{\underline{\tau}} = (\tau_3, \sigma_1, \sigma_2)$$

$$\tilde{\underline{\Sigma}} = (\Sigma_3, \Lambda_1, \Lambda_2) \ ; \quad \tilde{\underline{\Lambda}} = (\Lambda_3, \Sigma_1, \Sigma_2)$$

The spinors $\psi(0, \mathcal{L})$ are labelled by the quantum numbers $\mathcal{L}$. $\mathcal{L}$, in general, represent γ_3, $\tilde{J}$, J_3 and $\tilde{\Sigma}$; where $\tilde{J}$, J_3 and $\tilde{\Sigma}$ are the eigenvalues of $(\underline{\tilde{J}})^2 = (\underline{\tilde{\Sigma}} + \frac{1}{2}\underline{\tilde{\sigma}})^2$, J_3 and $(\underline{\tilde{\Sigma}})^2 = (\Sigma_3^2 - \Lambda_1^2 - \Lambda_2^2)$ respectively. Rewriting (IV.8) as,

$$\tilde{m} = (m_0 - m_1)\gamma_3 + m_1\gamma_3\left\{\left(\tfrac{1}{2} + \underline{\tilde{\sigma}}\cdot\underline{\tilde{\Sigma}}\right) - \left(\underline{\tilde{\Sigma}}\cdot\underline{\tilde{\Lambda}} - 1/2\right)\right\} \tag{IV.8'}$$

we obtain,

$$\tilde{m} = \begin{pmatrix} i\,(m_0 - m_1) + i m_1\sqrt{\mathcal{C} + \frac{1}{4}} & i m_1\left[\sqrt{\mathcal{C} + \frac{1}{4}} - 1/2\right] \\ -i m_1\left[\sqrt{\mathcal{C} + \frac{1}{4}} + 1/2\right] & -i\,(m_0 - m_1) + i m_1\sqrt{\mathcal{C} + \frac{1}{4}} \end{pmatrix}, \tag{IV.9}$$

(for a detailed derivation of (IV.9), *vide* Appendix A). Diagonalizing the above matrix, we obtain

$$\tilde{m}_J = i\left[m_1\left(\mathcal{C} + \tfrac{1}{4}\right)^{1/2} \pm \sqrt{m_1^2\left(\mathcal{C} + \tfrac{1}{4}\right) + (m_0 - m_1)^2 - \tfrac{1}{4}m_1^2}\;\right] \tag{IV.10}$$

using $\quad \mathcal{C} = \tilde{\Sigma}(\tilde{\Sigma} + 1)$, we obtain

$$(\tilde{m}_J)^2 = -2m_1^2\left(\tilde{\Sigma} + \tfrac{1}{2}\right)^2 - \left\{(m_0 - m_1)^2 - \tfrac{1}{4}m_1^2\right\}$$
$$\mp 2m_1\left(\tilde{\Sigma} + \tfrac{1}{2}\right)\sqrt{(m_0 - m_1)^2 - \tfrac{1}{4}m_1^2 + m_1^2\left(\tilde{\Sigma} + \tfrac{1}{2}\right)^2}\;, \tag{IV.11}$$

writing $\quad \tilde{\Sigma} = -\frac{1}{2} + i\nu$, we find from eq. (IV.11)

$$(m_\nu)^2 = 2m_1^2\nu^2 - \left\{(m_0 - m_1)^2 - \tfrac{1}{4}m_1^2\right\} \pm 2m_1\nu\sqrt{m_1^2\nu^2 - \left\{(m_0 - m_1)^2 - \tfrac{1}{4}m_1^2\right\}} \tag{IV.11'}$$

For the mass $\tilde{m}$ to be purely imaginary, we obtain the ranges of N: ν;

$$(m_0 - m_1)^2 - \frac{1}{4} m_1^2 \leq m_1^2 \nu^2 < \infty$$

or

$$\left[(m_0/m_1 - 1)^2 - \frac{1}{4} \right] \leq \nu^2 < \infty$$

ie,

$$\left[\left(\frac{m_0}{m_1} - 1 \right)^2 - 1/4 \right]^{\frac{1}{2}} \leq \nu < \infty \quad ; \qquad \text{(IV.12a)}$$

or

$$-\infty < \nu \leq -\left[(m_0/m_1 - 1)^2 - \frac{1}{4} \right]^{\frac{1}{2}} \quad , \qquad \text{(IV.12b)}$$

Class III (Light-like solutions)

Let us choose the frame,

$$p_\mu = (p ; 0, 0, p) \qquad \text{(IV.13)}$$

The 'little' group which leaves this configuration invariant is generated by J_3 $K_2 - J_1$ and $K_1 + J_2$. These generators satisfy the commutation relations of E(2) algebra namely,

$$[J_3, E_1] = i E_2 \ , \ [J_3, E_2] = -i E_1 \ , \ [E_1, E_2] = 0 \qquad \text{(IV.14)}$$

where

$$\mathcal{E} = (E_1, E_2), J_3 ; \quad E_1 = K_1 + J_2 = (\Lambda_1 + \tau_1/2) + (\Sigma_2 + \sigma_2/2)$$

$$E_2 = K_2 - J_1 = (\Lambda_2 + \tau_2/2) - (\Sigma_1 + \sigma_1/2) \qquad \text{(IV.15)}$$

and the invariant Casimir-operator is given by

$$\mathcal{E}^2 = E_1^2 + E_2^2 \qquad \text{(IV.16)}$$

Let us consider the wave-equation,

$$E\psi = \left[\underline{\alpha} \cdot \underline{p} + m_0 \gamma_0 + m_1 \gamma_0 (\underline{\sigma} \cdot \underline{\Sigma} - \underline{\tau} \cdot \underline{\Delta}) \right] \psi$$

Substituting (IV.13) in the above equation we have,

$$p - \alpha_3 p = m_0 \gamma_0 + m_1 \gamma_0 \left(\underline{\sigma} . \underline{\Xi} - \underline{\tau} \cdot \underline{\Delta} \right) \qquad \text{(IV.17)}$$

Multiplying Eq. (IV.17) from the left by $(1 + \alpha_3)$,

$$0 = m_0 (1 + \alpha_3) \gamma_0 + m_1 (1 + \alpha_3) \gamma_0 \left(\underline{\sigma} . \underline{\Xi} - \underline{\tau} \cdot \underline{\Delta} \right)$$

$$\text{or,} \quad 0 = m_0 (\gamma_0 + \gamma_3) + m_1 (\gamma_0 + \gamma_3) \left(\underline{\sigma} . \underline{\Xi} - \underline{\tau} \cdot \underline{\Delta} \right) \qquad \text{(IV.18)}$$

Multiplying (IV.18) by $(\gamma_0 - \gamma_3)$ from the left throughout,

$$0 = 2 m_0 + 2 m_1 \left(\underline{\sigma} . \underline{\Xi} - \underline{\tau} \cdot \underline{\Delta} \right)$$

ie,

$$m_0 + m_1 \left(\underline{\sigma} . \underline{\Xi} - \underline{\tau} \cdot \underline{\Delta} \right) = 0$$

or

$$m_0 \gamma_0 = - m_1 \gamma_0 \left(\underline{\sigma} . \underline{\Xi} - \underline{\tau} \cdot \underline{\Delta} \right) \qquad \text{(IV.19)}$$

We can proceed exactly as in the previous cases, and obtain the mass-matrix as,

$$- m_0 \gamma_0 = \begin{pmatrix} - m_1 (J + 3/2) & -i m_1 [J (J+1)]^{1/2} \\ i m_1 [J(J+1)]^{\frac{1}{2}} & - m_1 (J - 1/2) \end{pmatrix}$$

or diagonalising the right-hand side, we have;

$$- m_0 \gamma_0 = \begin{pmatrix} m_1 \left\{ (J + 1/2) + \sqrt{(J+1/2)^2 + 3/4} \right\} & 0 \\ 0 & m_1 \left\{ (J + 1/2) - \sqrt{(J+\frac{1}{2})^2 + 3/4} \right\} \end{pmatrix}$$

or,

$$- m_0 = m_1 (J + \tfrac{1}{2}) + m_1 \sqrt{(J + \tfrac{1}{2})^2 + 3/4} \qquad \text{(IV.19)}$$

and
$$m_0 = m_1 (J+\tfrac{1}{2}) - m_1 \sqrt{(J+\tfrac{1}{2})^2 + 3/4} \tag{IV.20}$$

Equations (IV.19) and (IV.20) are simultaneously true, iff

$$m_0 \equiv 0 \, , \quad \text{and} \quad m_1 \equiv 0$$

Thus, there are no light-like solutions of our wave equation.

Some further discussions on the mass-spectra:

a) Time-like case:

From equation (IV.5), we have;

$$\left[m_J - m_1 (J+\tfrac{1}{2})\right]^2 = (m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 + m_1^2 (J+\tfrac{1}{2})^2$$

or,
$$m_J^2 - 2 m_1 m_J (J+\tfrac{1}{2}) - \left[(m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right] = 0$$

or,
$$\left\{ m_J^2 - \left[(m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right] \right\}^2 = 4 m_1^2 m_J^2 (J+\tfrac{1}{2})^2 \; ; \tag{IV.21}$$

Let $y = m_J^2$; and $x = (J + 1/2)^2$

Then, we have from (IV.21)

$$\left\{ y - \left[(m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right] \right\}^2 = 4 m_1^2 y x$$

or ,

$$y^2 - 2y \left\{ (m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right\} + \left\{ (m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right\}^2 = 4 m_1^2 y x \, . \tag{IV.22}$$

Equation (IV.22) gives the equation of a hyperbola.

(i) If $m_1 = 0$, we obtain from (IV.22)

$$y^2 - 2 m_0^2 y + m_0^4 = 0$$

or
$$(y - m_0^2)^2 = 0 \, , \quad \text{i.e. } y = m_0^2 \, , \text{ or } \quad m_J = \pm m_0 \, .$$

This gives the equation of a st. line $\parallel$ to the x - axis (pure Dirac case).

ii) Rewriting the Eq. (IV.22),

$$y\left[y-2\left\{(m_0-m_1)^2-\tfrac{1}{4}m_1^2\right\}-4m_1^2x\right] = \left[\tfrac{1}{4}m_1^2-(m_0-m_1)^2\right]^2$$

we obtain for the equations of the asymptotics as,

$$y\left[y-2\left\{(m_0-m_1)^2-\tfrac{1}{4}m_1^2\right\}-4m_1^2x\right] = 0$$

ie,

$$y_1 = 0 \tag{IV.23a}$$

$$y_2 = 4m_1^2x+2\left\{(m_0-m_1)^2-\tfrac{1}{4}m_1^2\right\} \tag{IV.23b}$$

Equations (IV.23) define the boundary of the time-like curve.

iii) for $x \to \infty$,

a) $y \to 0$

b) $y \to 4m_1^2x$

ie, One branch of the mass-curve monotomically rises to the infinity and the other branch goes to zero. Thus, there is no discrete lowest mass, the mass-spectrum has only an accumulation point.

b) Space-like case

The interpretation of the mass-spectrum can be carried out in a complete analogous fashion to that of our previous case by studying the variation of $(m_{\tilde{J}})^2$ vs. ν^2. We take $(m_{\tilde{J}})^2$ along the negative ordinate and ν^2 along the negative abscissa. We will just discuss in this section some of the distinct features.

(i) $m_1 = 0$; Then from Eq. (IV.11) we get,

$$(m_{\mathcal{J}})^2 = -(m_{\tilde{\mathcal{J}}})^2 = m_o^2$$

$$\text{or} \qquad m_{\mathcal{J}} = \pm m_o$$

This corresponds to the familiar Dirac case, the particle having constant mass m_o($\pm m_o$ are interpreted as the rest masses of the particle and anti-particle respectively).

(ii) $m_o = 0$; This leads to the mass-spectrum arising purely from the symmetry breaking term $\frac{1}{2} m_1 \sigma_{\mu\nu} \Gamma^{\mu\nu}$ namely,

$$(m_{\tilde{\mathcal{J}}})^2 = -2m_1^2 (\tilde{\Sigma}+\tfrac{1}{2})^2 - 3/4\, m_1^2 \mp 2m_1^2 (\tilde{\Sigma}+\tfrac{1}{2}) \sqrt{(\tilde{\Sigma}+\tfrac{1}{2})^2 + 3/4}$$

or,
$$(m_\nu)^2 = 2m_1^2 \nu^2 - 3/4\, m_1^2 \pm 2m_1^2 \nu \sqrt{\nu^2 - 3/4} \tag{IV.24}$$

Equation (IV.24) admits selections of the wave-equation <u>only</u> in the range

$$\frac{3}{4} \leq \nu^2 < \infty \tag{IV.25}$$

$$\text{ie,} \quad \sqrt{\frac{3}{4}} \leq \nu < \infty \tag{IV.26a}$$

$$-\infty < \nu \leq -\sqrt{\frac{3}{4}} \tag{IV.26b}$$

(iii) $m_o = 0$; $m_1 = 0$;

This case does not admit any solution of the wave equation (vide the light-like solution).

(iv) Finally, the mass spectrum (IV.11) is degenerate in ν. This is evident from the fact that we get the same mass-spectrum for each of the ranges (IV.12a) and (IV.12b).

IV. Construction of the basis vectors, the normalization and the completeness of the Solutions of the field-equation:

Construction of the basis vectors:

a) Time-like case:

We have noted earlier that in the representation space $\mathcal{H}_G = \mathcal{H}_{SL(2,c)} \oplus \mathcal{H}_D$ the fields transform as double-indexed infinite-component column vectors; i.e., we label each field component by the total spin J, the spin-production J_3 and Σ. Then in an arbitrary frame, the spinor-wave functions are given by,

$$\Psi(\underline{p}, J J_3, \Sigma) = \chi_D(\sigma, \underline{p}) \otimes \psi(\underline{p}, \Sigma, \Sigma_3)$$

$$= \sum_{\sigma} C(\tfrac{1}{2}, \Sigma; \sigma, \Sigma_3 | J J_3)\, \chi_D(\sigma, \underline{p})\, \psi(\underline{p}, \Sigma, \Sigma_3) \tag{V.1}$$

Where, $C(\frac{1}{2}, \Sigma; \sigma, \Sigma_3 \mid J, J_3)$ is the usual Wigner,coefficient and $\chi_D(\sigma, \underline{p})$ and $\psi(\underline{p}, \Sigma, \Sigma_3)$ represent respectively the Dirac-spinor and the Majorana spinor wave-functions. $\chi_D(\sigma, \underline{p})$ and $\psi(\underline{p}, \Sigma, \Sigma_3)$ are obtained from their rest-states by applying the Lorentz-boosters:

ie,
$$\Psi(\underline{p}, J, J_3, \Sigma) = e^{i \underset{\sim}{\zeta} \cdot \underset{\sim}{K}}\, \psi(J, J_3, \Sigma)$$

$$= e^{i \underset{\sim}{\zeta} (\underset{\sim}{\Lambda} + \underset{\sim}{\Sigma}/2)}\, \psi(J, J_3, \Sigma)$$

$$= \sum_{\sigma} C(\tfrac{1}{2}, \Sigma; \sigma, \Sigma_3 | J J_3) \times$$

$$\left[e^{i \underset{\sim}{\zeta}/2 \cdot \underset{\sim}{\Sigma}}\, \chi_D \right] \left[e^{i \underset{\sim}{\zeta} \cdot \underset{\sim}{\Lambda}}\, \psi(\Sigma, \Sigma_3) \right] \tag{V.2}$$

where
$$\underset{\sim}{\zeta} = \hat{\zeta} \text{ arc tanh } (\underset{\sim}{p}/E_J)$$

and
$$E_J = \left[\underline{p}^2 + m_J^2 \right]^{1/2} \tag{V.3}$$

We have to note further here that for each value of J , there are two values of the masses; so we will introduçe an additional index ϵ to distinguish the two branches of the mass-spectrum.

We have,

$$e^{i\underset{\sim}{\zeta}\cdot\underset{\sim}{\Sigma}/2}\,\chi_D(\sigma) = \left(\cosh\zeta/2 - \underset{\sim}{\alpha}\cdot\hat{\zeta}\sinh\zeta/2\right)\chi_D(\sigma)$$

$$= \left\{\left[\frac{E_J+m_J}{2m_J}\right]^{1/2} - \frac{\underset{\sim}{\alpha}\cdot\underset{\sim}{p}}{\left[2m_J\,(E_J+m_J)\right]^{1/2}}\right\}\chi_\sigma(0)$$

$$= \frac{E_J+m_J - \underset{\sim}{\alpha}\cdot\underset{\sim}{p}}{\left[2m_J\,(E_J+m_J)\right]^{1/2}}\,\chi_\sigma(0) \qquad \text{(V.4)}$$

Again,

$$\psi(\underset{\sim}{p},\Sigma,\Sigma_3) = e^{i\underset{\sim}{\zeta}\cdot\underset{\sim}{\Lambda}}\,\psi(\Sigma,\Sigma_3)$$

$$= V(B_p)^{\Sigma'\Sigma_3'}_{\Sigma\,\Sigma_3}\,, \qquad \text{(V.5)}$$

Becuase of the unitarity of V, the spinors (V.5) are orthonormalized for all $\underset{\sim}{p}$;

$$\text{ie,}\quad \left(\psi(\underset{\sim}{p},\Sigma,\Sigma_3)\,,\ \psi(\underset{\sim}{p},\Sigma',\Sigma_3')\right) = \delta_{\Sigma\Sigma'}\ \delta_{\Sigma_3\Sigma_3'} \qquad \text{(V.6)}$$

Without any loss of generality, we can choose the frame $\underset{\sim}{p} = e_z\,p$. Then (V.5) assumes a very familiar form:

$$V(B_p)^{\Sigma'\Sigma_3'}_{\Sigma\,\Sigma_3} = \delta_{\Sigma_3\Sigma_3'}\ v^{\Sigma_3}_{\Sigma\Sigma'}(\zeta) \qquad \text{(V.7)}$$

where,

$$v_{\Sigma\Sigma'}(\zeta) = \Theta_{\Sigma\Sigma'}\left(\cosh\frac{\zeta}{2}\right)^{-(\Sigma+\Sigma')}\left(\sinh\frac{\zeta}{2}\right)^{\Sigma-\Sigma'}\times$$

$$F\left(\Sigma_3-\Sigma,\ 1-\Sigma-\Sigma_3,\ 1+\Sigma+\Sigma';\ -\sinh^2\frac{\zeta}{2}\right)$$

and

$$\Theta_{\Sigma\Sigma'} = \frac{1}{(\Sigma-\Sigma')!}\left[\frac{\Gamma(\Sigma'-\Sigma_3+1)\ \Gamma(\Sigma'+\Sigma_3)}{\Gamma(\Sigma-\Sigma_3+1)\ \Gamma(\Sigma+\Sigma_3)}\right]^{1/2} \qquad \text{(V.8)}$$

For $\Sigma' > \Sigma$, replace $\zeta \rightarrow -\zeta$

To obtain the expression for the arbitrary Lorentz-transformation, we can make a spatial rotation on the state-vectors and compute the corresponding matrix elements $V(B_p)$.

Thus, we obtain from Eqs. (V.2), (V.4), (V.7) and (V.8):

$$\psi_\varepsilon(p, J, J_3, \Sigma) = \sum_{\Sigma', \Sigma_3'} C\left(\tfrac{1}{2}, \Sigma' ; \sigma, \Sigma_3' \mid J\, J_3\right)$$

$$\left[\frac{E_J + m_J - \alpha_3 p}{[2m_J(E_J + m_J)]^{1/2}}\right] \chi_\sigma(0) \cdot \delta_{\Sigma_3 \Sigma_3'} \times$$

$$\left[\Theta_{\Sigma\Sigma'} \left(\cosh\frac{\zeta}{2}\right)^{-(\Sigma+\Sigma')} \left(\sinh\frac{\zeta}{2}\right)^{\Sigma-\Sigma'} \times F\left(\Sigma_3 - \Sigma, 1 - \Sigma - \Sigma_3, 1 + \Sigma + \Sigma' ; -\sinh^2\frac{\zeta}{2}\right)\right] \tag{V.9}$$

where, $\varepsilon = \pm$ denotes the upper or lower branch of the mass-spectrum (for $\Sigma < \Sigma'$, replace $\zeta \rightarrow -\zeta$ and $\Sigma \leftrightarrow \Sigma'$).

In terms of the Jacobi-polynomials the expression (V.9) can be re-written as,

$$\psi_\varepsilon(p, J, J_3, \Sigma) = \sum_{\Sigma' \Sigma_3'} C\left(\tfrac{1}{2}, \Sigma' ; \sigma, \Sigma_3' \mid J, J_3\right) \times$$

$$\frac{(E_J + m_J - \alpha_3 p)\ \chi_\sigma(0)}{[2m_J(E_J + m_J)]^{1/2}} \cdot \delta_{\Sigma_3 \Sigma_3'} \times$$

$$\left[\frac{(\Sigma - |\Sigma_3|)!\ (\Sigma' + |\Sigma_3|)!}{(\Sigma + |\Sigma_3|)!\ (\Sigma' - |\Sigma_3|)!}\right]^{1/2} \times$$

$$\frac{\left(-\tanh\frac{\zeta}{2}\right)^{\Sigma' - \Sigma}}{(\cosh \zeta/2)^{2|\Sigma_3|+1}} P^{(\Sigma' - \Sigma,\, 2|\Sigma_3|)}_{\Sigma - |\Sigma_3|}\left(1/\cosh\zeta\right) \tag{V.10}$$

where $$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{s=0}^{n} \binom{n+\alpha}{s} \binom{n+\beta}{n-s} (x-1)^{n-s} (x+1)^{s}$$

or,

$$\psi_\varepsilon(p, J, J_3, \Sigma) = \sum_{\Sigma', \Sigma_3'} C\left(\tfrac{1}{2}, \Sigma'; \sigma, \Sigma_3' \mid J, J_3\right) \times$$

$$\left[\frac{(E_J + m_J - \alpha_3 p)\ \chi_\sigma(0)}{\left[2m_J (E_J + m_J)\right]^{1/2}}\right] \delta_{\Sigma_3 \Sigma_3'} \times$$

$$\left[\frac{(\Sigma - |\Sigma_3|)!\ (\Sigma' + |\Sigma_3|)!}{(\Sigma + |\Sigma_3|)!\ (\Sigma' - |\Sigma_3|)!}\right]^{1/2} \times$$

$$\left[\frac{(2m_J)^{-2|\Sigma_3|-1}\ (-p)}{(E_J + m_J)^{\Sigma' - \Sigma + 2|\Sigma_3| + 1}}\right] P_{\Sigma - |\Sigma_3|}^{(\Sigma' - \Sigma,\ 2|\Sigma_3|)}\left(m_J / E_J\right)$$

(V.11)

b) Space-like case

We label the spinor wave functions by $\tilde{J}$, J_3, and $\tilde{\Sigma}$. Then in an arbitrary frame, we have;

$$\psi_\varepsilon(p, \tilde{J}, J_3, \tilde{\Sigma}) = \chi_D(\underline{p}, \sigma, \varepsilon) \otimes \psi(\underline{p}, \tilde{\Sigma}, \Sigma_3, \varepsilon)$$

$$= \sum C\left(\tfrac{1}{2}, \tilde{\Sigma}; \sigma, \Sigma_3 \mid \tilde{J}, J_3\right) \chi_\sigma(\underline{p}, \varepsilon) \times \psi(\underline{p}, \tilde{\Sigma}, \Sigma_3)$$

(V.12)

As usual, we will assume $\underline{p} = p\, e_3$.

Then,

$$\psi_{\varepsilon}(p, \tilde{J}, J_3, \tilde{\Sigma}) = e^{i\zeta K_3}\ \chi_\sigma(0, \varepsilon) \otimes \psi(\tilde{\Sigma}, \Sigma_3, \varepsilon)$$

$$= \sum c(\tfrac{1}{2}, \tilde{\Sigma}; \sigma, \Sigma_3 | \tilde{J}\, J_3)\ \left[e^{i\zeta \cdot \tau_3/2} \chi_\sigma(0, \varepsilon)\right] \left[e^{i\zeta\Lambda_3} \psi(\tilde{\Sigma}, \Sigma_3, \varepsilon)\right], \tag{V.13}$$

where ζ = arc tanh $(E_{\tilde{J}}/p)$, and $(E_{\tilde{J}})^2 = \underline{p}^2 - m_\nu^2$; (V.14)

$$m_\nu^2 = 2 m_1^2 \nu^2 - \left[(m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right] \pm 2 m_1 \nu \left[m_1^2 \nu^2 - \left\{ (m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right\}\right]^{1/2}$$

$$\frac{1}{m_1^2} \left\{ (m_0 - m_1)^2 - \tfrac{1}{4} m_1^2 \right\} \leq \nu^2 < \infty \tag{V.15}$$

we have further, $$e^{i\zeta\tau_3/2} \chi_\sigma(0) = \left(\cosh \frac{\zeta}{2} - \alpha_3 \sinh \frac{\zeta}{2}\right) \chi_\sigma(0)$$

$$= \frac{(p + m_{\tilde{J}}) + \alpha_3 E_{\tilde{J}}}{\left[2 m_{\tilde{J}} (p + m_{\tilde{J}})\right]^{1/2}}\ \chi_\sigma(0) \tag{V.16}$$

and $$e^{i\zeta\Lambda_3} \psi(\tilde{\Sigma}, \Sigma_3) = V_{\tilde{\Sigma}, \tilde{\Sigma}'}(\zeta)\ \delta_{\Sigma_3 \Sigma_3'} \tag{V.17}$$

where $V_{\tilde{\Sigma}\tilde{\Sigma}'}(\zeta)$ are those given by Bargmann. Explicitly, they can be written as, (V.8) [We have to just replace $\Sigma, \Sigma' \to \tilde{\Sigma}, \tilde{\Sigma}'$ in (V.8)].

Thus, we obtain the expression for spinor wave functions as,

$$\psi_\varepsilon(p, \tilde{J}, J_3, \tilde{\Sigma}) = \sum_{\Sigma', \Sigma_3'} C(\tfrac{1}{2}, \tilde{\Sigma}'; \sigma, \Sigma_3' \mid \tilde{J}, J_3) \times$$

$$\left\{ \frac{(p + m_{\tilde{J}}) + \alpha_3 E_{\tilde{J}}}{[2m_{\tilde{J}}\,(p + m_{\tilde{J}})]^{1/2}} \right\} \chi_\sigma(0) \times$$

$$\left[\frac{(\tilde{\Sigma} - |\Sigma_3|)!\ (\tilde{\Sigma}' - |\Sigma_3|)!}{(\tilde{\Sigma} + |\Sigma_3|)!\ (\tilde{\Sigma}' - |\Sigma_3|)!} \right]^{1/2} \times$$

$$\left[\frac{(2m_{\tilde{J}})^{-2|\Sigma_3|-1}\ (-E_{\tilde{J}})^{\tilde{\Sigma}' - \tilde{\Sigma}}}{(p + m_{\tilde{J}})^{\tilde{\Sigma}' - \tilde{\Sigma} + 2|\Sigma_3| + 1}} \right] \times$$

$$P_{\tilde{\Sigma} - |\Sigma_3|}^{(\tilde{\Sigma}' - \tilde{\Sigma},\ 2|\Sigma_3|)} \left(\frac{m_{\tilde{J}}}{p} \right)$$

(V.18)

As usual $\varepsilon = \pm$ denoting the two branches of the mass-spectrum for each value of $(\tilde{\Sigma} + 1/2)$ or ν.

To summarize our results of this section, we explicitly constructed spinor wave-functions for the time-like and space-like solutions of our wave-equation. The spinors are orthonormalized as,

$$\Big(\psi_\varepsilon(p, J, J_3),\ \psi_{\varepsilon'}(p, J', J_3') \Big) = \left(\frac{E_J}{m_J} \right) \delta_{J_3 J_3'}\ \delta_{J J'}\ \delta_{\varepsilon \varepsilon'} \tag{V.19}$$

$$\Big(\psi_\varepsilon(p, \tilde{J}, J_3),\ \psi_{\varepsilon'}(p, \tilde{J}', J_3') \Big) = \left(\frac{E_{\tilde{J}}}{m_{\tilde{J}}} \right) \delta_{J_3 J_3'}\ \delta(\nu - \nu')\ \delta_{\varepsilon \varepsilon'}$$

(V.20)

[Note that these spinors have been normalized in the continuum] ;

and
$$\Big(\psi_\varepsilon(p, J, J_3),\ \psi_{\varepsilon'}(p, \tilde{J}, J_3) \Big) = 0 \tag{V.21}$$

We have to note further that each of the orthonormal conditions are separately satisfied by the $+ve$ and $-ve$ energy spinor wave-functions. As usual ε takes values $\varepsilon = \pm$ denoting the upper and lower branch of the mass-spectra.

Then, the completeness relation for any eigen-vector ψ in the reducible representation space of $SL(2,C) \otimes D \left[X = \sum_{\alpha} \int_{\oplus} d\tilde{\alpha} \{\alpha, \tilde{\alpha}\} \right]$ may be written as,

$$\psi(p) = \sum_{\alpha} \left(\bar{\psi}_{\alpha}(p), \psi(p) \right) \psi_{\alpha}(p) + \int_{-i\infty}^{+i\infty} d\tilde{\alpha} \left(\bar{\psi}_{\tilde{\alpha}}(p), \psi(p) \right) \psi_{\tilde{\alpha}}(p) \tag{V.22}$$

where α represents the total spin and takes the discrete values and $\tilde{\alpha}$ represents the continuous spin of the system.

VI. Quantization of the infinite-component generalized-fields:

As shown in Section II, the generalized-field equation,

$$(i \gamma_{\mu} \partial^{\mu} - M) \psi(x) = 0, \quad M = m_0 + \tfrac{1}{2} m_1 \sigma_{\mu\nu} \Gamma^{\mu\nu} \tag{VI.1}$$

follows from the Lagrangian density

$$\mathcal{L} = \bar{\psi}(x) \left(-i \gamma_{\mu} \partial^{\mu} + M \right) \psi(x)$$

with

$$\bar{\psi}(x) = \psi^{+}(x) \gamma_0 \tag{VI.2}$$

The equation conjugate to (V.1) is given by

$$\left[i \partial_{\mu} \bar{\psi} \gamma_{\mu} + \bar{\psi} M \right] = 0 \tag{VI.3}$$

Thus, we define a conserved density

$$J_{\mu}(x) = \bar{\psi}(x) \gamma_{\mu} \psi(x) \tag{VI.4}$$

The momentum conjugate to ψ is given by,

$$\pi_\sigma = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma} = i\,\psi_\sigma^+ \tag{VI.5}$$

Then, the Hamiltonian density $\mathcal{H}$ is obtained from (VI.1) and (VI.3) as,

$$\begin{aligned}\mathcal{H} &= \pi\dot{\psi} - \mathcal{L}\\ &= \psi^+(x)\,(-i\,\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla} + \gamma_0 M)\,\psi(x)\\ &= \psi^+(x)\,\left(i\,\frac{\partial}{\partial t}\right)\psi(x)\end{aligned} \tag{VI.6}$$

We can obtain the expressions for the energy-momentum four-vector, the generalized angular momentum tensor in the standard manner from the conservation principle, namely;

$$\underset{\sim}{P} = :\int d^3x\;\bar{\psi}\,\gamma_0\,(-i\,\underset{\sim}{\nabla})\,\psi: \tag{VI.7}$$

$$H = :\int \mathcal{H}(x)\,d^3x = :\int d^3x\;\psi^+\,(-i\,\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla} + M)\,\psi: \tag{VI.8}$$

$$J_{\mu\nu} = :\int d^3x\;\psi^+\left[\Gamma_{\mu\nu} + \tfrac{1}{2}\sigma_{\mu\nu} + i\,(x_\mu\partial_\nu - x_\nu\partial_\mu)\right]\psi: \tag{VI.9}$$

For the space-components of $J_{\mu\nu}$ in particular we have;

$$\underline{J} = (J_{23}, J_{31}, J_{12}) = :\int d^3x\;\psi^+\left[-i\,\vec{r}\times\vec{\nabla} + \vec{\Sigma} + \tfrac{1}{2}\vec{\sigma}\right]\psi: \tag{VI.10}$$

The double dots on both sides of the above expressions denote as usual that the normal ordered products are obtained by moving all destruction operators to the right. From (V.4) we obtain an additional conserved quantity the charge Q :

$$Q = :\int d^3x\ \Psi^{+}(x)\ \Psi(x): \tag{VI.11}$$

To establish the second quantized theory for the generalized fields, we make use of the relations obtained in Section V. We define the general solution of Eq. (VI.1) as,

$$\begin{aligned}\Psi_\varepsilon(x) = \sum_{J,J_3}\int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_J}{E_J}\right)^{1/2}\Big[&\exp\{-i(\underset{\sim}{p}\cdot\underset{\sim}{x}-\sqrt{\underset{\sim}{p}^2+m_J^2}\cdot t)\}\\ &\cdot u(\underset{\sim}{p},J,J_3,\varepsilon)\,b(\underset{\sim}{p},J,J_3,\varepsilon)\\ &+\exp\{i(\underset{\sim}{p}\cdot\underset{\sim}{x}-\sqrt{\underset{\sim}{p}^2+m_J^2}\cdot t)\}\cdot v(\underset{\sim}{p},J,J_3,\varepsilon)\,d^{+}(\underset{\sim}{p},J,J_3,\varepsilon)\Big]\\ +\sum_{J_3}\int d\nu\int\frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_\nu}{E_\nu}\right)^{\frac{1}{2}}\Big[&\exp\{i(\underset{\sim}{p}\cdot\underset{\sim}{x}-\sqrt{\underset{\sim}{p}^2-m_\nu^2}\cdot t)\}\cdot\\ &\cdot u(\underset{\sim}{p},\nu,J_3,\varepsilon)\,b(\underset{\sim}{p},\nu,J_3,\varepsilon)\\ &+\exp\{-i(\underset{\sim}{p}\cdot\underset{\sim}{x}-\sqrt{\underset{\sim}{p}^2-m_\nu^2}\cdot t)\}\cdot v(\underset{\sim}{p},\nu,J_3,\varepsilon)\,d^{+}(\underset{\sim}{p},\nu,J_3,\varepsilon)\Big]\end{aligned} \tag{V.12}$$

Similarly, the adjoint field $\overset{+}{\Psi}(x)$ is given by,

$$\begin{aligned}\overset{+}{\Psi}_\varepsilon(x) = \sum_{J,J_3}\int\frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_J}{E_J}\right)^{1/2}\Big[&\exp\{i(\underset{\sim}{p}\cdot\underset{\sim}{x}-\sqrt{\underset{\sim}{p}^2+m_J^2}\cdot t)\}\times\\ &u^{+}(\underset{\sim}{p},J,J_3,\varepsilon)\,b^{+}(\underset{\sim}{p},J,J_3,\varepsilon)\\ &+\exp\{-i(\underset{\sim}{p}\cdot\underset{\sim}{x}-\sqrt{\underset{\sim}{p}^2+m_J^2}\cdot t)\}\times\\ &v^{+}(\underset{\sim}{p},J,J_3,\varepsilon)\,d(\underset{\sim}{p},J,J_3,\varepsilon)\Big]+\end{aligned}$$

$$+ \sum_{J_3} \int d\nu \int \frac{d^3p}{(2\pi)^{3/2}} \cdot \left(\frac{m_\nu}{E_\nu}\right)^{\frac{1}{2}} \Big[\exp\{i\,(\underset{\sim}{p}\cdot\underset{\sim}{x} - \sqrt{\underset{\sim}{p}^2 - m_\nu^2}\cdot t)\}$$

$$u^+(\underset{\sim}{p},\nu,J_3,\varepsilon)\, b^+(\underset{\sim}{p},\nu,J_3,\varepsilon)$$

$$+ \exp\{-i\,(\underset{\sim}{p}\cdot\underset{\sim}{x} - \sqrt{\underset{\sim}{p}^2 - m_\nu^2}\cdot t)\} \times$$

$$v^+(\underset{\sim}{p},\nu,J_3,\varepsilon)\, d\,(\underset{\sim}{p},\nu,J_3,\varepsilon)\Big]$$

(VI.13)

with $\qquad E_J = p_J^0 = (\underset{\sim}{p}^2 + m_J^2)^{1/2}$

and $\qquad (E_{\tilde{J}})^2 = (\underset{\sim}{p}^2 + m_{\tilde{J}}^2)$

or $\qquad E_\nu^2 = (\underset{\sim}{p}^2 - m_\nu^2)$

We then postulate the canonical anticommutation relations between b_α, b_α^+, d_α and d_α^+ as

$$[b(\underset{\sim}{p},J,J_3,\varepsilon),\ b^+(\underset{\sim}{p}',J',J_3',\varepsilon')]_+ = \delta^3(\underset{\sim}{p}-\underset{\sim}{p}')\,\delta_{\varepsilon\varepsilon'}\,\delta_{JJ'}\,\delta_{J_3J_3'}$$

$$[d(\underset{\sim}{p},J,J_3,\varepsilon),\ d^+(\underset{\sim}{p}',J',J_3',\varepsilon')]_+ = \delta^3(\underset{\sim}{p}-\underset{\sim}{p}')\,\delta_{\varepsilon\varepsilon'}\,\delta_{JJ'}\,\delta_{J_3J_3'}$$

$$[b(\underset{\sim}{p},\nu,J_3,\varepsilon),\ b^+(\underset{\sim}{p}',\nu',J_3',\varepsilon')]_+ = \delta^3(\underset{\sim}{p}-\underset{\sim}{p}')\,\delta_{\varepsilon\varepsilon'}\,\delta(\nu-\nu')\,\delta_{J_3J_3'}$$

$$[d(\underset{\sim}{p},\nu,J_3,\varepsilon),\ d^+(\underset{\sim}{p}',\nu',J_3',\varepsilon')]_+ = \delta^3(\underset{\sim}{p}-\underset{\sim}{p}')\,\delta_{\varepsilon\varepsilon'}\,\delta(\nu-\nu')\,\delta_{J_3J_3'}$$

(VI.14)

and all other commutators vanish. From the relations (VI.14) and the completeness relation (V.22), we obtain the local commutation relations for the generalized fields as:

$$[\Psi_\alpha(t, \underset{\sim}{x}), \Psi_\beta^+(t, \underset{\sim}{x}')]_+ = \delta^3(\underset{\sim}{x} - \underset{\sim}{x}') \cdot \delta_{\alpha\beta} \qquad \text{(VI.15)}$$

As observed by some authors, since the 'spectral conditions' are no more true for generalized infinite-component fields, we have constructed a local field $\Psi(x)$, which annihilates the vacuum.[2] We have to note further that contrary to earlier observations, our fields constructed in the above fashion are local. This is born out of the fact that our mass spectra is no more bounded from below. We just have an accumulation point at the minima.

Another interesting feature is the expression for the charge Q. We can after a little manipulation derive the charge-operator as,

$$\begin{aligned} Q &= \int d^3x : \Psi^+ \Psi : \\ &= \sum_{J, J_3} \int d^3p : [b^+(p, J, J_3, \varepsilon)\, b(p, J, J_3, \varepsilon) \\ &\qquad + d^+(p, J, J_3, \varepsilon)\, d(p, J, J_3, \varepsilon)] : \\ &\quad + \sum_{J_3} \int d\nu \int d^3p : [b^+(p, \nu, J_3, \varepsilon)\, b(p, \nu, J_3, \varepsilon) \\ &\qquad + d^+(p, \nu, J_3, \varepsilon)\, d(p, \nu, J_3, \varepsilon)] : \end{aligned} \qquad \text{(VI.16)}$$

Better still, we can express Q in terms of particle number-operators.

Define

$$\begin{aligned} N_+(p, \alpha, J_3, \varepsilon) &= b^+(p, \alpha, J_3, \varepsilon)\, b(p, \alpha, J_3, \varepsilon) \\ N_-(p, \alpha, J_3, \varepsilon) &= d^+(p, \alpha, J_3, \varepsilon)\, d(p, \alpha, J_3, \varepsilon) \end{aligned} \qquad \text{(VI.17)}$$

where 'α' denotes either J or ν. Then

$$Q = \sum_{J, J_3} \int d^3p \left[N_+(p, J, J_3, \varepsilon) - N_-(p, J, J_3, \varepsilon) \right]$$

$$+ \sum_{J_3} \int d\nu \int d^3p \left[N_+(p, \nu, J_3, \varepsilon) - N_-(p, \nu, J_3, \varepsilon) \right] \qquad \text{(VI.18)}$$

N_+ and N_- are respectively interpreted as number operators for +ve energy particles and antiparticles respectively. Note further that in the case $m_1 = 0$, the fields only contain the time-like parts and correspondingly the charge Q has the first terms in the r.h.s. of Eq. (VI.18).

We would like to bring out some salient features of our fields. In the general solution of (VI.1), the fields $\Psi(x)$ contain both the +ve and -ve frequency solutions which in turn are associated with annihilation and creation operators for particles and antiparticles respectively. We have explicitly displayed the +ve frequency solutions in Section V. To obtain the -ve frequency solutions, we just have to replace $\Psi(x)$ by $\gamma_5 \Psi(x)$. Thus the fields $\Psi(x)$ explicitly contain $\Psi(x)$ and $\gamma_5 \Psi(x)$ parts. This is very similar to the familiar pure Dirac fields. In either case we note, the S-principle is automatically satisfied[10]. Contrary to the pure Dirac fields or the generalized Dirac fields of our present discussion, the Majorana fields $\Psi(x)$ contain only the annihilation operators. Hence, it necessitates the introduction of the conjugate fields with the creation operators. However, the quantized fields so constructed do not possess any symmetry between the +ve and -ve frequency solutions. To redress this difficulty, one rather demands that (i) $\Psi(x)$ and $\Psi^{+T}(x)$ to be treated on the same footing; (ii) the action is invariant under the interchange of $\Psi(x)$ and $V\Psi^{+T}(x)$, where $V = \exp(i\pi J_2)$. By constructing the fields in the above manner, one then resotres the usual spin-statistics relation (for a detailed discussion vide references (1) and (10)).

We make no secret of the fact that we have been able to formulate the second quantized theory of the infinite- component fermi fields in accordance with the substitution principle and satisfying the usual spin-statistics relations. The pathologies diagonised by earlier work[8], have been redressed in our present discussion. We note further that an identical procedure can be carried out for the quantization of the infinite-component Bose -fields.

VII. CONCLUSIONS:

To conclude our discussions, we constructed quantum theory of the infinite-component generalized fields satisfying local commutativity. Since the fields explicitly contain particle and anti-particle solutions, the conventional TCP invariance is also preserved! Like other infinite-component field theories, our field equation possesses time-like and space-like solutions. The former gives rise to discrete spin-spectra and the latter to continuous-spin spectra for the masses. The continuous-spin spectrum which is rather a peculiar characteristic of infinite-component field theories satisfying linear covariant field equations, gives rise to an entirely new kind of radiation involving 'space-like-particles.[1,10] The special features of this phenomena has been professed by Sudarshan and the possible implications have been also discussed from the point of view of finite-component field theories.[10]

Another feature of the mass-spectrum is that, for each value of the 'spin', there are two values for the masses. One is an ascending-branch and the other asymptotically goes to zero. In fact, the former one is rather well-coming for the hadron-spectra. To be more optimistic, the two branches of the discrete spectra can be interpreted as the mass-spectrum of the 'electron' and the 'muon' by cleverly adjusting the parameters m_o and m_1.

Finally, we believe that such a formulation of the field equation has some added advantage over the pure Majorana wave-equation. Some more interesting cases of field equations and a systematic study of their solutions and quantization schemes will be reported elsewhere.

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APPENDIX - A

a) Time-like Case

We wish to derive the expression for the mass-matrix

$$\mathcal{M} = \begin{pmatrix} m_0 - m_1 (J+3/2) & -i m_1 [J(J+1)]^{1/2} \\ \\ i m_1 [J(J+1)]^{1/2} & -m_0 - m_1 (J-1/2) \end{pmatrix}$$

(IV.4)

Let us consider the mass operator

$$\mathcal{M} = m_0 \gamma_0 + m_1 \gamma_0 \left(\underline{\sigma} \cdot \underline{\Sigma} - \underline{\tau} \cdot \underline{\Lambda} \right) \qquad \text{(IV.3)}$$

We have seen that the basis vectors can be labelled by the eigenvalues of γ_0, σ_3, $\underline{\Sigma}^2$ and Σ_3 or alternatively by $|\beta, J, \lambda, \Sigma\rangle$, where β, J, λ, are the eigenvalues of γ_0, the total spin $\underline{J}^2 = (\underline{\Sigma} + \underline{\sigma}/2)^2$ and $J_3 = (\frac{1}{2}\sigma_3 + \Sigma_3)$ respectively. We note further that, since γ_0 does not commute with $\mathcal{M}$, these do not furnish the eigenstates of mass. To write down the mass-matrix, we find that since for a unitary representation $\underline{\Lambda}$ behaves like a vector under $\underline{\Sigma}$, $\underline{\tau} \cdot \underline{\Lambda}$ term in the mass-matrix will contribute to the off-diagonal matrix elements whereas $\underline{\sigma} \cdot \underline{\Sigma}$ will contribute for the diagonal ones.

We have, $\qquad \underline{J} = \underline{\Sigma} + \underline{\sigma}/2$

squaring both sides, we get

$$\underline{J}^2 = J(J+1) = \underline{\Sigma}^2 + 3/4 + \underline{\sigma} \cdot \underline{\Sigma}$$

or

$$\underline{\sigma} \cdot \underline{\Sigma} = J(J+1) - \Sigma(\Sigma+1) - 3/4 \qquad \text{(A.1)}$$

For

$$\Sigma = J + \tfrac{1}{2}, \quad \underline{\sigma} \cdot \underline{\Sigma} = -(J + 3/2)$$

$$= J - 1/2, \quad \underline{\sigma} \cdot \underline{\Sigma} = J - 1/2$$

Now,

$$(\underline{\tau}\cdot\underline{\Lambda})(\underline{\tau}\cdot\underline{\Lambda}) = -(\underline{\alpha}\cdot\underline{\Lambda})(\underline{\alpha}\cdot\underline{\Lambda})$$

$$= -\underline{\Lambda}^2 - \underline{\sigma}\cdot\underline{\Sigma}$$

$$= -\underline{\Lambda}^2 - (\underline{J}^2 - \underline{\Sigma}^2 - 3/4)$$

$$= (\underline{\Sigma}^2 - \underline{\Lambda}^2) - \underline{J}^2 + 3/4 = -\underline{J}^2 = -J(J+1) \tag{A.2}$$

Thus,

$$\mathfrak{M} = \begin{pmatrix} m_0 - m_1(J-3/2) & -im_1[J(J+1)]^{1/2} \\ im_1[J(J+1)]^{1/2} & -m_0 - m_1(J-1/2) \end{pmatrix}$$

b) <u>Space-like Case</u>

In this case, the basic vectors are labelled by γ_3, $\tilde{J}$, λ, $\tilde{\Sigma}$, where, γ_3, $\tilde{J}$, λ, $\tilde{\Sigma}$ are the eigenvalues of γ_3, $\underline{\tilde{J}} = (\tilde{\underline{\Sigma}} + \tilde{\underline{\sigma}}/2)^2$, $J_3 = (\Sigma_3 + \sigma_3/2)$ and $\mathcal{C} = (\Sigma_3{}^2 - \Lambda_1{}^2 - \Lambda_2{}^2) = \tilde{\underline{\Sigma}}^2$ respectively.

Defining
$$J_3 = \Sigma_3 + \sigma_3/2,$$
$$K_1 = \Lambda_1 + \tau_1/2$$
$$K_2 = \Lambda_2 + \tau_2/2$$

We find the expression for the second order Casimir operator

$$\begin{aligned} Q &= J_3{}^2 - K_1{}^2 - K_2{}^2 \\ &= (\Sigma_3 + \tfrac{1}{2}\sigma_3)^2 - (\Lambda_1 + \tau_1/2)^2 - (\Lambda_2 + \tau_2/2)^2 \\ &= (\Sigma_3{}^2 - \Lambda_1{}^2 - \Lambda_2{}^2) + \tilde{\underline{\sigma}}\cdot\tilde{\underline{\Sigma}} + 3/4 \\ &= \mathcal{C} + 3/4 + \tilde{\underline{\sigma}}\cdot\tilde{\underline{\Sigma}} \end{aligned} \tag{A.3}$$

where $\underset{\sim}{\tilde{\sigma}} = (\sigma_3, \tau_1, \tau_2)$, $\underset{\sim}{\tilde{\tau}} = (\tau_3, \sigma_1, \sigma_2)$

$\underset{\sim}{\tilde{\Sigma}} = (\Sigma_3, \Lambda_1, \Lambda_2)$, $\underset{\sim}{\tilde{\Lambda}} = (\Lambda_3, \Sigma_1, \Sigma_2)$

Note uder $\underset{\sim}{\tilde{\Sigma}}$ and $\underset{\sim}{\tilde{\sigma}}$, $\underset{\sim}{\tilde{\Lambda}}$ and $\underset{\sim}{\tilde{\tau}}$ transform respectively like vectors. Further, we have,

$$\left(\underset{\sim}{\tilde{\sigma}} \cdot \underset{\sim}{\tilde{\Sigma}} - \underset{\sim}{\tilde{\tau}} \cdot \underset{\sim}{\tilde{\Lambda}}\right) = \left(\sigma_3 \Sigma_3 - \tau_1 \Lambda_1 - \tau_2 \Lambda_2\right) - \left(\tau_3 \Lambda_3 - \sigma_1 \Sigma_1 - \sigma_2 \Sigma_2\right)$$

$$= \left(\underset{\sim}{\sigma} \cdot \underset{\sim}{\Sigma} - \underset{\sim}{\tau} \cdot \underset{\sim}{\Lambda}\right)$$

as it should.

From (A.3), we have

$$\left(Q - \mathcal{C} - 3/4\right) = \underset{\sim}{\tilde{\sigma}} \cdot \underset{\sim}{\tilde{\Sigma}} \tag{A.4}$$

squaring both sides of (A.4) and after simplification, we obtain,

$$(Q - \mathcal{C} - 3/4)^2 = 2\mathcal{C} - Q + 3/4$$

or, solving for Q_1

$$Q = (\mathcal{C} + \tfrac{1}{4}) \pm \left(\mathcal{C} + \tfrac{1}{4}\right)^{1/2} \tag{A.5}$$

Further, from (A.3) we have,

$$\underset{\sim}{\tilde{\sigma}} \cdot \underset{\sim}{\tilde{\Sigma}} = Q - \mathcal{C} - 3/4$$

Substituting for Q, we obtain,

$$\underset{\sim}{\tilde{\sigma}} \cdot \underset{\sim}{\tilde{\Sigma}} = -\frac{1}{2} \pm \left(\mathcal{C} + \frac{1}{4}\right)^{1/2}$$

$$\text{or} \quad \left(\frac{1}{2} + \underset{\sim}{\tilde{\sigma}} \cdot \underset{\sim}{\tilde{\Sigma}}\right) = \pm \left(\mathcal{C} + \frac{1}{4}\right)^{1/2} \tag{A.6}$$

again,

$$\left(\underset{\sim}{\tilde{\tau}} \cdot \underset{\sim}{\tilde{\Lambda}}\right)\left(\underset{\sim}{\tilde{\tau}} \cdot \underset{\sim}{\tilde{\Lambda}}\right) = \left(\Sigma_1^2 + \Sigma_2^2 - \Lambda_3^2\right) - \underset{\sim}{\tilde{\sigma}} \cdot \underset{\sim}{\tilde{\Sigma}}$$

$$= \left(-3/4 - \mathcal{C}\right) - \left(Q - \mathcal{C} - 3/4\right)$$

$$\left[\because\ C_0 = -3/4 = \underset{\sim}{\Sigma}^2 - \underset{\sim}{\Lambda}^2 = \left(\Sigma_3^2 - \Lambda_1^2 - \Lambda_2^2\right) - \left(\Lambda_3^2 - \Sigma_1^2 - \Sigma_2^2\right)\right]$$

$$\text{or} \quad \left(\underset{\sim}{\tilde{\tau}} \cdot \underset{\sim}{\tilde{\Lambda}}\right)\left(\underset{\sim}{\tilde{\tau}} \cdot \underset{\sim}{\tilde{\Lambda}}\right) = -Q \tag{A.7}$$

$$\therefore \quad \left(-\tfrac{1}{2} + \tilde{\underset{=}{\Sigma}}\cdot\tilde{\underset{-}{\Delta}}\right)\left(-\tfrac{1}{2} - \tilde{\underset{=}{\Sigma}}\cdot\tilde{\underset{-}{\Delta}}\right) = \tfrac{1}{4} - (\tilde{\underset{=}{\Sigma}}\cdot\tilde{\underset{-}{\Delta}})(\tilde{\underset{=}{\Sigma}}\cdot\tilde{\underset{-}{\Delta}})$$

$$= \tfrac{1}{4} + Q$$

$$= \left(\sqrt{Q+\tfrac{1}{4}} \pm \tfrac{1}{2}\right)^2 \qquad \text{(A.8)}$$

Note further that ($\frac{1}{2} + \tilde{\underset{-}{\sigma}}\cdot\tilde{\underset{=}{\Sigma}}$) contributes to the diagonal elements of the mass-matrix, where as, ($-\frac{1}{2} + \tilde{\underset{=}{\Sigma}}\cdot\tilde{\underset{-}{\Delta}}$) to the off-diagonal ones. Thus the matrix $\tilde{\mathfrak{m}}$ can be written as,

$$\tilde{\mathfrak{m}} = \begin{pmatrix} i\,(m_0 - m_1) + i m_1 \left(Q + \frac{1}{4}\right)^{\frac{1}{2}} & i m_1 \left(\sqrt{Q+\frac{1}{4}} - \frac{1}{2}\right) \\ -i m_1 \left(\sqrt{Q+\frac{1}{4}} + \frac{1}{2}\right) & -i\,(m_0 - m_1) + i m_1 \left(Q + \frac{1}{4}\right)^{\frac{1}{2}} \end{pmatrix}$$

—